CLOSED LOOP PROPERTIES AND BLOCK RELATIVE GAIN

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Abstract: Block Relative Gain (BRG) is a useful method for screening alternatives for block decentralized control at the design stage. In this paper, we establish the connection between the BRG and closed loop properties like stability, input output controllability, block diagonal dominance and interactions. Based on these results, simple rules for pairing of variables for block decentralized control are proposed.

Keywords: Control System Design, Decentralized control, Interaction.

1. INTRODUCTION

Manousiouthakis *et al.* (1986) generalized the concept of Relative Gain Array (RGA) (Bristol, 1966) to Block Relative Gain (BRG). It is a powerful technique for input-output controllability analysis and screening alternatives for block decentralized control quickly at the design stage. During the past few years, RGA has been studied extensively (Grosdidier *et al.*, 1985; Hovd and Skogestad, 1992) and its properties are well understood, but BRG has largely been overlooked. Some researchers (Nett and Manousiouthakis, 1987; Chen, 1992) have found relations between BRG and Euclidian condition number. It is shown that generally, a system is difficult to control, if the maximum singular value of BRG is large. Despite these studies, BRG has not gained widespread popularity and block pairings are selected primarily based on heuristics (Castro and Doyle, 2002). This can be attributed to lack of studies showing that, similar to RGA, information regarding closed loop properties can be obtained using BRG. This motivates the present work.

In this paper, we establish the connection between BRG and closed loop properties like stability, input output controllability, block diagonal dominance and interactions. We show that the common conjecture that a system is *weakly* interacting, if BRG is close to the identity matrix, is not true. Further, a system can have large interactions despite BRG being exactly the identity matrix. Based on these insights, simple rules for pairing of variables are proposed.

This paper focuses on extracting useful feedback properties from gain information, since it is often the only reliable information available at design stage (Grosdidier *et al.*, 1985). The discussion is limited to square, stable and linear time invariant (LTI) systems represented as $\mathbf{G}(s)$. The steady state gain matrix is represented as $\mathbf{G}(0)$ or simply $\mathbf{G} \in \mathbb{R}^{n\times n}$ and its individual elements as g_{ij}. A vector of variables is denoted by a boldface letter (e.g. $\mathbf{y}$, $\mathbf{u}$). The objective is to decompose the original system into a set of M non-overlapping square subsystems such that, $\mathbf{G}_{ii} \in \mathbb{R}^{m_i\times m_i}$; $i = 1, 2\cdots M$, $\sum_i m_i = n$. The pair $(\mathbf{y}_i, \mathbf{u}_j)$ denotes the variables related by $\mathbf{G}_{ij}(s)$, which is the ij^{th} block of $\mathbf{G}(s)$.

2. PRELIMINARIES

Let the system be partitioned as shown in Figure 1. The steady state BRG between $(\mathbf{y}_1, \mathbf{u}_1)$ is defined as (Manousiouthakis *et al.*, 1986),

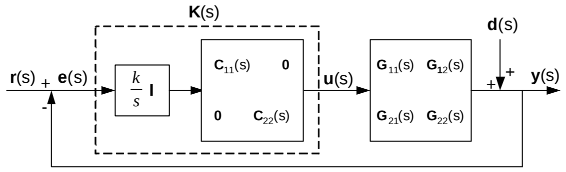

Fig. 1. Block Diagram of Closed loop system

$$[\mathbf{\Lambda}_B]_{11} = \mathbf{G}_{11}[\mathbf{G}^{-1}]_{11} \tag{1}$$

where $\mathbf{G}_{11}$ and $[\mathbf{G}^{-1}]_{11}$ are the first $m_1 \times m_1$ blocks of $\mathbf{G}$ and $\mathbf{G}^{-1}$ respectively. If, $\mathbf{G}_{22}$ is non-singular, then recognizing that $[\mathbf{G}^{-1}]_{11} = \bar{\mathbf{G}}_{11}^{-1}$, $[\bar{\mathbf{G}}]_{11} = \mathbf{G}_{11} - \mathbf{G}_{12}\mathbf{G}_{22}^{-1}\mathbf{G}_{21}$ (Horn and Johnson, 1990), the BRG between $(\mathbf{y}_1, \mathbf{u}_1)$ can be alternatively calculated as

$$[\mathbf{\Lambda}_B]_{11} = \mathbf{G}_{11}\bar{\mathbf{G}}_{11}^{-1} \tag{2}$$

