A TOOL TO ANALYZE ROBUST STABILITY FOR CONSTRAINED MPC

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Abstract: A sufficient condition for robust stability of nonlinear constrained Model Predictive Control (MPC) with respect to plant/model mismatch is derived. This work is an extension of a previous study on the unconstrained nonlinear MPC problem, and is based on Nonlinear Programming sensitivity concepts. It addresses the discrete time state feedback problem with all states measured. A strategy to estimate bounds on the plant/model mismatch is proposed, that can be used off-line as a tool to assess the extent of model mismatch that can be tolerated to guarantee robust stability.

Keywords: Model-based control, Modeling errors, Nonlinear models, Nonlinear programming, Predictive control, Robust stability, Sensitivity analysis

1. INTRODUCTION

A prominent aspect of the research in the nonlinear Model Predictive Control (MPC) field is the development of a theoretical analysis framework to study the stability and robustness properties of the closed loop system in the presence of disturbances and modeling errors. A broad review by Mayne et al. (2000) on constrained MPC points out that while research on stability has reached a relatively mature stage, further research is required to develop implementable robust MPC for nonlinear systems.

In this work we develop a framework that can be used to evaluate off-line, the closed-loop robustness of a system with constrained MPC in the presence of plant/model mismatch. It is a direct extension of a previous work on the unconstrained case by Santos and Biegler (1999) for the discrete state feedback problem. Both the plant and model are simulated using nonlinear state space models.

The paper is organized as follows. Section 2 is devoted to preliminary definitions and assumptions on the modeling errors, and to a brief description of the MPC problem. In Section 3, we analyze the convergence of the optimal control problem solution for both the perfect and model mismatch cases, by exploiting the properties of the exact penalty function, and we establish a sufficient condition for robust stability. In Section 4, using nonlinear programming sensitivity concepts, we characterize this sufficient condition for the MPC problem with a general cost function. We further detail this characterization for the case of a quadratic cost function, and we obtain a bound on the plant/model uncertainty. This bound can be estimated through off-line calculations using a procedure that constitutes a tool to analyze robust stability for constrained MPC. These results are illustrated in Section 5 with a simple example. Finally, concluding remarks regarding the analysis of conditions for robust stability of

MPC in the presence of plant/model mismatch are given in Section 6.

2. DEFINITIONS AND NOTATION

For this study we treat only the state feedback case and assume that at every time index k all the states can be measured accurately. We assume the state dynamics of the plant are described by the following nonlinear, continuous-time set of equations:

$$\dot{x}^{\mathrm{p}} = \boldsymbol{f}^{\mathrm{p}}(x^{\mathrm{p}}, u^{\mathrm{p}}) , \qquad (1)$$

where $x^{\mathrm{p}} \in \mathbf{R}^{n_{\mathrm{s}}}$ is the vector of states and $u^{\mathrm{p}} \in \mathbf{R}^{n_{\mathrm{m}}}$ is the vector of inputs, with $\boldsymbol{f}^{\mathrm{p}} : \mathbf{R}^{n_{\mathrm{s}}} \times \mathbf{R}^{n_{\mathrm{m}}} \rightarrow \mathbf{R}^{n_{\mathrm{s}}}$.

The stationary discrete-time counterpart of (1) is

$$x^{\mathrm{p}}_{k+1} = f^{\mathrm{p}}(\Delta t; x^{\mathrm{p}}_k, u^{\mathrm{p}}_k) , \qquad (2)$$

where Δt is the sampling period and $f^{\mathrm{p}} : \mathbf{R}^{n_{\mathrm{s}}} \times \mathbf{R}^{n_{\mathrm{m}}} \rightarrow \mathbf{R}^{n_{\mathrm{s}}}$. We will drop the Δt for convenience. A model with the same dimension as (2) is considered for the MPC framework:

$$x_{k+1} = f(x_k, u_k) , \qquad (3)$$

where $x_k \in \mathbf{R}^{n_{\mathrm{s}}}$ is the vector of *nominal* states, u_k is the same vector of inputs as in (2), with $f : \mathbf{R}^{n_{\mathrm{s}}} \times \mathbf{R}^{n_{\mathrm{m}}} \rightarrow \mathbf{R}^{n_{\mathrm{s}}}$. We consider $(x^{\mathrm{p}}_k, u^{\mathrm{p}}_k) = (x_k, u_k) = (0, 0)$ the point at which both the plant and the model operate at steady state, such that $f(0, 0) = f^{\mathrm{p}}(0, 0) = 0$.

