IDENTIFICATION OF MULTIRATE SAMPLED-DATA SYSTEMS

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Abstract: This paper studies identification of a general single-input and single-output (SISO) multirate sampled-data system. Using the lifting technique, we associate the multirate system with an equivalent linear time-invariant lifted system, from which a fast-rate discrete-time system is extracted. Uniqueness of the fast-rate system, controllability and observability of the lifted system, and other related issues are discussed. The effectiveness is demonstrated through simulation and a real-time implementation.

Keywords: Multirate sampled-data system, System identification, Controllability and observability, Lifting technique

1. INTRODUCTION

The term Multirate Sampled-Data (MRSD) Systems describes a common phenomena existing in the industry that different variables are sampled at different rates for some reasons (Chen and Qiu, 1994), e.g., a high-purity distillation column (Lee, *et al.*, 1992) and a bioreactor (Gudi, *et al.*, 1995) and CCR octane quality control (Li, *et al.*, 2003). Fig. 1 depicts a SISO MRSD system, where G_c is a continuous-time linear time-invariant (LTI) and causal system with or without a time-delay; H is a zero-order hold with an updating period mh and S is a sampler with period nh, where m, n are different positive integers and h is a positive real number called the base period; discrete-time signals u and y are the system input and output respectively; a continuous-time signal v_c is the unmeasured disturbance. Essentially, it is a linear periodically time-varying (LPTV) system (Kranc, 1957), to which many system identification algorithms cannot be applied directly.

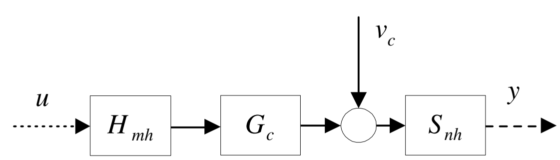

Fig. 1. A SISO multirate sampled-data system

Under such a framework, Lu and Fisher (1988,1989) used an output error method and a least-squares method to estimate intersample outputs based on the fast sampled inputs and slow sampled outputs. Verhaegen and Yu (1994) extended a Multivariable Output Error State Space (MOESP) class of algorithms to identify P subsystems of an LPTV process with period P. Gudi, *et al.* (1995) generated frequent estimates of the primary output based on the secondary outputs and the regular measurement of inputs by an adaptive inferential strategy. Li, *et al.* (2001) identified a fast single-rate model with period mh from multirate input and output data, with an assumption that $m < n$. This work motivates us: Could we do better?

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Doing better implies two things: first, a fast-rate model with period h instead of mh will be identified; second, a general MRSD system is treated without the assumption $m < n$. Note that our objective includes that of Li, *et al.* (2001), since a model with period mh is readily obtained from a model with period h. The improvement is significant: technically, we need to use additional conditions such as observability of lifted models and coprimeness of the integers m and n (to be clarified later); in terms of applications, the availability of the fast-rate model with period h broadens the choices for multirate control design; the relaxation of assumptions makes identification of fast-rate models for more general MRSD systems possible.

The question states precisely as follows: For a sampling period h, the unknown continuous time system G_c has a discrete time counterpart realized by the step-invariant-transformation, $G_d := S_h G_c H_h$, represented by a state-space model:

$$D + C\left(zI - A\right)^{-1} B = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \qquad (1)$$

Given the multirate sampled-data system in Fig. 1, how to identify the so-called fast-rate system G_d?

To answer this question, we start in Section 2 with using the lifting technique to associate such an LPTV system with an LTI system, the so-called lifted system. The uniqueness of recovering the fast-rate system from the lifted system is shown in Section 3. Section 4 analyzes controllability and observability of the lifted system, which are essential to the identifiability issues. Section 5 presents two approaches to compute a fast-rate model. Section 6 illustrates the effectiveness of the proposed methods through two examples. We end with some conclusions in Section 7.

2. LIFTING SIGNALS AND SYSTEMS

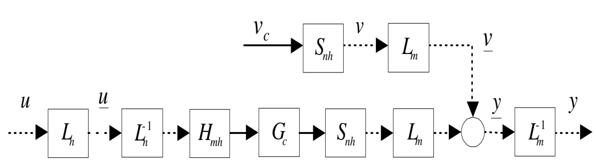

Fig. 2. The lifted multirate sampled-data system

Henceforth, we will focus our discussion on the SISO MRSD system depicted in Fig. 1. Let ψ be a discrete-time signal defined on Z_+ and n be some positive integer. The n-fold lifting operator L_n is defined as the mapping from ψ to $\underline{\psi}$:

$$\{\psi(0), \psi(1), \cdots\} \mapsto \left\{ \begin{bmatrix} \psi(0) \\ \psi(1) \\ \vdots \\ \psi(n-1) \end{bmatrix}, \begin{bmatrix} \psi(n) \\ \psi(n+1) \\ \vdots \\ \psi(2n-1) \end{bmatrix}, \cdots \right\}.$$