Similarly, the BRG between $(\mathbf{y}_i, \mathbf{u}_i)$ can be defined as

$$[\mathbf{\Lambda}_B]_{ii} = \mathbf{G}_{ii}[\mathbf{G}^{-1}]_{ii} = \mathbf{G}_{ii}\bar{\mathbf{G}}_{ii}^{-1} \tag{3}$$

Manousiouthakis *et al.* (1986) have suggested choosing the pairings such that the eigenvalues of $[\mathbf{\Lambda}_B]_{ii}$ are close to 1 for all i. This rule is based on the incorrect conjecture that a system is *weakly* interacting if the BRG is close to the identity matrix. Due to this limitation, this rule can lead to pairings with significant interactions in many cases.

3. CLOSED LOOP PROPERTIES

3.1 Stability

In this section, we establish the connection between the BRG and simultaneous stabilization of the closed loop system and individual loops. It is based on a similar result for RGA shown to be true by Grosdidier *et al.* (1985).

Let the system $\mathbf{G(s)}$ be partitioned as shown in Figure 1. If the controller contains an integrating element to give asymptotically zero tracking error, $\mathbf{K}_{ii}(s)$ can be expressed as $\frac{k}{s}\mathbf{C}_{ii}(s)$, $k > 0$. It is assumed that $\mathbf{C}_{ii}(s)$ and $\mathbf{G}_{ii}(s)$ are stable, contain no transmission zeros and $\mathbf{G}(s)\mathbf{C}(s)$ is proper. Defining, $\mathbf{L}(s) = \mathbf{G}(s)\mathbf{K}(s)$, the closed loop system is given by

$$\mathbf{y}(s) = [\mathbf{I} + \mathbf{L}(s)]^{-1}\mathbf{L}(s)\mathbf{r}(s) + [\mathbf{I} + \mathbf{L}(s)]^{-1}\mathbf{d}(s)$$

and $\mathbf{y}_1(s)$ is given by

$$\begin{aligned}\mathbf{y_1}(s) = &\left[\hat{\mathbf{L}}_{11}(s)\mathbf{L}_{11}(s) + \hat{\mathbf{L}}_{12}(s)\mathbf{L}_{21}(s)\right]\mathbf{r_1}(s)\\ &+ [\hat{\mathbf{L}}_{11}(s)\mathbf{L}_{12}(s) + \hat{\mathbf{L}}_{12}(s)\mathbf{L}_{22}(s)]\mathbf{r_2}(s)\\ &+ \hat{\mathbf{L}}_{11}(s)\mathbf{d_1}(s) + \hat{\mathbf{L}}_{12}(s)\mathbf{d_2}(s)\end{aligned} \tag{4}$$

where $\hat{\mathbf{L}}_{ij}(s) = [\mathbf{L}^{-1}(s)]_{ij}$. Using the property of partitioned matrices (Horn and Johnson, 1990):

$$\begin{aligned}\hat{\mathbf{L}}_{11}(s) &= [(\mathbf{I} + \mathbf{L}_{11}(s)) - \mathbf{L}_{12}(s)\\ &\qquad (\mathbf{I} + \mathbf{L}_{22}(s))^{-1}\mathbf{L}_{21}(s)]^{-1}\\ \hat{\mathbf{L}}_{12}(s) &= (\mathbf{I} + \mathbf{L}_{11}(s))^{-1}\mathbf{L}_{12}(s)[\mathbf{L}_{21}(s)(\mathbf{I}+\\ &\quad + \mathbf{L}_{11}(s))^{-1}\mathbf{L}_{12}(s) - (\mathbf{I} + \mathbf{L}_{22}(s))]^{-1}\end{aligned}$$

At low frequencies, $[\mathbf{I} + \mathbf{L}_{ij}(s)]^{-1} \approx \mathbf{L}_{ij}^{-1}(s)$ (Hovd and Skogestad, 1992). This approximation is valid, when the controller contains integral action. Using this approximation,

$$\begin{aligned}\hat{\mathbf{L}}_{11}(s) &\approx \left[\mathbf{I} + \bar{\mathbf{G}}_{11}(s)\mathbf{K}_{11}(s)\right]^{-1}\\ \hat{\mathbf{L}}_{12}(s) &\approx -\mathbf{L}_{11}^{-1}(s)\mathbf{L}_{12}\left[\mathbf{I} + \bar{\mathbf{G}}_{22}(s)\mathbf{K}_{22}(s)\right]^{-1}\end{aligned}$$