As in Keerthi and Gilbert (1988) we also apply the definition of a function belonging to class $\mathcal{K}_\infty$, along with related assumptions. A function $W(r) : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$, $r \in \mathbf{R}_+$, belongs to class $\mathcal{K}_\infty$ if: a)it is continuous; b)$W(r) = 0 \Leftrightarrow r = 0$; c)it is nondecreasing; d)$W(r) \longrightarrow \infty$ when $r \longrightarrow \infty$. We define $\|\cdot\|$ as the Euclidean norm and assume there exists a modeling bound function $W_{\mathrm{m}}(\|x\|) \in \mathcal{K}_\infty$ such that $\left\|f^{\mathrm{p}}(x^{\mathrm{p}}_k, u^{\mathrm{p}}_k) - f(x_k, u_k)\right\| \leq W_m(\|x_k\|)$, and positive constants K_{m} and γ such that

$$W_{\mathrm{m}}(\|x_k\|) = K_{\mathrm{m}} \|x_k\|^\gamma . \qquad (4)$$

The MPC problem minimizes

$$\Psi(x_i, \boldsymbol{s}_i) = \sum_{k=i}^{i+p-1} h(x_k, u_k) + h(x_{i+p}) , \qquad (5)$$

where $\Psi : \mathbf{R}^{n_{\mathrm{s}}} \times \mathbf{R}^{n_{\mathrm{m}}} \rightarrow \mathbf{R}$, $\Psi(0, 0) = 0$. Here $h(x, u) \in \mathcal{K}_\infty$ is a general cost function, x_i is the initial state vector at the time index i, $i \geq 0$, and $\boldsymbol{s}_i$ is the solution vector over the predictive horizon, given by

$$\boldsymbol{s}_i^{\mathrm{T}} = \left[s_i^{\mathrm{T}}\ s_{i+1}^{\mathrm{T}}\ \cdots\ s_{i+p}^{\mathrm{T}}\right] , \qquad (6)$$

where $s_{i+k}^{\mathrm{T}} = \left[x_{i+k}^{\mathrm{T}}\ u_{i+k}^{\mathrm{T}}\right]$, $k = 0, 1, \cdots, p$. This formulation allows a shorter input horizon m, with $m \leq p$ and $u_k = u_{i+m-1}, k = i + m, \ldots, i + p$. Traditionally, the decision variables of the MPC problem are the control profiles. In the optimization framework used in this study the state profiles are decision variables as well. It uses a multiple shooting method to solve (3) over the predictive horizon (e.g., Santos et al. (1995); Santos (2001)). State and control constraints over this horizon are included in the MPC formulation, set as lower and upper bounds – subscripts $_{\mathrm{L}}$ and $_{\mathrm{U}}$ –,

$$b(x_k) = \begin{bmatrix} x_k - x_{\mathrm{U}\,k} \\ -x_k + x_{\mathrm{L}\,k} \end{bmatrix} \leq 0 , \qquad (7)$$

with $k = i + 1, \ldots, i + p$, and

$$b(u_k) = \begin{bmatrix} u_k - u_{\mathrm{U}\,k} \\ -u_k + u_{\mathrm{L}\,k} \end{bmatrix} \leq 0 , \qquad (8)$$

with $k = i, \ldots, i + m - 1$. We define the vector of inequality constraints of the problem at i as

$$\boldsymbol{b}(\boldsymbol{s}_i)^{\mathrm{T}} = \Big[b(x_{i+1})^{\mathrm{T}} \cdots\ b(x_{i+p-1})^{\mathrm{T}}\ b(u_i)^{\mathrm{T}} \cdots\ b(u_{i+m-1})^{\mathrm{T}}\Big] . \qquad (9)$$

Finally, we impose terminal state constraints $x_{i+p} = 0$, or if we allow $p \rightarrow \infty$ then this constraint is automatically satisfied for a finite value of (5).

We denote by $\mathcal{P}(x_i)$ the MPC problem solved at every time index i, $i \geq 0$, given by

$$\min_{\boldsymbol{s}_i} \Psi(x_i, \boldsymbol{s}_i) \qquad (10)$$

$$\text{s.t.} \quad \boldsymbol{c}(x_i, \boldsymbol{s}_i) = 0 \qquad (11)$$

$$\boldsymbol{b}(\boldsymbol{s}_i) \leq 0 , \qquad (12)$$

$$\text{where} \quad \boldsymbol{c}(x_i, \boldsymbol{s}_i) = \begin{bmatrix} x_{k+1} - f(x_k, u_k) , \\ k = i, \ldots, i + p - 1 \\ x_{i+p} \end{bmatrix} ,$$

with optional constraints added for a shorter input horizon, $m \leq p$. We assume in this analysis that $\boldsymbol{s}_i$ is a feasible solution for (10–12) and that there exists a sufficiently long (and possibly infinite) horizon that insures an admissible trajectory to satisfy the terminal state constraints and (12).