We lift u by L_n into $\underline{u}$, and lift y by L_m into $\underline{y}$. The disturbance v_c is fictitiously sampled into v with period nh, same as the output sampling period, and v is lifted by L_m into $\underline{v}$ (see Fig. 2). Thus, $\underline{u}$, $\underline{y}$ and $\underline{v}$ share the same period mnh, and form a discrete-time LTI system (Francis and Georgiou, 1988):

$$\underline{y} = \underline{G_d}\, \underline{u} + \underline{v} \qquad (2)$$

Here $\underline{G_d}$ is the so-called lifted system from $\underline{u}$ to $\underline{y}$; it has a state space representation by matrices $\underline{A}$, $\underline{B}$, $\underline{C}$ and $\underline{D}$, which are related to A, B, C and D of (1) as shown in Chen and Qiu (1994):

$$\left[\begin{array}{c|c} \underline{A} & \underline{B} \\ \hline \underline{C} & \underline{D} \end{array}\right] := \qquad (3)$$

$$\left[\begin{array}{c|cccc} A^{mn} & \sum_{i=mn-m}^{mn-1} A^i B & \sum_{i=mn-2m}^{mn-m-1} A^i B & \cdots & \sum_{i=0}^{m-1} A^i B \\ \hline C & D_{00} & D_{01} & \cdots & D_{0,n-1} \\ CA^n & D_{10} & D_{11} & \cdots & D_{1,n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ CA^{mn-n} & D_{m-1,0} & D_{m-1,1} & \cdots & D_{m-1,n-1} \end{array}\right]$$

where

$$D_{ij} = D\chi_{[jm,(j+1)m)}(in) + \sum_{r=jm}^{(j+1)m-1} CA^{in-1-r} B\chi_{[0,in)}(r)$$

and a characteristic function on integers is defined:

$$\chi_{[a,b)}(r) = \begin{cases} 1, & a \le r < b \\ 0, & \text{otherwise.} \end{cases}$$

A noise model can be used to further describe the character of the noise term $\underline{v}$ in (2), but it is not within our current objective. Hence, we adopt an output error model structure, since for open loop systems, output error models will give consistent estimates, even if the additive noise is not white (Ljung, 1999). An innovation form of the state-space model with the Kalman filter gain $\underline{K} = 0$ represents the overall discrete-time lifted system:

$$\dot{x} = \underline{A}\, x + \underline{B}\, \underline{u} + \underline{K}\, \underline{e}, \qquad (4)$$

$$\underline{y} = \underline{C}\, x + \underline{D}\, \underline{u} + \underline{e}. \qquad (5)$$

Here overdot denotes one sample advance, $\underline{e}$ is a white noise vector and x is a state vector. If p is the order of G_d, then the dimensions of A, B, C, D are $p \times p$, $p \times 1$, $1 \times p$, and 1×1, respectively, and those of $\underline{A}, \underline{B}, \underline{C}, \underline{D}, \underline{K}$ are $p \times p$, $p \times n$, $m \times p$, $m \times n$, and $p \times m$, respectively. Note that $\underline{A}$ and A share the same dimension.

3. UNIQUENESS OF FAST-RATE SYSTEMS

Before starting the exploration of recovering the fast-rate system from the lifted one, a question arises naturally: Is the recovery of G_d from $\underline{G_d}$ unique? The answer is affirmative if m and n are coprime. We observe:

$$\begin{aligned}\underline{G_d} &= L_m S_{nh} G_c H_{mh} L_n^{-1} \\ &= L_m S_n \left(S_h G_c H_h\right) H_m L_n^{-1} \\ &= L_m S_n G_d H_m L_n^{-1},\end{aligned} \tag{6}$$

by properties $S_{nh} = S_n S_h$ and $H_{mh} = H_h H_m$, where S_n and H_m are the discrete-time downsampler and the discrete-time zero-order-hold type upsampler respectively. Since the lifting is one-to-one, the problem of recovery of a unique G_d from $\underline{G_d}$ is equivalent to answering a question: Is the mapping $G_d \longmapsto S_n G_d H_m$ one-to-one?