where $\bar{\mathbf{G}}_{11}(s)$ and $\bar{\mathbf{G}}_{22}(s)$ represent Schur complements of $\mathbf{G}_{22}(s)$ and $\mathbf{G}_{11}(s)$ in $\mathbf{G}(s)$ respectively. If any of the zeros of $[\mathbf{I} + \bar{\mathbf{G}}_{11}(s)\mathbf{K}_{11}(s)]$ lie in the right half plane and no pole-zero cancellations occur in (4), then the closed loop system is unstable. Similarly, it can be shown that if the system is to be decomposed into M blocks, the stability of system depends on the location of zeros of $[\mathbf{I} + \bar{\mathbf{G}}_{ii}(s)\mathbf{K}_{ii}(s)]$, $i = 1, 2 \cdots M$. Now, we can relate this finding to BRG using the concept of integral controllability (Grosdidier *et al.*, 1985).

Lemma 1. If $\mathrm{Re}\{\lambda_j(\bar{\mathbf{G}}_{ii}(0)\mathbf{C}_{ii}(0))\} < 0;\ j = 1, 2 \cdots m_i$ for some i, then the closed loop system is not integral controllable.

Sketch of Proof. Since $\mathbf{G}(s)$ is stable and $\mathbf{G}_{ii}(s)$ non-singular for all i by assumption, $\bar{\mathbf{G}}_{ii}(s)$ is also stable. Now, Lemma 1 can be shown to be true by following the proof of Theorem 7 of Grosdidier *et al.* (1985).

It should be noted that the low frequency approximation has little effect on the applicability of Lemma 1, since the maximum value of $\bar{\mathbf{G}}_{ii}(s)\mathbf{K}_{ii}(s)$ is seen at origin of s-plane.

Lemma 2. If $\mathrm{Re}\{\lambda_j(\mathbf{G}_{ii}(0)\mathbf{C}_{ii}(0))\} < 0;\ j = 1, 2 \cdots m_i$, then the subsystem $(\mathbf{y}_i(s), \mathbf{r}_i(s))$, considered in isolation, is not integral controllable.

PROOF. If all other loops are open, the stability of subsystem $(\mathbf{y}_i(s), \mathbf{r}_i(s))$ depends on the zeros of $[\mathbf{I} + \mathbf{G}_{ii}(s)\mathbf{K}_{ii}(s)]$. The proof follows by replacing $\bar{\mathbf{G}}_{ii}$ by $\mathbf{G}_{ii}$ in Lemma 1. □

Proposition 3. If $\det([\mathbf{\Lambda}_B(0)]_{ii}) < 0$, then one of the following is true,

(1) The i^{th} loop by itself is unstable or
(2) The closed loop system is unstable.

PROOF. Using (3),

$$\det([\mathbf{\Lambda}_B(0)]_{ii}) = \frac{\det(\mathbf{G}_{ii}(0))}{\det(\bar{\mathbf{G}}_{ii}(0))} = \frac{\det(\mathbf{G}_{ii}(0)\mathbf{C}_{ii}(0))}{\det(\bar{\mathbf{G}}_{ii}(0)\mathbf{C}_{ii}(0))}$$

Thus, $\det([\mathbf{\Lambda}_B(0)]_{ii}) < 0$, if $\det(\mathbf{G}_{ii}(0)\mathbf{C}_{ii}(0)) < 0$ or $\det(\bar{\mathbf{G}}_{ii}(0)\mathbf{C}_{ii}(0)) < 0$. If $\det(\mathbf{G}_{ii}(0)\mathbf{C}_{ii}(0)) < 0$, then at least one of the eigenvalues of $\mathbf{G}_{ii}(0)\mathbf{C}_{ii}(0)$ is negative since,

$$\det(\bar{\mathbf{G}}_{ii}(0)\mathbf{C}_{ii}(0)) = \prod_{j=1}^{m_i} \lambda_j(\bar{\mathbf{G}}_{ii}(0)\mathbf{C}_{ii}(0))$$

The closed loop system is unstable, if any eigenvalue of $\bar{\mathbf{G}}_{ii}(0)\mathbf{C}_{ii}(0)$ is negative (see Lemma 1). Similarly, the i^{th} loop, considered in isolation with other loops, is unstable if $\det(\mathbf{G}_{ii}(0)\mathbf{C}_{ii}(0)) < 0$ (see Lemma 2). □

Proposition 3 can be interpreted on similar terms as Theorem 6 of Grosdidier *et al.* (1985), where the implications of negative RGA elements were drawn. If $\det([\mathbf{\Lambda}_B(0)]_{ii}) < 0$ for some i and a controller exists, which stabilizes the individual loops, then the closed loop system is unstable. If the controller is designed such that the closed loop system is stable, then the i^{th} loop is unstable. In this case, the system is loop failure sensitive.