3. STABILITY ANALYSIS

To extend the analysis made for the unconstrained case (Santos and Biegler, 1999) to (10 – 12) we use an exact penalty formulation as developed by Oliveira and Biegler (1994). This approach converts (10 – 12) to the problem $\mathcal{P}_\rho(x_i)$:

$$\min_{\boldsymbol{s}_i} \Upsilon(x_i, \boldsymbol{s}_i, \rho_i) \qquad (13)$$

$$\text{s.t.} \quad \boldsymbol{c}(x_i, \boldsymbol{s}_i) = 0 , \qquad (14)$$

$$\text{with} \quad \Upsilon(x_i, \boldsymbol{s}_i, \rho_i) = \Psi(x_i, \boldsymbol{s}_i) + P(\boldsymbol{s}_i, \rho_i) , \qquad (15)$$

$\Upsilon : \mathbf{R}^{n_{\mathrm{s}}} \times \mathbf{R}^{n_{\mathrm{m}}} \rightarrow \mathbf{R}$, $\Upsilon(0, 0, 0) = 0$, where

$$P(\boldsymbol{s}_i, \rho_i) = \rho_i \cdot \left\{ \sum_{k=i+1}^{i+p-1} \max\{0, b(x_k)\} + \sum_{k=i}^{i+m-1} \max\{0, b(u_k)\} \right\} , \quad (16)$$

and ρ_i is the penalty parameter. We remark that $P(\boldsymbol{s}_i, \rho_i)$ is bounded from below by zero as well. An important property that motivates the use of the exact penalty function, is that a sufficient condition to recover the original optimal constrained solution, $\boldsymbol{s}_i^*$, is to have a finite penalty parameter with $\rho_i > \|\omega_i^*\|_\infty$, where ω_i^* is the vector of the Lagrange multipliers associated to the inequality constraints from the original problem (Fletcher, 1987). Thus this condition on ρ_i ensures that the control and state profiles do not exceed the region delimited by (7) and (8) over p. We will assume that the parameter ρ_i can be chosen in advance to be sufficiently large, i. e.,

$$\rho \geq \max\{\rho_i\} \, . \quad (17)$$

Note that if ρ_i cannot be bounded, then $\mathcal{P}(x_i)$ has no feasible solution. Of course, feasible solutions of $\mathcal{P}(x_i)$ cannot be guaranteed and for this reason, a 'reasonable' value can be chosen for ρ so that solutions of $\mathcal{P}_\rho(x_i)$ can be considered even if they cannot always satisfy the bound constraints. To simplify the notation we set

$$\Upsilon^*(x_i) = \Upsilon(x_i, \boldsymbol{s}_i^*, \rho) \, . \quad (18)$$

3.1 *Perfect model case*

The essence of our stability analysis follows from familiar arguments developed by Muske and Rawlings (see Mayne et al. (2000)). We first consider the case where the model is perfect and there is no source of disturbances. From the assumptions stated in Section 2, the solution of $\mathcal{P}_\rho(x_i)$ satisfies $(x_k, u_k) = (0, 0)$ for $k \geq i + p$. Hence the locally optimal solution gives

$$\Upsilon^*(x_i) = \sum_{k=i}^{i+p-1} h(x_k^*, u_k^*) + \underbrace{h(x_{i+p}^*)}_{=0} + P(\boldsymbol{s}_i^*, \rho) \, . \quad (19)$$

Note that we assume the point $(x_k^{\mathrm{p}}, u_k^{\mathrm{p}}) = (x_k, u_k) = (0, 0)$ is within the state and control bound constraints.

Consider now the problem at the next time index, $\mathcal{P}_\rho(x_{i+1})$. Because the model is perfect and there is no source of disturbances, the resulting optimal sequence of $\mathcal{P}_\rho(x_i)$ is a feasible solution for $\mathcal{P}_\rho(x_{i+1})$. Moreover, the objective function at the solution of $\mathcal{P}_\rho(x_{i+1})$ can be no greater than the solution $\mathcal{P}_\rho(x_i)$; the solution of $\mathcal{P}_\rho(x_{i+1})$ can not be worse because now the terminal constraint is only enforced one interval ahead. Therefore

$$\Upsilon^*(x_i) - \Upsilon^*(x_{i+1}) \geq h(x_i, u_i^*, \rho) \, , \quad (20)$$

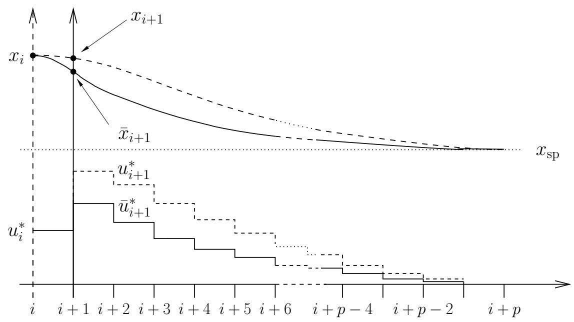

Fig. 1. *Plant/Model state trajectory mismatch.*

where $h(x_i, u_i^*, \rho) = h(x_i, u_i^*) + \rho \cdot \max\{0, b(x_i)\} + \rho \cdot \max\{0, b(u_i^*)\}$. Note also that $h(x_i, u_i^*, \rho) \in \mathcal{K}_\infty$. Thus the sequence $\{\Upsilon^*(x_i)\}$ over N time indices decreases, and because (5) and (16) are bounded from below by zero it converges. Taking the sum of (20) over N we obtain