Proposition 1. Assume G_d is LTI and causal. Then, the mapping $G_d \longmapsto S_n G_d H_m$ is one-to-one if and only if m and n are coprime.

Proof:

For sufficiency, it suffices to show that $S_n G_d H_m = 0$ implies $G_d = 0$. Let us assume $S_n G_d H_m = 0$ and let μ be the impulse response of G_d, i.e., $\mu = G_d \delta$, where δ is the discrete-time unit impulse signal. It follows that for any integer i, $S_n G_d H_m U^i \delta = 0$, where U is the unit time-delay operator. This implies, by the definition of H_m,

$$S_n G_d \left(U^{im} + U^{im+1} + \cdots + U^{im+m-1}\right) \delta = 0.$$

The time invariance of G_d and the definition of S_n imply

$$\begin{aligned}&\mu\left(im + jn\right) + \mu\left(im + jn + 1\right) + \cdots \\ &\quad + \mu\left(im + jn + m - 1\right) = 0, \ \forall i, \ j.\end{aligned} \tag{7}$$

Since m and n are coprime, there exist integers m' and n' such that $mm' + nn' = 1$. Thus, for any k, there always exist $i = km'$ and $j = kn'$ in (7) to get $im + jn = k$. Hence,

$$\mu\left(k\right) + \mu\left(k+1\right) + \cdots + \mu\left(k + m - 1\right) = 0, \forall k. \tag{8}$$

By causality of $\mu\left(k\right)$, (8) implies that $\mu\left(k\right) = 0$, $\forall k$, e.g., if $k = -\left(m-1\right)$, then $\mu\left(0\right) = 0$; if $k = -\left(m-2\right)$, then $\mu\left(1\right) = 0$ and so on. Hence, $G_d = 0$.

The necessity is proved as follows. If m and n are not coprime, there exists a common factor k: $m = km', n = kn'$, where m' and n' are coprime. It follows from (6) that $S_n G_d H_m = S_{n'} G_{kd} H_{m'}$ where $G_{kd} = S_{kh} G_c H_{kh}$, i.e., a discrete-time counterpart of G_c with period kh. Thus, the mapping $G_d \longmapsto S_n G_d H_m$ is not one-to-one, since the mapping $G_d \longmapsto G_{kd} = S_k G_d H_k$ is known to be not injective. □

Therefore, in order to get a unique fast-rate system we assume that m and n are coprime. Note that any common factor of m and n can be absorbed into h.

4. LIFTED SYSTEMS

4.1 *Controllability and Observability*

For a state space system to be identifiable, the lifted system $\underline{G_d}$ generally needs to be controllable and observable (Ljung *et al.*, 1999). If the continuous-time system G_c is controllable and observable and the sampling period is non-pathological, then the discrete-time system G_d is also controllable and observable (Kalman, *et al.*, 1963), which is still valid if a continuous time delay exists. Francis and Georgiou (1988) have proved that if G_d is stabilizable and detectable, and satisfies an additional condition $(*)$: For every eigenvalue λ of A, none of the $mn-1$ points

$$\lambda e^{j\frac{2\pi k}{mn}}, k = 1, 2, \cdots, mn-1$$

is an eigenvalue of A, then $\left(A^{mn}, A^i B\right)$ is stabilizable and $\left(CA^i, A^{mn}\right)$ is detectable, for any positive integer i. Based on these, we reach:

Proposition 2. Assume A satisfies the condition $(*)$. If (C, A) is observable, so is $(\underline{C}, \underline{A})$; If (A, B) is controllable and A has no eigenvalues on the unit circle, $(\underline{A}, \underline{B})$ is also controllable.

Proof: The first part follows with some trivial modifications from Francis and Georgiou (1998) in which $\left(CA^i, \underline{A}\right)$ was shown detectable. We prove the second part by showing $\left(\underline{A}, \sum_{i=0}^{m-1} A^i B\right)$ is controllable, i.e., all eigenvalues of $\underline{A}$ are controllable. Now each eigenvalue of $\underline{A}$ has the form λ^{mn}, where λ is an eigenvalue of A. Define functions:

$$g\left(s\right) := \frac{s^{mn} - \lambda^{mn}}{s - \lambda},$$

$$f\left(s\right) := \sum_{i=0}^{m-1} s^i.$$

By non-pathological sampling, $g\left(A\right)$ is invertible (Chen and Francis, 1995). If A has no eigenvalues on the unit circle, then $\sum_{i=0}^{m-1} \lambda^i \neq 0$. Thus $f\left(A\right)$ is invertible. Therefore,