Proposition 3 provides only a necessary condition for stability. Consider the case when the sum of the number of negative eigenvalues of $\bar{\mathbf{G}}_{ii}(0)\mathbf{C}_{ii}(0)$ and $\mathbf{G}_{ii}(0)\mathbf{C}_{ii}(0))$ is even. Then, $\det([\mathbf{\Lambda}_B]_{ii})$ will be positive, despite the closed loop system and the individual loop being unstable.

Remark 4. The elements of the BRG with only 1×1 blocks are same as the diagonal elements of the RGA (Manousiouthakis *et al.*, 1986). Thus, Proposition 3 generalizes Bristol's pairing rule of avoiding pairing on negative RGA elements to block pairings.

3.2 Input-Output Controllability

It is well known that Right Half Plane (RHP) zeros pose a limitation on the achievable performance of the system. Hovd and Skogestad (1992) have shown that the frequency dependent RGA can be used to detect the presence of RHP zeros. The applicability of their result is limited to the individual elements of the system and $(n-1) \times (n-1)$ subsystems of $\mathbf{G}(s)$. The next proposition complements their result for subsystems having different dimensions.

Proposition 5. Consider the partition of the system matrix $\mathbf{G}(s)$ as shown in Figure 1. Then $[\mathbf{\Lambda}_B(s)]_{11}$ is an $m_1 \times m_1$ transfer function matrix. If there exists m_1, $2 \le m_1 \le n-2$, such that $\lim_{s\to j\infty} \det([\mathbf{\Lambda}_B(s)]_{11})$ is nonzero, finite and has a different sign from $\det([\mathbf{\Lambda}_B(0)]_{11})$, then at least one of the following is true,

(1) $\mathbf{G}_{11}(s)$ has an RHP transmission zero.
(2) $\mathbf{G}_{22}(s)$ has an RHP transmission zero.

PROOF. For a given partitioning of the system, $2 \le m_1 \le n-2$, consider that $\lim_{s\to j\infty} \det([\mathbf{\Lambda}_B(s)]_{11})$ is nonzero and finite. If the signs of $\det([\mathbf{\Lambda}_B(0)]_{11})$ and $\lim_{s\to j\infty} \det([\mathbf{\Lambda}_B(s)]_{11})$ are different, then there exists a frequency ω_o, $\omega_o > 0$, such that $\det([\mathbf{\Lambda}_B(j\omega_o)]_{11}) = 0$.

The equality, $\det([\mathbf{\Lambda}_B(s)]_{11}) = 0$, is satisfied, iff one or both of $\det(\mathbf{G}_{11}(j\omega_o))$ and $\det(\bar{\mathbf{G}}_{11}^{-1}(j\omega_o))$ are zero. Now, $\det(\mathbf{G}_{11}(j\omega_o))$ being zero implies the presence of an RHP transmission zero in $\mathbf{G}_{11}(s)$ at that frequency. If $\det(\bar{\mathbf{G}}_{11}^{-1}(j\omega_o)) = 0$, then $\bar{\mathbf{G}}_{11}^{-1}(s)$ contains an RHP transmission zero and $\bar{\mathbf{G}}_{11}(s)$ contains an RHP pole at that frequency. Due to stability assumptions, an RHP pole in $\bar{\mathbf{G}}_{11}(s)$ at $s = j\omega_o$ can arise only due to an RHP zero in $\mathbf{G}_{22}(s)$ at $s = j\omega_o$. □

Manousiouthakis *et al.* (1986) have shown that BRG is input scaling independent. Thus, if an input channel of $\mathbf{G}(s)$ contains an RHP zero, the signs of $\det([\mathbf{\Lambda}_B(j\infty)]_{11})$ and $\det([\mathbf{\Lambda}_B(0)]_{11})$ will remain unchanged. The change of sign of $\det([\mathbf{\Lambda}_B(s)]_{11})$ is only a sufficient, but not a necessary condition for the presence of RHP zeros in the subsystems of $\mathbf{G}(s)$.

Proposition 5 excludes the case in which any subsystem contains a zero at the origin, $(s = 0)$. Should a subsystem contain a zero at the origin, it would be extremely difficult to control the system. The relation between zeros at the origin and the steady state BRG is established in the next corollary. The proof of this corollary follows directly from the proof of Proposition 5.

