DESIGN OF SUB-OPTIMAL ROBUST GAIN-SCHEDULED PI CONTROLLERS

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Abstract**:** A methodology is proposed for the analysis and design of a robust gain-scheduled PI controller for nonlinear chemical processes. The stability and performance tests can be formulated as a finite set of linear matrix inequalities (LMI) and hence, the resulting problem is numerically tractable. Input saturation and model error are explicitly incorporated into the analysis. A simulation study of a nonlinear CSTR (continuous stirred tank reactor) process indicates that this approach can provide useful sub-optimal robust controllers.

Keywords: robust control, nonlinear systems.

1. INTRODUCTION

This paper derives LMI-based tests, to test the closed-loop stability and performance of gain-scheduled Proportional-Integral (PI) controllers, when applied to nonlinear processes.

The design of gain-scheduled controllers for Linear Parameter Varying (LPV) systems has been reported in a number of publications (e.g. Shamma and Athans, 1992) and software is available, e.g. Matlab, to design these controllers using LMI. Two main problems in the application of these techniques to chemical engineering processes are: i- models of chemical systems are often not available in LPV form ready for the LMI's tests, ii- The LMI-based methodology results in controller structures that are significantly more complex than the PI or PID control forms, which are widely accepted by the chemical industry.

Following these, Knapp and Budman (2001) have proposed to model nonlinear processes with a special class of state-affine nonlinear discrete model. These state-affine models are in LPV form where the manipulated variable fulfills the role of the time-varying parameter. They showed that by using these models in combination with a discrete PI controller, the analysis of the closed loop system can be reduced to the solution of a set of LMI. These models are nonlinear with respect to the manipulated variables and then, this input nonlinearity is treated as model uncertainty with respect to a linear nominal model. Then, the robust stability and performance of the closed loop system can be analyzed with respect to this model uncertainty.

Using these state-affine models in combination with the proposed gain-scheduled PI controller, the closed-loop system can be represented by a class of discrete-time systems state-space equations with a state vector η.

For time-varying real uncertainty, a quadratic stability test seeks a fixed quadratic Lyapunov function $V(t)=\eta(t)^T P\eta(t)$ that proves stability for all admissible uncertainties. It is shown that finding an adequate P, amounts to solving a

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convex problem involving a system of LMI. This system of LMI can be extended to test robust performance as well.

In the current paper we have expanded the work of Knapp (2001) by considering a special class of scheduled PI controllers, defined in section 2, where the tuning coefficients of the controller are linear functions of the manipulated variable. These linear functions are defined in terms of 4 parameters. Then, this work also addresses the optimization of these parameters. The parameterization of the controller in terms of a small number of parameters greatly facilitates the optimization step.

The paper is organized as follows. Section 2 presents the state-affine model realization and the gain-scheduled PI controller structure. Section 3 derives LMI-based stability condition. Section 4 develops the performance condition and addresses the performance optimization problem. Section 5 integrates input saturation and modeling error into the analysis. Section 6 illustrates the validity of the design approach by a case study example. Section 7 summarizes the conclusions and future work.

2. STATE-AFFINE MODEL AND GAIN-SCHEDULED PI CONTROLLER

Based on Knapp and Budman's (2000, 2001) work, a state-affine model for a nonlinear process is obtained as follows

$$x(t+1)=\{F_0+\sum_{i=1}^{n}F_i u(t)^i\}x(t)+\{G_1+\sum_{i=1}^{n}G_{i+1}u(t)^i\}u(t)$$
$$y(t)=H_0x(t)+d(t) \qquad (1)$$

where F,G,H are polynomial matrices. Disturbances of infinite frequencies can not be effectively rejected unless an infinite closed-loop bandwidth is used, because of robust stability limitations. Therefore, the actual disturbance $v(t)$ is filtered through a low-pass filter as follows:

$$d(t+1)=BWd(t)+(1-BW)v(t) \qquad (2)$$

Where $0\le BW\le 1$, which is a bandwidth related weight.