$$\begin{aligned}&\operatorname{rank}\left(\left[\begin{array}{cc}\left(A^{mn} - \lambda^{mn} I\right) & \sum_{i=0}^{m-1} A^i B\end{array}\right]\right) \\ &= \operatorname{rank}\left(f\left(A\right)\left[\begin{array}{cc} A - \lambda I & B\end{array}\right]\left[\begin{array}{cc} f^{-1}\left(A\right) g\left(A\right) & 0 \\ 0 & I\end{array}\right]\right) \\ &= \operatorname{rank}\left(\left[\begin{array}{cc} A - \lambda I & B\end{array}\right]\right).\end{aligned}$$

Thus, $(\underline{A}, \underline{B})$ is controllable. $\square$

4.2 *Effect of Time Delays*

If there exists a continuous time delay τ larger than h, A has at least two poles at $z = 0$ (Åström and Wittenmark, 1997). Thus, the condition $(*)$ is not satisfied. Observability has been shown to be lost and a remedy is proposed by Li, *et al.* (2001), which is summarized below:

First, we can identify an $m \times n$ time-delay matrix Γ from $\underline{u}$, $\underline{y}$ using correlation analysis (Ljung *et al.*, 1999):

$$\Gamma = \begin{bmatrix} l_{00} & l_{01} & \cdots & l_{0,n-1} \\ l_{10} & l_{11} & \cdots & l_{1,n-1} \\ \vdots & \vdots & & \vdots \\ l_{m-1,0} & l_{m-1,1} & \cdots & l_{m-1,n-1} \end{bmatrix}$$

where l_{ij} is the estimated time delay from the j-th input $\underline{u}_j$ to the i-th output $\underline{y}_i$, $i = 0, 1, \cdots, m-1$ and $j = 0, 1, \cdots, n-1$. The relation between l_{ij} and τ is (Li, *et al.*, 2001):

$$(l_{ij} - 1)\, mnh < \tau + jmh - inh \le l_{ij} mnh. \quad (9)$$

Second, there exists a one-to-one correspondence between Γ and a positive integer k such that τ is estimated as $kh < \hat{\tau} \le kh + h$ (Sheng, *et al.*, 2003).

Third, since m and n are coprime, there exist integers k_1 and k_2 such that

$$k = k_1 m + k_2 n. \quad (10)$$

Then, we shift the measured input data: $u_s[l] = u[l - k_1]$ and shift the measured output data: $y_s[l] = y[l + k_2]$, so that, the time delay between u_s and y_s is not larger than h. Hence, controllability and observability will be preserved.

4.3 *Causality Constraint*

Lifting causes a causality constraint, i.e., $\underline{D}$ in (3) is lower triangular. How to identify a model under such a constraint? A modified sub-space identification algorithm was proposed by Li, *et al.* (2001). As an easier alternative, a structured state-space model with free parameters (Ljung, 2001) can be used to deal with the constraint. For instance, if $m = 2$ and $n = 3$, $\underline{D}$ will be parameterized as:

$$\begin{bmatrix} 0 & 0 & 0 \\ \times & \times & 0 \end{bmatrix},$$

where $\times$ marks an adjustable parameter.

5. FAST-RATE MODEL COMPUTATION

Once $\underline{G_d}$ is estimated, how to extract matrices A, B, C? Note $D = 0$ if G_c is causal. The difficulty lies in that in general A cannot be determined by taking the mn-th roots of $\underline{A}$. Once $\hat{A}$, an estimation of A, is known, B and C can be determined as:

$$\hat{C} = C_1, \hat{B} = \left(\sum_{i=0}^{m-1} \hat{A}^i \right)^{-1} B_n$$

where $\underline{B}$, $\underline{C}$ are partitioned as:

$$\underline{B} = \left[B_1 \; B_2 \; \cdots \; B_n \right], \quad (11)$$

$$\underline{C} = \left[C_1^T \; C_2^T \; \cdots \; C_m^T \right]^T. \quad (12)$$

Here the dimensions of B_1, B_2, $\cdots$, B_n are $p \times 1$ and those of C_1, C_2, $\cdots$, C_m are $1 \times p$ and p is the order of the estimated fast-rate model. Note that the proof of *Proposition 2* shows the existence of the inverse.