$$\Upsilon^*(x_1) - \Upsilon^*(x_{N+1}) = \sum_{i=1}^{N} \big(\Upsilon^*(x_i) - \Upsilon^*(x_{i+1})\big) \geq \sum_{i=1}^{N} h(x_i, u_i^*, \rho) \, . \quad (21)$$

Also, because $\{\Upsilon^*(x_i)\}$ is decreasing, then as $N \to \infty$, $h(x_i, u_i^*, \rho) \to 0$ and $x_i \to 0$.

3.2 *Model mismatch case*

Consider now the case with plant/model mismatch. Again, suppose that the solution of $\mathcal{P}_\rho(x_i)$ gives (19). Now to solve the problem at time index $i + 1$ there are available two initial state conditions to solve (3). One is the prediction made at time index i for $i + 1$, $\bar{x}_{i+1}$ from (19), and the other one is defined by the state measurements at $i + 1$, x_{i+1} from (2). This leads to two MPC problems we denote here by $\mathcal{P}_\rho(\bar{x}_{i+1})$ and $\mathcal{P}_\rho(x_{i+1})$, respectively. Note that both problems are solved with the same model, and the difference between their solutions reflects the degree of plant/model mismatch – Figure 1:

- $\mathcal{P}_\rho(\bar{x}_{i+1})$ – using ρ and $\bar{x}_{i+1}$, we obtain:

$$\Upsilon^*(\bar{x}_{i+1}) = \sum_{k=i+1}^{i+p} h(\bar{x}_k^*, \bar{u}_k^*) + \underbrace{h(\bar{x}_{i+p+1}^*)}_{=0} + P(\bar{\boldsymbol{s}}_{i+1}^*, \rho) \, . \quad (22)$$

  From the perfect model case we assume that ρ is large enough in order to obtain feasible solutions to $\mathcal{P}(\bar{x}_{i+1})$ if they exist. Thus the arguments for the perfect model case are also valid for this case.

- $\mathcal{P}_\rho(x_{i+1})$ – using ρ and x_{i+1}, we obtain:

$$\Upsilon^*(x_{i+1}) = \sum_{k=i+1}^{i+p} h(x_k^*, u_k^*) + \underbrace{h(x_{i+p+1}^*)}_{=0}$$

$$+\ P(\boldsymbol{s}^*_{i+1},\rho) = \sum_{k=i+1}^{i+p+1} h(x^*_k, u^*_k, \rho)\,. \tag{23}$$

Since x_{i+1} can be different from $\bar{x}_{i+1}$, we may not have $P(\boldsymbol{s}^*_{i+1},\rho)=0$.

3.2.1. *Sufficient condition for robust stability* To account for the existence of mismatch we consider the addition and subtraction of $\Upsilon^*(\bar{x}_{i+1})$ to the difference $\Upsilon^*(x_i) - \Upsilon^*(x_{i+1})$,

$$\begin{aligned}\Upsilon^*(x_i) - \Upsilon^*(x_{i+1}) &= \Upsilon^*(x_i) - \Upsilon^*(\bar{x}_{i+1}) \\ &\quad - \big(\Upsilon^*(x_{i+1}) - \Upsilon^*(\bar{x}_{i+1})\big)\,. \end{aligned} \tag{24}$$

The term $\Upsilon^*(x_i) - \Upsilon^*(\bar{x}_{i+1})$ represents the difference between the optimal objective functions as in (20). Thus it follows that

$$\begin{aligned}&\Upsilon^*(x_i) - \Upsilon^*(x_{i+1}) \geq \\ &\quad h(x_i, u^*_i, \rho) - \big(\Upsilon^*(x_{i+1}) - \Upsilon^*(\bar{x}_{i+1})\big)\,. \end{aligned} \tag{25}$$

To ensure a closed loop stable system, we force the right hand side to be bounded by a positive function $W\big(\|x_i\|\big)$ of class $\mathcal{K}_\infty$. This ensures that the sequence $\{\Upsilon^*(x_i)\}$ is decreasing, that is,

$$h(x_i, u^*_i, \rho) - \big(\Upsilon^*(x_{i+1}) - \Upsilon^*(\bar{x}_{i+1})\big) \geq W\big(\|x_i\|\big)\,, \tag{26}$$

with $W\big(\|x_i\|\big) \to 0$ as $\|x_i\| \to 0$, for all i, $i \geq 0$. The difference $\Upsilon^*(x_{i+1}) - \Upsilon^*(\bar{x}_{i+1})$ is a measure of the plant/model mismatch and henceforth we will refer to it as the *mismatch term*.