A gain-scheduled PI controller of the form given by (3) is used. When $W_c=W_d=0$, the control law $\hat{u}$ reduces to a conventional discrete PI controller with proportional gain K_c and reset time τ_I. Thus the coefficients C_c and D_c of the PI controller are augmented in equation (3) by a linear dependency with respect to the manipulated variable u to allow for scheduling as a function of u. $\hat{u}(t)$ stands for the control action calculated without saturation whereas $u(t)$ is computed with saturation limits.

$$\begin{aligned}&\xi(t+1)=A_c\xi(t)+B_ce(t)\\&\hat{u}(t)=(C_c+W_cu(t))\xi(t)+(D_c+W_du(t))e(t)\\&e(t)=y_d(t)-y(t),\,y_d(t)=0\\&A_c=1,B_c=1,C_c=\frac{K_c}{\tau_I},D_c=K_c+\frac{K_c}{\tau_I}\end{aligned} \qquad (3)$$

For a process represented by the state-affine model (1), at the nominal operating point, it is valid to assume that the process can be accurately modeled by the linear part of the state-affine model given by (4). It is also assumed that most of the model uncertainty is due to the time-varying nonlinearity of the state-affine model around this operating point. It is therefore possible to describe the model uncertainty δ_i in the form of (5).

$$\begin{aligned}x(t+1)&=F_0x(t)+G_1u(t)\\y(t)&=H_0x(t)\end{aligned} \qquad (4)$$

$$\delta_i=u(t)^i,i=1,2,\ldots,n \qquad (5)$$

(5) represents the key advantage of the methodology used here. In general it is very difficult to quantify the uncertainty, δ_i, from mechanistic first-principle models (Doyle, 1990). In our case, since δ_i is equal to the powers of the input, it can be easily quantified. Each input in a process is known to lie between a lower and an upper limit known during the design stage due to, for example, actuator constraints or economic considerations. According to (5):

$$u(t)\in\left[\underline{u}\quad\overline{u}\right]\rightarrow\delta_i\in\left[\underline{\delta_i}\quad\overline{\delta_i}\right] \qquad (6)$$
$$S:=\{(\omega_1,\omega_2,\cdots,\omega_n):\omega_i\in\{\underline{\delta_i},\overline{\delta_i}\}\}$$

Rewriting (1) using (5) gives:

$$\begin{aligned}x(t+1)&=\{F_0+\sum_{i=1}^{n}F_i\delta_i\}x(t)+\{G_1+\sum_{i=1}^{n}G_{i+1}\delta_i\}u(t)\\y(t)&=H_0x(t)+d(t)\end{aligned} \qquad (7)$$

The closed-loop system of (7), (2) and (3) is then put into a form given by (8) suitable for analysis. The state matrix $A(\delta)$ depends on the uncertainties defined by (6).

$$
\begin{aligned}
&A_{11} = F_0 + \sum_i F_i \delta_i - (G_1 + \sum_i G_{i+1}\delta_i)(D_c + W_d \delta_1) H_0 \psi \\
&A_{12} = (G_1 + \sum_i G_{i+1}\delta_i)(C_c + W_c \delta_1)\psi \\
&A_{13} = -(G_1 + \sum_i G_{i+1}\delta_i)(D_c + W_d \delta_1)\psi \\
&A(\delta) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ -B_c H_0 \psi & A_c \psi & -B_c \psi \\ 0 & 0 & BW \end{bmatrix} \\
&\begin{bmatrix} \eta(t+1) \\ e(t) \end{bmatrix} = \begin{bmatrix} A(\delta) & B \\ C & D \end{bmatrix} \begin{bmatrix} \eta(t) \\ \nu(t) \end{bmatrix} \Leftrightarrow \\
&\begin{bmatrix} x(t+1) \\ \xi(t+1) \\ d(t+1) \\ \hline e(t) \end{bmatrix} = \begin{bmatrix} A(\delta) & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \\ d(t) \\ \hline \nu(t) \end{bmatrix} \\
&B = \begin{bmatrix} 0^T & 0^T & (1 - BW)^T \end{bmatrix}^T \quad C = \begin{bmatrix} -H_0 & 0 & -1 \end{bmatrix} \\
&D = [0], \eta(0) = \eta_0 \qquad (8)
\end{aligned}
$$

3. QUADRATIC STABILITY

Consider the uncertain nonlinear system (8). This system is quadratic stable if there exists a positive-definite quadratic Lyapunov function

$$V(t) = \eta(t)^T P \eta(t), \; P > 0 \qquad (9)$$

such that $V(t) > 0$ and $V(t+1) - V(t) < 0$ for all admissible uncertainties and for all initial conditions η_0.