Corollary 6. If there exists m_1, $1 < m_1 < n-1$, such that $\det([\mathbf{\Lambda}_B(0)]_{11}) = 0$, then one or both of the subsystems, $\mathbf{G}_{11}(s)$ and $\mathbf{G}_{22}(s)$ contain a zero or a transmission zero at the origin.

Either of these conditions is undesirable, as it makes the subsystem uncontrollable using a controller with integral action. The system may also contain zeros close to the origin in the open left half plane (LHP). Presence of such poorly damped zeros also affects the controllability.

The gain of a multivariate system depends on the input direction. Let the gain of $(\mathbf{y_1}(s), \mathbf{u_1}(s))$ be $\|\mathbf{G}_{11}(0)\mathbf{v}\|_2$, $\|\mathbf{v}\|_2 = 1$. Similarly, let the apparent gain of this loop, when all other loops are closed be $\|\bar{\mathbf{G}}_{11}(0)\mathbf{w}\|_2$, $\|\mathbf{w}\|_2 = 1$.

Proposition 7. The worst case gain mismatch between $\mathbf{G}_{11}(0)$ and $\bar{\mathbf{G}}_{11}(0)$ is bounded as follows,

$$\bar{\sigma}([\mathbf{\Lambda}_B(0)]_{11}) \leq \max_{\substack{\|\mathbf{v}\|_2=1\\ \|\mathbf{w}\|_2=1}} \frac{\|\mathbf{G}_{11}(0)\mathbf{v}\|_2}{\|\bar{\mathbf{G}}_{11}(0)\mathbf{w}\|_2} \quad (5)$$

$$\frac{1}{\underline{\sigma}([\mathbf{\Lambda}_B(0)]_{11})} \leq \max_{\substack{\|\mathbf{v}\|_2=1\\ \|\mathbf{w}\|_2=1}} \frac{\|\bar{\mathbf{G}}_{11}(0)\mathbf{w}\|_2}{\|\mathbf{G}_{11}(0)\mathbf{v}\|_2} \quad (6)$$

PROOF. For (5),

$$\begin{aligned}\max \frac{\|\mathbf{G}_{11}(0)\mathbf{v}\|_2}{\|\bar{\mathbf{G}}_{11}(0)\mathbf{w}\|_2} &= \frac{\bar{\sigma}(\mathbf{G}_{11}(0))}{\underline{\sigma}(\bar{\mathbf{G}}_{11}(0))}\\ &= \bar{\sigma}(\mathbf{G}_{11}(0))\bar{\sigma}(\bar{\mathbf{G}}_{11}^{-1}(0))\\ &\geq \bar{\sigma}([\mathbf{\Lambda}_B(0)]_{11})\end{aligned}$$

For (6),

$$\begin{aligned}\max \frac{\|\bar{\mathbf{G}}_{11}(0)\mathbf{w}\|_2}{\|\mathbf{G}_{11}(0)\mathbf{v}\|_2} &= \frac{\bar{\sigma}(\bar{\mathbf{G}}_{11}(0))}{\underline{\sigma}(\mathbf{G}_{11}(0))}\\ &= \bar{\sigma}(\bar{\mathbf{G}}_{11}(0))\bar{\sigma}(\mathbf{G}_{11}^{-1}(0))\\ &\geq \bar{\sigma}([\mathbf{\Lambda}_B(0)]_{11}^{-1})\\ &\geq \frac{1}{\underline{\sigma}([\mathbf{\Lambda}_B(0)]_{11})} \qquad \square\end{aligned}$$

Proposition 7 suggests that if at least one of the conditions, $\bar{\sigma}([\mathbf{\Lambda}_B(0)]_{11}) \gg 1$ and $\underline{\sigma}([\mathbf{\Lambda}_B(0)]_{11}) \ll 1$, is satisfied, then the gain of the $(\mathbf{y_1}(s), \mathbf{u_1}(s))$ loop changes considerably due to closure of all the other loops. If $\bar{\sigma}([\mathbf{\Lambda}_B(0)]_{11}) \approx \underline{\sigma}([\mathbf{\Lambda}_B(0)]_{11}) \approx 1$, the change in gain may still be large, as (5) and (6) are lower bounds on the gain mismatch with one of the loops open. This affirms our earlier assertion that if the BRG is far from the identity matrix, the system has large interactions, but the converse is not true. This is further discussed in §3.4.