4. THE MISMATCH TERM

To characterize the mismatch term we start by invoking the mean value theorem to derive an expression for the mismatch term as a function of the difference between the two problem solutions. Then we consider the optimality conditions of both problems to derive a bound on the mismatch term, which leads to a sufficient condition for closed loop stability under the presence of plant/model mismatch.

4.1 *Preliminaries*

First of all, we assume that a value of ρ can be chosen that is sufficiently large. By invoking the mean value theorem it follows that

$$\begin{aligned}&\Upsilon^*(x_{i+1}) - \Upsilon^*(\bar{x}_{i+1}) = \\ &\int_0^1 \left\{ \frac{\mathrm{d}}{\mathrm{d}x_{i+1}} \Big[\Upsilon^*\big(\bar{x}_{i+1} + \tau(x_{i+1} - \bar{x}_{i+1})\big)\Big]^{\mathrm{T}} \right\} \\ &\qquad\qquad \cdot (x_{i+1} - \bar{x}_{i+1})\, \mathrm{d}\tau\,. \end{aligned} \tag{27}$$

This is done assuming (15) is differentiable. However, because of the exact penalty terms it is not. To overcome this we apply a smoothing function (Balakrishna and Biegler, 1992) to every element of (16); e.g., for a scalar x,

$$\max\big\{0, b(x)\big\} \approx b(x,\xi) = \frac{\big(b(x)^2 + \xi^2\big)^{1/2}}{2} + \frac{b(x)}{2} \tag{28}$$

with small $\xi > 0$. Henceforth (15) is replaced by

$$\Upsilon(x_i, \boldsymbol{s}_i, \rho_i, \xi) = \Psi(x_i, \boldsymbol{s}_i)\ +\ P(\boldsymbol{s}_i, \rho_i, \xi)\,, \tag{29}$$

which is continuous and at least twice differentiable with respect to (6). For the forthcoming developments it is convenient to keep notation (18), and to define $k = 1, \ldots, p$:

$$\varepsilon^*_{i+k} = \big[s^*_{i+k} - \bar{s}^*_{i+k}\big] = \begin{bmatrix} x^*_{i+k} - \bar{x}^*_{i+k} \\ u^*_{i+k} - \bar{u}^*_{i+k} \end{bmatrix}. \tag{30}$$

From (5), (28), (29) and (30), (27) becomes

$$\begin{aligned}&\Upsilon^*(x_{i+1}) - \Upsilon^*(\bar{x}_{i+1}) = \\ &\sum_{k=1}^{p} \int_0^1 \nabla_{s_{i+k}} h(\bar{s}^*_{i+k} + \tau\,\varepsilon^*_{i+k}, \rho, \xi)^{\mathrm{T}}\, \varepsilon^*_{i+k}\, \mathrm{d}\tau\,. \end{aligned}$$

4.2 *Derivation of a bound on the mismatch term*

We start by considering the optimality conditions of problem $\mathcal{P}_\rho(x_{i+1})$. The Lagrangian for this problem is $\mathcal{L}(\boldsymbol{s}_{i+1}, \lambda) = \Upsilon(x_{i+1}) + \lambda^{\mathrm{T}}\, \boldsymbol{c}(x_{i+1}, \boldsymbol{s}_{i+1})$, where λ is the Lagrange multiplier vector. The optimality conditions are:

$$\begin{bmatrix} \nabla_{\boldsymbol{s}} \Upsilon^*(x_{i+1}) + \nabla_{\boldsymbol{s}} \boldsymbol{c}(x_{i+1}, \boldsymbol{s}^*_{i+1})^{\mathrm{T}}\, \lambda^* \\ \boldsymbol{c}(x_{i+1}, \boldsymbol{s}^*_{i+1}) \end{bmatrix} = 0\,. \tag{31}$$

We also assume that $\nabla_{\boldsymbol{s}} \boldsymbol{c}(x_{i+1}, \boldsymbol{s}^*_{i+1})$ has full row rank and we define a basis, Z, for the null space of this matrix, i.e., $\nabla_{\boldsymbol{s}} \boldsymbol{c}(x_{i+1}, \boldsymbol{s}^*_{i+1}) \cdot Z = 0$. By taking the projection of $\nabla_{\boldsymbol{s}} \Upsilon^*(x_{i+1}) + \nabla_{\boldsymbol{s}} \boldsymbol{c}(x_{i+1}, \boldsymbol{s}^*_{i+1})^{\mathrm{T}}\, \lambda^*$ on the null space of $\nabla_{\boldsymbol{s}} \boldsymbol{c}(x_{i+1}, \boldsymbol{s}^*_{i+1})$, (31) becomes

$$\begin{bmatrix} Z^{\mathrm{T}} \cdot \nabla_{\boldsymbol{s}} \Upsilon^*(x_{i+1}) \\ \boldsymbol{c}(x_{i+1}, \boldsymbol{s}^*_{i+1}) \end{bmatrix} = 0\,. \tag{32}$$