Definition 3.1(quadratic stability): The system

$$\eta(t+1) = A(\delta)\eta(t), \eta(0) = \eta_0 \qquad (10)$$

is quadratically stable if there exists a symmetric matrix P such that

$$P > 0 \qquad (11)$$

$$A(\delta)^T P A(\delta) - P < 0 \qquad (12)$$

hold for all admissible uncertainties.

When δ ranges in a polytope with vertices in S, it suffices to enforce (12) at the vertices, and so (12) is equivalent to the following convex LMI problem

$$A(\omega)^T P A(\omega) - P < 0, \; for \quad all \quad \omega \in S \qquad (13)$$

A complete summary of LMI theory is given by Boyd et al. (1994).

4. QUADRATIC H_∞ PERFORMANCE

Definition 4.1 (quadratic H_∞ performance): System (1) with zero initial state has quadratic H_∞ performance γ if there exists a symmetric matrix P such that

$$P > 0 \qquad (14)$$

$$\begin{bmatrix} A(\delta)^T P A(\delta) - P & A(\delta)^T P B & C^T \\ B^T P A(\delta) & B^T P B - \gamma^2 I & D^T \\ C & D & -I \end{bmatrix} < 0 \qquad (15)$$

is satisfied for all admissible uncertainties.

This condition establishes that the closed-loop system defined by (8) satisfies $\|e\|_{L_2} < \gamma \|v\|_{L_2}$ for all L_2-bounded input ν, that is (15) guarantees

$$V(t+1) - V(t) + e^T(t)e(t) - \gamma^2 v^T(t)v(t) < 0 \qquad (16)$$

(15) is equivalent to the finite LMI as follows

$$\begin{bmatrix} A(\omega)^T P A(\omega) - P & A(\omega)^T P B & C^T \\ B^T P A(\omega) & B^T P B - \gamma^2 I & D^T \\ C & D & -I \end{bmatrix} < 0$$
$$for \quad all \quad \omega \in S \qquad (17)$$

Equation (17) is solved as a generalized eigenvalue problem (GEVP), to optimize γ.

5. INPUT SATURATION AND MODELING ERROR

Input saturation would occur when the controller outputs $\hat{u}(t)$ exceeded the limits. The gain-scheduled PI controller can be reformulated using a variable gain $\widetilde{K}_c$. Define:

$$\psi = \frac{1}{|\hat{u}|} = \frac{1}{K_c} \frac{1}{\left| \frac{1}{\tau_I}\xi + (1 + \frac{1}{\tau_I})e \right|} \tag{18}$$

Then the gain of the controller is given by:

if $0 \le \psi \le 1 \quad \widetilde{K}_c = K_c \psi$

else $\psi > 1 \quad \widetilde{K}_c = K_c = \text{constant}$

These definitions ensure that $|u|$ never exceeds the saturation limit of 1 whereas $|\hat{u}|$ can exceed the limit.

A lumped error δ_t in the output is considered as the modeling error so that the H matrix can be rewritten as follows:

$$H = H_0 \xrightarrow{\delta_t} H = H_0 + W_t \delta_t \tag{19}$$

δ_t can be easily calculated from the difference between the model prediction and the actual data from the process (Budman and Knapp, 2000 and 2001). Limits of ψ and δ_t need to be taken into account in the stability and performance analysis.

6. DESIGN CASE STUDY: CSTR

The case study under investigation is a CSTR from Doyle et al. (1989). A state-affine mode is first obtained, see (Budman and Knapp, 2000 and 2001). Input saturation with $\psi \in [0.4 \quad 1]$ and modeling error with $\delta_t = [-1 \quad 1]$ and $W_t = 0.025$, will also be considered. In principle, the lower limit of ψ should have been assumed to be equal to zero for the case that the calculated control action is infinite. When a lower limit of zero was assumed, robustness could not be achieved. Fortunately, the output in a real process is always bounded due to sensor saturation or the physical limitation of the process, e.g. conversion cannot be larger than 1. Accordingly, following equation (3), a finite upper limit for the control action exists and consequently a lower bound of ψ larger than zero can be assumed.