3.3 Block diagonal dominance

An advantage of block decentralized controllers is that if the blocks are weakly interacting, then the individual controllers can be tuned independently of each other. The concept of block diagonal dominance can be used to assess this property of the partitioned system. In this section, the relation between block diagonal dominance and BRG is established.

Let the system matrix $\mathbf{G}(s)$ be split into a block diagonal part, $\mathbf{G}_{bd}(s)$ and an off-block diagonal part, $\mathbf{G}(s) - \mathbf{G}_{bd}(s)$. Furthermore, assume that the controller $\mathbf{K}(s)$ has a block diagonal structure same as $\mathbf{G}_{bd}(s)$. Define $\mathbf{E}(s) = (\mathbf{G}(s) - \mathbf{G}_{bd}(s))\mathbf{G}_{bd}(s)^{-1}$. Then, a system is block diagonal dominant (Grosdidier and Morari, 1986), if

$$\mu_{\mathbf{\Delta}}(\mathbf{E}(s)) < 1 \quad (7)$$

where $\mu_{\mathbf{\Delta}}$ is the structured singular value (Skogestad and Postlethwaite, 1996) with $\mathbf{\Delta}$ having same structure as $\mathbf{G}_{bd}(s)$. Next we show that information regarding block diagonal dominance can be obtained using the BRG.

Proposition 8. For a system partitioned into 2 blocks,

$$\mu_{\mathbf{\Delta}}(\mathbf{E}(s)) \geq \sqrt{\left|\frac{1}{\underline{\sigma}([\mathbf{\Lambda}_B(s)]_{ii})} - 1\right|} \quad (8)$$

PROOF. Consider the system being partitioned as shown in Figure 1. Then,

$$\mathbf{E}(s) = \begin{bmatrix} \mathbf{0} & \mathbf{G}_{12}(s)\mathbf{G}_{22}(s)^{-1} \\ \mathbf{G}_{21}(s)\mathbf{G}_{11}(s)^{-1} & \mathbf{0} \end{bmatrix}$$

Using Theorem 2 of Skogestad and Morari (1988),

$$\begin{aligned}\mu_{\mathbf{\Delta}}^2(\mathbf{E}(s)) &= \bar{\sigma}(\mathbf{G}_{12}(s)\mathbf{G}_{22}^{-1}(s))\bar{\sigma}(\mathbf{G}_{21}(s)\mathbf{G}_{11}^{-1}(s))\\ \mu_{\mathbf{\Delta}}^2(\mathbf{E}(s)) &\geq \bar{\sigma}(\mathbf{G}_{12}(s)\mathbf{G}_{22}^{-1}(s)\mathbf{G}_{21}(s)\mathbf{G}_{11}^{-1}(s)) \quad (9)\end{aligned}$$

Using (2),

$$\frac{1}{\underline{\sigma}([\mathbf{\Lambda}_B(s)]_{11})} \leq 1 + \bar{\sigma}(\mathbf{G}_{12}(s)\mathbf{G}_{22}^{-1}(s)\mathbf{G}_{21}(s)\mathbf{G}_{11}^{-1}(s)) \quad (10)$$

Substituting (9) in (10) and rearranging,

$$\mu_{\mathbf{\Delta}}(\mathbf{E}(s)) \geq \sqrt{\left|\frac{1}{\underline{\sigma}([\mathbf{\Lambda}_B(s)]_{11})} - 1\right|} \quad (11)$$

Similarly, (8) can be shown to be true for $[\mathbf{\Lambda}_B(s)]_{22}$. $\square$

Using (8), it can be shown that,

$$\lim_{\underline{\sigma}([\mathbf{\Lambda}_B(0)]_{ii}) \to 0} \mu_{\mathbf{\Delta}}(\mathbf{E}(0)) = \infty$$

Thus the system in Figure 1 cannot be block diagonal dominant if $\underline{\sigma}([\mathbf{\Lambda}_B(0)]_{ii}) \ll 1$. Though this result is proven for the case, when the system is partitioned into two blocks, numerical evidence suggests that it is true for any partitioning.