Thus proceeding as in the unconstrained case study (Santos and Biegler, 1999) we derive a bound for stability on the mismatch term,

$$\begin{aligned}&\big|\Upsilon^*(x_{i+1}) - \Upsilon^*(\bar{x}_{i+1})\big| \leq \\ &\sum_{k=1}^{p} \left\| \int_0^1 \nabla_{s_{i+k}} h(\bar{s}^*_{i+1} + \tau\,\varepsilon^*_{i+k}, \rho, \xi)^{\mathrm{T}}\, \mathrm{d}\tau \right\| \\ &\qquad\qquad \cdot \Gamma \cdot W_{\mathrm{m}}\big(\|x_i\|\big)\,, \end{aligned} \tag{33}$$

that provides a sufficient stability condition for a general cost function $h(x,u)$, where Γ is derived from sensitivity information from (32) (see Santos and Biegler (1999)).

4.3 *Constrained MPC with quadratic function and finite horizon*

Typically, in MPC formulations (5) is defined with

$$h(s_{i+k}) = s^{\mathrm{T}}_{i+k}\, Q_{i+k}\, s_{i+k}\,, \tag{34}$$

where $Q_{i+k} = \mathrm{diag}\{Q_{\mathrm{x}\,i+k},\, Q_{\mathrm{u}\,i+k}\}$, and $Q_{\mathrm{x}\,i+k} \in \mathbf{R}^{n_\mathrm{s}\times n_\mathrm{s}}$ and $Q_{\mathrm{u}\,i+k} \in \mathbf{R}^{n_\mathrm{m}\times n_\mathrm{m}}$ are diagonal matrices corresponding to the state and input weighting matrices at predictive horizon time index $i+k$, respectively. From (34), the analytical form of the integral term in (33) is

$$\int_0^1 \nabla_{s_{i+k}} h(\bar{s}^*_{i+k} + \tau\, \varepsilon^*_{i+k}, \rho, \xi)^\mathrm{T}\, \mathrm{d}\tau =$$
$$\left(2\, \bar{s}^*_{i+k} + \varepsilon^*_{i+k}\right)^\mathrm{T} Q_{i+k} + \rho \cdot r\left(\bar{s}^*_{i+k}, \varepsilon^*_{i+k}, \xi\right), \quad (35)$$

where $r\left(\bar{s}^*_{i+k}, \varepsilon^*_{i+k}, \xi\right)$ denotes a vector whose elements are nonlinear functions of $\bar{s}^*_{i+k}$, ε^*_{i+k} and ξ. Following the same developments as in Santos and Biegler (1999) we obtain

$$\left|\Upsilon^*(x_{i+1}) - \Upsilon^*(\bar{x}_{i+1})\right| \leq \sum_{k=1}^{p} \Big\{ \left(\left\|2\, \bar{s}^*_{i+k}\right\| + \left\|\varepsilon^*_{i+k}\right\|\right) \left\|Q_{i+k}\right\| + \rho \cdot \left\| r\left(\bar{s}^*_{i+k}, \varepsilon^*_{i+k}, \xi\right)\right\| \Big\} \cdot \Gamma \cdot W_\mathrm{m}\left(\|x_i\|\right). \quad (36)$$

We assume there are positive constants Q, α_1 and α_2, such that for all $i \geq 0$ and $k \leq p$

$$\left\| r\left(\bar{s}^*_{i+k}, \varepsilon^*_{i+k}, \xi\right)\right\| \leq \alpha_1 \left\|2\, \bar{s}^*_{i+k}\right\| + \alpha_2 \left\|\varepsilon^*_{i+k}\right\| \quad (37)$$

and $\left\|Q_{i+k}\right\| \leq Q$. Moreover, since $\bar{s}^*_{i+k}$, $k = 1, \ldots, p$, depends on x_i, we set

$$\left\|2\, \bar{s}^*_{i+k}\right\| \leq \hat{K}\, \|x_i\|, \quad (38)$$

where $\hat{K}$ is a positive constant. From (4), with $\gamma = 1$ (see Santos and Biegler (1999)),

$$\left\|\varepsilon^*_{i+k}\right\| \leq \Gamma \cdot W_\mathrm{m}\left(\|x_i\|\right) \leq \Gamma\, K_\mathrm{m} \|x_i\|, \quad (39)$$

for every k, $k \leq p$. Finally, substituting (37), (38) and (39) in (36) leads to

$$\left|\Upsilon^*(x_{i+1}) - \Upsilon^*(\bar{x}_{i+1})\right| \leq K_\mathrm{B}\, \|x_i\|^2, \quad (40)$$

where $K_\mathrm{B} = p \Big\{ \left(\hat{K} + \Gamma\, K_\mathrm{m}\right) Q + \rho \cdot (\alpha_1\, \hat{K} + \alpha_2\, \Gamma\, K_\mathrm{m}) \Big\}\, \Gamma\, K_\mathrm{m}. \quad (41)$