Fig.1 shows the robust stability and robust performance $(\gamma = 1)$ regions for linear PI controllers, i.e. with W_c and W_d equal to zero, defined in terms of the proportional gain and reset time.

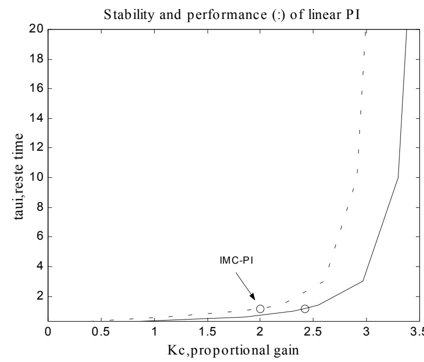

Fig.1. Stability and performance regions of linear PI controller parameters. Stability region is the area above the solid line including the solid line as limit. Performance region for a $\gamma = 1$ is the area above the dotted line including the dotted line as limit.

For the purpose of comparison with the gain-scheduled controller, a set of PI controller parameters was selected in the neighborhood of the robust performance boundary shown in Fig. 1. as follows: $K_c = 2$ and $\tau_I = 1.1545$. This point corresponds approximately to the Internal Model Control (IMC) tuning parameters around the nominal operating point based on the rules available in the literature (Morari and Zafiriou, 1989). Using these linear PI controller parameters in equation (3), gain-scheduled PI controller weights W_c, W_d can be calculated according to the stability and performance tests presented above. Accordingly, regions of robust stability and robust performance are computed in terms of different combinations of the weights and the results are shown in Fig.2 and 3. The circles shown in Fig.2 and 3 represent the linear PI controllers selected on the limit of robust stability and performance, respectively, i.e. $W_c = 0, W_d = 0$, also shown on the curves in Fig.1.

In order to improve upon the performance of the linear PI controller, a pair of gain scheduling weight values can be sought inside the robust performance region, corresponding to a point indicated by a star in Fig.3, that will provide a better performance. Since the performance of the controller is directly related to the parameter γ as shown by equation (17) the objective is to minimize this value.

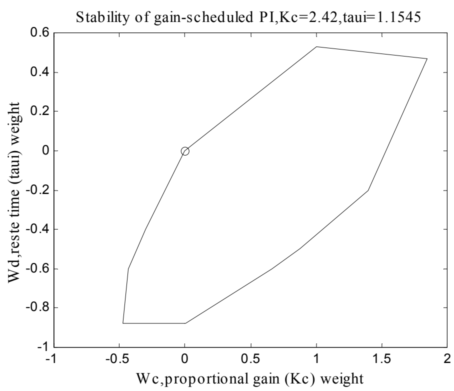

Fig.2. Stability region of gain-scheduled PI controller weights, that is, the area inside the solid box including the solid circle as limit.

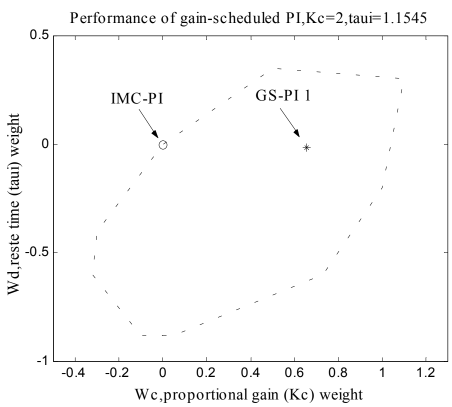

Fig.3. Performance region of gain-scheduled PI controller weights, that is, the area inside the dotted box including the dotted circle as limit.