3.4 Closed loop Interactions

If $\mathbf{G}(s) = \mathbf{G}_{bd}(s)$ or the system itself is block diagonal, it is trivially *non-interacting*. In this section, such a system is referred to as an *ideal* system. When the controller contains integral action, the sensitivity functions of the actual and ideal systems are related as,

$$\mathbf{S}(s) \approx \mathbf{S}_{bd}(s)\mathbf{\Gamma}(s) \quad (12)$$
$$\mathbf{S}(s) = (\mathbf{I} + \mathbf{G}(s)\mathbf{K}(s))^{-1}$$
$$\mathbf{S}_{bd}(s) = (\mathbf{I} + \mathbf{G}_{bd}(s)\mathbf{K}(s))^{-1}$$

where $\mathbf{\Gamma}(s) = \mathbf{G}_{bd}(s)\mathbf{G}(s)^{-1}$ is the Performance Relative Gain Array (PRGA) (Hovd and Skogestad, 1992). Let $\mathbf{\Gamma}(s)$ be expressed through its singular value decomposition as, $\mathbf{\Gamma}(s) = \mathbf{U}(s)\mathbf{\Sigma}(s)\mathbf{V}(s)^T$. Then,

$$\mathbf{\Gamma}(s)\mathbf{v}_i(s) = \sigma_i(s)\mathbf{u}_i(s), \quad \forall i = 1 \cdots n$$

where $\sigma_i(s)$ is the i^{th} singular value and $\mathbf{u}_i(s)$ and $\mathbf{v}_i(s)$ are the corresponding left and right singular vectors, calculated at a particular frequency. Grosdidier (1990) has argued that the exogenous signals oriented in the direction of singular vectors associated with $\bar{\sigma}(\mathbf{\Gamma}(s))$ most adversely affect the closed loop performance and vice versa. Then, a necessary condition for interactions to be minimum is that $\sigma_i(\mathbf{\Gamma}(s)) \approx 1$ for all $i = 1, \cdots n$. Recognizing that $[\mathbf{\Lambda}_B(s)]_{ii} = [\mathbf{\Gamma}(s)]_{ii}$,

$$\bar{\sigma}([\mathbf{\Lambda}_B(s)]_{ii}) \leq \bar{\sigma}(\mathbf{\Gamma}(s)) \quad (13)$$

Therefore, if $\bar{\sigma}([\mathbf{\Lambda}_B(s)]_{ii}) \gg 1$, then $\bar{\sigma}(\mathbf{\Gamma}(s)) \gg 1$. When $[\mathbf{\Lambda}_B(s)]_{ii} = \mathbf{I}$, then $\sigma_j([\mathbf{\Lambda}_B(s)]_{ii}) = 1$ for all $j = 1, \cdots m_i$. Then (13) suggests that $\bar{\sigma}(\mathbf{\Gamma}(s))$ can still be large, despite BRG being exactly the identity matrix.

Based on these observations and Proposition 7, we conclude that the system has large interactions if $\bar{\sigma}(\mathbf{\Lambda}_B(s)) \gg 1$ and $\underline{\sigma}(\mathbf{\Lambda}_B(s)) \ll 1$ or in other words, BRG *is very different from the identity matrix*, but the converse is not true. Use of PRGA is necessary for drawing any conclusions regarding closed loop interactions. Note that due to the approximation involved (see (12)), this result holds only at low frequencies.

4. ALTERNATE PAIRING RULES

In earlier sections, it was shown that useful information regarding closed loop properties can be extracted using BRG. In this section, we summarize those results in the form of pairing rules.

Pairing Rule 1. Avoid pairing on variables with $\det([\mathbf{\Lambda}_B(0)]_{ii}) \leq 0$ (Proposition 3 and Corollary 6).

Pairing Rule 2. Avoid pairing on variables if $\underline{\sigma}([\mathbf{\Lambda}_B(0)]_{ii}) \ll 1$ for some $i = 1, \cdots M$ (Propositions 7 and 8).

Pairing Rule 3. Prefer pairing on variables for which $\sum_i |\sigma_i(\mathbf{\Gamma}(0)) - 1|$ is small, provided Rules 1 and 2 are satisfied (see §3.4).

These rules are based on gain information only and may suggest inferior pairings for systems containing large time delays. In such cases, if a reliable dynamic model is available, then ensuring that $\sum_i |\sigma_i(\mathbf{\Gamma}(s)) - 1|$ is small up to the crossover frequency is helpful. In addition,

Pairing Rule 4. Avoid pairing on variables with different signs of $\det([\mathbf{\Lambda}_B(0)]_{ii})$ and $\det([\mathbf{\Lambda}_B(j\infty)]_{ii})$ (Proposition 5).