Note that the first term of the sum on the right hand side of (41) is the expression of K_B obtained for the unconstrained case. Therefore, when there are no active constraints $\rho = 0$ and (40) is equal to the unconstrained case sufficient stability bound. Also, from (26) and (40) it follows that

$$h(x_i, u^*_i, \rho) - \left|\Upsilon^*(x_{i+1}) - \Upsilon^*(\bar{x}_{i+1})\right| \geq h(x_i, u^*_i) - K_\mathrm{B}\, \|x_i\|^2 = W\left(\|x_i\|\right). \quad (42)$$

Suppose that $Q_{\mathrm{x}\,i} = \alpha_\mathrm{x} I$ and $Q_{\mathrm{u}\,i} = \alpha_\mathrm{u} I$, with constants $\alpha_\mathrm{x} > 0$ and $\alpha_\mathrm{u} \geq 0$. Because u^*_i is an implicit function of x_i we can write $\|u^*_i\|^2 = \beta\, \|x_i\|^2$, $\beta > 0$. Thus

$$h(x_i, u^*_i) = \alpha_\mathrm{x}\, x_i^\mathrm{T} x_i + \alpha_\mathrm{u}\, {u^*_i}^\mathrm{T} u^*_i = (\alpha_\mathrm{x} + \alpha_\mathrm{u}\, \beta)\, \|x_i\|^2 \quad (43)$$

For a given x_i, with no active constraints, and with $\alpha_\mathrm{u} = 0$, from (42) it follows that $K_\mathrm{B} < \alpha_\mathrm{x}$ to satisfy the sufficient condition for stability. When $\alpha_\mathrm{u} \neq 0$, this condition is relaxed to

$$K_\mathrm{B} < \alpha_\mathrm{x} + \alpha_\mathrm{u}\, \beta. \quad (44)$$

4.4 *A tool to analyze robust stability*

Because β in (44) depends on the optimization problem solution it is impossible to know a priori K_B. In Section 5 we illustrate that $K_\mathrm{B} < \alpha_\mathrm{x}$ provides a conservative sufficient condition for stability. In any case, when constraint violations occur a tighter value of the sufficient stability condition for the constrained case, K_B, can be estimated by exploiting the state-space region of interest from

$$K_\mathrm{B} \geq \max_{x_i} \frac{\left|\Upsilon^*(x_{i+1}) - \Upsilon^*(\bar{x}_{i+1})\right|}{\|x_i\|^2}. \quad (45)$$

This procedure involves the calculation off-line of K_B according to the following cycle:

1 *For a given x_i, $i \geq 0$, perform the following steps:*
2 *Solve $\mathcal{P}_\rho(x_i)$; save $\bar{x}_{i+1}$.*
3 *Implement u^*_i and set $i = i + 1$.*
   i *Using x_{i+1}, solve $\mathcal{P}_\rho(x_{i+1})$ to obtain $\Upsilon^*(x_{i+1})$.*
   ii *Using $\bar{x}_{i+1}$, solve $\mathcal{P}_\rho(\bar{x}_{i+1})$ to obtain $\Upsilon^*(\bar{x}_{i+1})$.*
   iii *Go to 1 and repeat steps with new values of x_i.*

Therefore for a nonzero x_i we can compute a lower bound for K_B.

5. ILLUSTRATIVE EXAMPLE

Consider an exothermic zero-order reaction system, A $\rightarrow$ B, with concentration and temperature dynamics described by

$$\frac{\mathrm{d}C_\mathrm{A}}{\mathrm{d}t} = \frac{F_0}{V}\,(C_{\mathrm{A}_0} - C_\mathrm{A}) - k_0\, \mathrm{e}^{-E_\mathrm{a}/(R\, T_\mathrm{r})}, \quad (46)$$
$$\frac{\mathrm{d}T_\mathrm{r}}{\mathrm{d}t} = \frac{1}{\rho_\mathrm{L} C_\mathrm{p} V}(-Q_\mathrm{R} + Q_\mathrm{G}), \quad (47)$$