We propose two approaches to compute A. The first approach, the controllability and observability approach, is based on assumptions that (A_{mh}, B_{mh}) is controllable and (C, A_{nh}) is observable, where

$$A_{mh} := A^m, A_{nh} := A^n,$$

$$B_{mh} := B_n = \sum_{i=0}^{m-1} A^i B.$$

Similar to the proof of *Proposition 2*, both assumptions can be shown to be valid if the conditions in *Proposition 2* are true.

Step 1: Given $\underline{A}$ and $\underline{B}$ in (11), (3) implies

$$A_{mh}^n = \underline{A},$$

$$B_{mh} = B_n, A_{mh} B_{mh} = B_{n-1}, \cdots, A_{mh}^{n-1} B = B_1.$$

Thus, $A_{mh}^k B_{mh}$ is known for any $k \ge 0$. We form the controllability matrix Γ_c of (A_{mh}, B_{mh}) and the shifted controllability matrix Γ:

$$\Gamma_c = \left[B_{mh} \; A_{mh} B_{mh} \; \cdots \; A_{mh}^{p-1} B_{mh} \right],$$

$$\Gamma = \left[A_{mh} B_{mh} \; A_{mh}^2 B_{mh} \; \cdots \; A_{mh}^p B_{mh} \right].$$

Since $A_{mh} \Gamma_c = \Gamma$ and the controllability assumption implies that Γ_c is full row rank, A_{mh} is uniquely determined by

$$\hat{A}_{mh} = \Gamma \Gamma_c^T \left(\Gamma_c \Gamma_c^T \right)^{-1}.$$

Step 2: Given $\underline{A}$ and $\underline{C}$ in (12), (3) implies

$$A_{nh}^m = \underline{A},$$

$$C = C_1, CA_{nh} = C_2, \cdots, CA_{nh}^{m-1} = C_m.$$

Thus, CA_{nh}^k is known for any $k \geq 0$. We form the observability matrix Ψ_o of (C, A_{nh}) and the shifted observability matrix Ψ:

$$\Psi_o = \begin{bmatrix} C \\ CA_{nh} \\ \cdots \\ CA_{nh}^{p-1} \end{bmatrix}, \Psi = \begin{bmatrix} CA_{nh} \\ CA_{nh}^2 \\ \cdots \\ CA_{nh}^p \end{bmatrix}.$$

Since $\Psi_o A_{nh} = \Psi$ and the observability assumption implies that Ψ_o is full column rank, A_{nh} is uniquely determined by

$$\hat{A}_{nh} = \left(\Psi_o^T \Psi_o\right)^{-1} \Psi_o^T \Psi.$$

Step 3: Now, $A_{mh} = A^m$ and $A_{nh} = A^n$ are estimated. Since m and n are coprime, there exist two integers $m^{'}, n^{'}$ such that

$$nn^{'} - mm^{'} = 1.$$

Thus, we have:

$$(A_{mh})^{m'} A = (A_{nh})^{n'}.$$

Therefore,

$$\hat{A} = \left(\hat{A}_{mh}^{m'}\right)^{\dagger} \left(\hat{A}_{nh}\right)^{n'}.$$

where † denotes a pseudo-inverse.

The second approach, the matrix roots approach, is based on a condition that $\underline{A}$ is diagonalizable, i.e.,

$$P^{-1}\underline{A}P = \text{diag}\left(\lambda_1, \lambda_2, \cdots, \lambda_p\right).$$

Since $\underline{A} = A^{mn}$, A and $\underline{A}$ share same eigenvectors. If $\rho_i = \alpha_i + j\beta_i$ is a pole of G_c, then

$$\lambda_i = e^{mnh\rho_i} = e^{mnh\alpha_i} e^{jmnh\beta_i}.$$

Assume $|mnh\beta_i| < \pi$ for $i = 1, \cdots, p$.

$$A = P \, \text{diag}\left(\lambda_1^{\frac{1}{mn}}, \lambda_2^{\frac{1}{mn}}, \cdots, \lambda_p^{\frac{1}{mn}}\right) P^{-1}$$

where $\lambda_i^{\frac{1}{mn}}$ is the principal n-th root of λ_i; if this condition is not true, A can be found by searching through all mn-th roots of $\underline{A}$.