Remark 9. Alternatives satisfying $\mu_{\mathbf{\Delta}}(\mathbf{E}(0)) < 1$ also possess the property of decentralized integral controllability (DIC) resulting in easier on-line controller tuning. However, the computational load for calculation of μ is substantial (Skogestad and Postlethwaite, 1996). Then, Pairing Rule 2 can be seen as a pre-screening step resulting in reduced computational load.

Remark 10. Since BRG and PRGA are output scaling dependent, so are its singular values. Therefore, prior to pairing selection, specification of a suitable scaling of system matrix is necessary to avoid ambiguity. A possible approach is to normalize the system matrix such that $\|y_i\| \leq 1$ for all $i = 1, \cdots n$.

Remark 11. These pairing rules equally hold for fully decentralized control structures. For many problems, $\sum_i |\sigma_i(\mathbf{\Gamma}(0)) - 1|$ is small, if the diagonal elements of RGA elements are close to 1. Thus, Bristol's rule of pairing on RGA elements close to 1 is implicit here, but, in general, it is neither necessary nor sufficient for the system to be weakly interacting.

5. NUMERICAL EXAMPLE

Example 12. Consider the 4×4 ALSTOM gasifier system (Dixon *et al.*, 2000). The gasifier is described by 3 linearized state space models of 25^{th} order at 100%, 50% and 0% load conditions. Prior to pairing selection, the outputs of the system are scaled such that $\|y_i\|_2 \leq 1$ at all load conditions.

Screening of various block decentralized alternatives for the system is done such that $\det([\mathbf{\Lambda}_B(0)]_{ii})$ > 0 and $\underline{\sigma}(\mathbf{\Lambda}_B(0)) > 0.1$ at different load conditions. A representative set of alternatives satisfying these conditions is presented in Table 1. The pairings $(1, 1)$ [1] and (1-4, 1-4) contain RHP zeros at $s = 3.3013$ and 3.2879 respectively at 100% load conditions making the use of these alternatives unattractive.

Based on steady state analysis, ((1-2-4, 1-3-4),(3-2)) seems to be the best structure. This was

[1] (1,1) represent the pairing $(\mathbf{y}_1, \mathbf{u}_1)$.

Pairing	100% load $\min_i \underline{\sigma}([\mathbf{\Lambda}_B(0)]_{ii})$	$\sum_i \lvert\sigma_i(\mathbf{\Gamma}(0)) - 1\rvert$	0% load $\min_i \underline{\sigma}([\mathbf{\Lambda}_B(0)]_{ii})$	$\sum_i \lvert\sigma_i(\mathbf{\Gamma}(0)) - 1\rvert$	*Remarks*
(1,1),(2,3),(3,2),(4,4)	0.33	2.96	0.48	3.28	RHP zero
(1-2,1-3),(3,2),(4,4)	0.73	2.41	0.48	1.56	
(1-4,1-4),(2,3),(3,2)	0.17	2.99	0.44	4.15	RHP zero
(1-2,1-3),(3-4,2-4)	0.86	1.83	0.53	1.73	
(1-2-3,1-2-3),(4-4)	0.30	2.99	0.43	1.16	
(1-2-4,1-3-4),(3-2)	0.80	1.86	0.75	1.12	

Table 1. Block decentralized pairings for ALSTOM gasifier system

further confirmed by using frequency-dependent PRGA. It is also seen that this alternative satisfies $\mu_{\mathbf{\Delta}}(\mathbf{E}(0)) < 1$ at all load conditions and thus is DIC.

This system has also been analyzed by Chin and Munro (2002) at 100% load conditions, where they have suggested the use of $((\text{1-3-4}, \text{2-3-4}), (\text{2-1}))$. This alternative satisfies Rules 1 and 2 at 100% load conditions, but the relative gain of the pairing (2-1) is negative at 0% load conditions. This shows that this alternative will be loop failure sensitive under varying operating conditions.

6. CONCLUSIONS

The main contributions of this paper include:

(i) an extension of Bristol's rule of avoiding pairing on negative RGA elements to block pairings (Proposition 3),
(ii) a connection between Grosdidier's interaction measure and BRG. (Proposition 8),
(iii) a correction and restatement of the common conjecture that a system is *weakly* interacting, if BRG is close to the identity matrix (§3.4).

The pairing rules proposed in this paper will be helpful in selecting block pairings for the system.

ACKNOWLEDGEMENTS

After submission of the manuscript, we found that Proposition 3 was shown to be true earlier by Grosdidier and Morari (Grosdidier and Morari, 1987). The financial support from *National Sciences and Engineering Research Council of Canada* is gratefully acknowledged.

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