with $Q_\mathrm{R} = -\rho_\mathrm{L} C_\mathrm{p} F_0 (T_0 - T_\mathrm{r}) + UA(T_\mathrm{r} - T_\mathrm{j})$, and $Q_\mathrm{G} = (-\Delta H_\mathrm{r}) V k_0 \mathrm{e}^{-E_\mathrm{a}/(R\, T_\mathrm{r})}$. Note that (47) does not depend on C_A. The system is open loop unstable for $T_{\mathrm{r},\,i} > 34\,°\mathrm{C}$. Data and a more detailed description of this system can be found in Santos (2001). The control objective is to control T_r – the set-point is $T_{\mathrm{r\,sp}} = 34\,°\mathrm{C}$ – by manipulating the cooling fluid temperature T_j subject to: $T_\mathrm{r} \geq 0\,°\mathrm{C}$ and $T_\mathrm{j} \geq 15\,°\mathrm{C}$. To satisfy these constraints the control problems are solved using (15) with $\rho = 1000$. We set $(\alpha_\mathrm{x}, \alpha_\mathrm{u}) = (1, 0)$, $(p, m) = (25, 1)$, $\Delta t = 0.5\,\mathrm{min}$ and we note that the plant and the model have the same steady state with $T_\mathrm{r} = T_\mathrm{j} = 34\,°\mathrm{C}$. To test for the sufficient stability condition parametric model mismatch on U is considered. Figure 2 shows the variation of K_B with $T_{\mathrm{r},i}$, varying from 24 to 44 °C, and for various mismatches: $U_\mathrm{m} =$

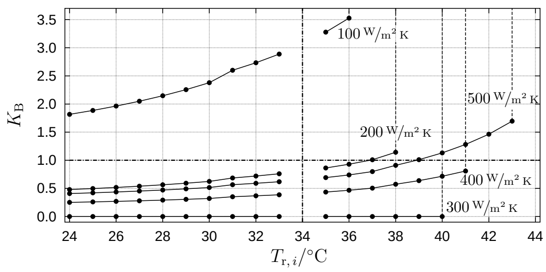

Fig. 2. *Variation of the constant bound.*

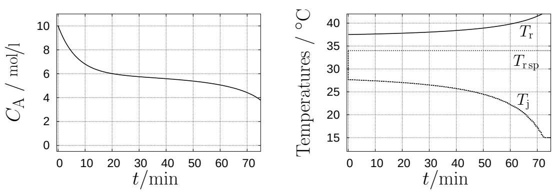

Fig. 3. *Unstable regulator control response.*

$100, 200, 300, 400$ and $500\,\mathrm{W/m^2\,K}$. The *true* plant value is $U_p = 300\,\mathrm{W/m^2\,K}$. Thus, we observe in Figure 2 that in the case of perfect model $K_B = 0$. We emphasize that when the constraints are not satisfied then (16) is not zero and K_B is very high. Here, the nonexistence of a feasible solution is overcome by increasing appropriately p to pursue the calculation of K_B.

For $(\alpha_x, \alpha_u) = (1, 0)$, a *sufficient* condition for robust stability from (42) requires $K_B < 1$. This can be seen in Figures 2 and 3. For $T_{r,i} < 34\,°C$, the profiles show $K_B < 1$ always. Under these conditions the system is closed loop stable in the sense that the state converges to the origin (set-point), $T_r = 34\,°C$. On the other hand, for $T_{r,i} > 34\,°C$, the profiles of K_B increase such that they tend to cross the *line* $K_B = 1$ as $T_{r,i}$ increases. Since (45) provides a lower bound on K_B it means the system can become closed loop unstable under these plant/model mismatch conditions – e.g., Figure 3 with $T_{r,i} = 37.5\,°C$ and $U_m = 400\,\mathrm{W/m^2\,K}$. On the other hand, with $U_m = 500\,\mathrm{W/m^2\,K}$, the system is closed loop unstable when $T_{r,i} > 34\,°C$. Again, from (45) this is consistent with the theory since $K_B > 1$ for $T_{r,i} \geq 39\,°C$.

On the other hand, stable performance may still be observed if (42) is violated because this condition is only *sufficient*. For instance, with $U_m = 100\,\mathrm{W/m^2\,K}$ the system is closed loop stable despite $K_B > 1$. The same result is observed for $U_m = 200\,\mathrm{W/m^2\,K}$ when $T_{r,i} \geq 37\,°C$. In these cases $U_p > U_m$, thus the control solution is favorable to the plant; i.e., the control system provides a cooling rate greater than the one really necessary. But for $U_m = 400$ and $500\,\mathrm{W/m^2\,K}$ the cooling rate calculated by the controller may not be sufficient to cool down the reactor liquid and therefore a temperature runaway may occur.

6. CONCLUSIONS

We develop a strategy based on nonlinear programming sensitivity that determines conditions under which the constrained model predictive control is robustly stable with respect to modeling errors. Here, a sufficient condition for robust stability is derived and an offline procedure is developed to evaluate constants which determine sufficient conditions for this property. These constants are available from bounds on the model mismatch and from the NLP solution of the receding horizon model. This procedure is applicable to both linear and nonlinear model predictive controllers in discrete time that satisfy nominal stability properties based on Lyapunov arguments.

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