6. EXAMPLES

Example 1:

For a system depicted in Fig. 3, take the process and noise model to be

$$G_c(s) = \frac{1}{20s^2 + 4s + 1} e^{-5s}, N_c(s) = \frac{1}{10s + 1}$$

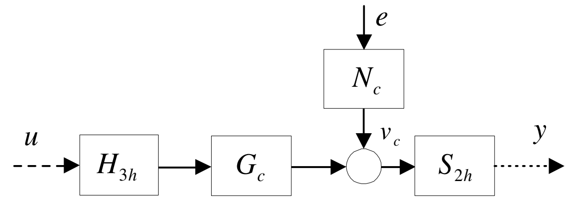

Fig. 3. A SISO MRSD system simulation diagram

and $m = 3$, $n = 2$, $h = 1$ sec. We generate a low frequency random binary signal (RBS) as the input signal u. e is a white noise. The signal-to-noise ratio (SNR) is 3 : 1. The identification procedure is: First, we estimate the time delay as 6 sec and shift the measured output and input data as described in Section 4.2; second, we lift the shifted data to form the lifted signals with a time delay no larger than h; next, based on the lifted signals, we choose a 2nd order lifted model $\underline{\hat{G}_d}$ and compute a fast-rate model $\hat{G}_d$ with period h; finally, we incorporate the estimated time delay. Fig. 4 compares step responses of the actual system G_d and the estimated fast-rate models $\hat{G}_d$. The models are obtained through the proposed approaches: the controllability and observability approach and the matrix roots approach. Both achieve satisfactory results.

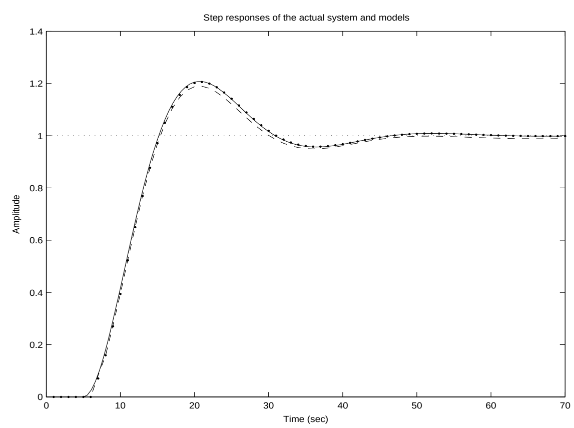

Fig. 4. Step responses of the actual system (solid) and the estimated fast-rate models by the controllability and observability approach (dash) and the matrix roots approach (dotted)

Example 2:

The experiment [2] is implemented on a pilot-scale process in the compute process control laboratory at the University of Alberta. It is a SISO system with the manipulated input u as the cold water valve position and the measured output y as the tank water level. Both are represented by currents (mA), which have linear relationships with the physical units. Around an operating point $u = 11$

[2] Data and Matlab programs are available online. http://www.ee.ualberta.ca/~jwang/paper.html

mA and $y = 10.3$ mA, a RBS input with a limiting magnitude of 0.4 mA is designed. The input updating period is 80 sec and the output sampling period is 120 sec. Thus, $m = 2$, $n = 3$ and $h = 40$ sec, a dual configuration to Example 1. With 'cheap' data acquisition, we simultaneously measure the input and output every 40 sec, say, u_f and y_f, to be used later for model validation. Following a similar procedure as Example 1, we choose a 2nd order fast-rate model with period 40 sec, using the matrix roots approach. To validate the model, we take u_f as the model input and estimate the model output, which is compared with y_f in Fig. 5. The model captures the process dynamics and steady states very well.

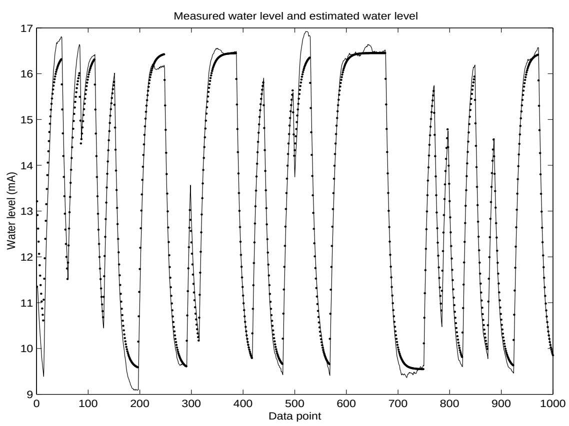

Fig. 5. Comparison of the measured water level (solid) and the estimated water level (dotted)

7. CONCLUSIONS

In this paper, we studied how to estimate a fast-rate model for a general multirate sampled-data system under some mild conditions. The idea is to associate the multirate sampled-data system with an equivalent lifted system, from which the fast-rate model is extracted. Some topics are still open, e.g., how exactly the noise would affect the estimation? how to get an explicit variance expression of the estimated model? These are left to the future investigation.

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