The problem of searching for a $\gamma_{optimal}$ is not convex in terms of the controller parameters. The conditions result in a nonlinear matrix inequality for the controller parameters. Branch and bound methods have been proposed to solve LMI's systems of this type (Fukuda and Kojima, 2001; Braatz, et al., 1997). For simplicity, it was decided to limit the search to a sub-optimal design in the neighborhood of the selected linear PI controller using the FMIN optimization function in Matlab. This was done by using K_c and τ_I computed by the IMC rules and by optimizing the values of the weights W_c and W_d. The objective is to assess the improvement in performance over that obtained with this IMC-PI controller. Subsequently, an additional optimization was conducted where all the parameters, i.e. K_c, τ_I and the weights, were allowed to change to minimize γ.

The optimization of the controller weights using the GEVP procedure produces the best robust gain-scheduled PI controller in the neighborhood of the IMC design, shown as a star in Fig.3. For this design $\gamma^*_{optimal}$ =0.5890 and this is an improvement of 38.9% over $\gamma^o_{optimal}$ =0.9634 in robust performance obtained with the IMC-PI design. When all the parameters are optimized, an additional improvement in performance is obtained with $\gamma_{optimal}$ =0.3894. Table 1 summarizes the optimization results.

Table 1 Optimization design results

	IMC-PI	G-S PI 1	G-S PI 2
K_c	2	2	1.3723
τ_I	1.1545	1.1545	2.949
W_c	0	0.6547	-0.004
W_d	0	-0.015	0.001
$\gamma_{optimal}$	0.9634	0.5890	0.3894
$\gamma_{simulation}$	0.3787	0.3495	0.202

To assess the conservatism of the analysis a simulation study is conducted for the CSTR using the different controllers synthesized in this work. The performance is tested by investigating through a large number of simulations how the system rejects a bounded disturbance. $\gamma_{simulation}$ is used to refer to the performance limit obtained from the simulation.

$\gamma_{simulation}$ calculated based $\|e\|_{L_2} < \gamma \|v\|_{L_2}$ is always bounded by $\gamma_{optimal}$ in each case, indicating that the analysis tests produce a worst-performance bound as expected and it is not exceeded. The difference between $\gamma_{optimal}$ and $\gamma_{simulation}$ shows that the design procedure is conservative to some degree.

Simulations were conducted for a large number of different disturbances. A disturbance was sought that would result in the worst performance for each controller. Then for the worst case found from simulation, $\gamma_{simulation}$ was calculated. Simulation results for the IMC-PI controller and for the sub-optimal gain-scheduled PI controller are shown in Fig.4. These simulations correspond to a spike type disturbance also shown in Fig.4. Worse performance than the one shown in Fig.4 may be also possible but there is no systematic way to find the specific disturbance function that will lead to it.

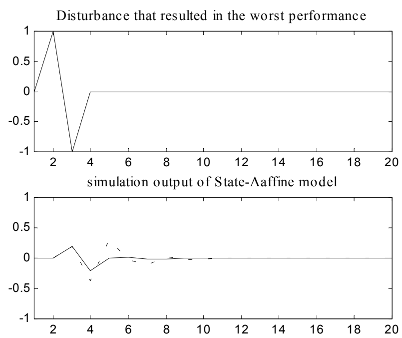

Fig.4. Closed-loop simulations of state-affine model (lower two curves).
Linear PI controller (dotted line), $K_c = 2, \tau_I = 1.1545, \gamma_{simulation} = 0.3787$.
Gain-scheduled PI controller (solid line), $K_c = 1.3723, \tau_I = 2.949, W_c = -0.004$, $W_d = 0.001, \gamma_{simulation} = 0.202$.

Conservatism associated with the design approach comes from two main facts. First, a possible source of this conservatism is that simulation can only be done on a limited period of time, while the calculation of the performance condition requires an infinite simulation interval. Second, conservatism is obviously inherent to the robust control approach where several scenarios included in the analysis will not occur during actual closed-loop operation.

7. CONCLUSIONS

An approach is proposed to design gain-scheduled PI controllers for nonlinear processes using process data. It is based on empirical state-affine models of the process. Gain-scheduled PI controller with sub-optimal performance is obtained using a GEVP based optimization algorithm. Simulations show that the gain-scheduled controller provides better performance than a conventional PI controller found for robustness with IMC rules. A performance index γ, although conservative, has been found to be a good indicator of the relative performance of the different controllers.

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