Robust tuning of feedback linearizing controllers via bifurcation analysis

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Abstract

Feedback linearization is a nonlinear controller design strategy that results in an explicit formulation of the feedback control law. This method can result in excellent performance if an accurate dynamic process model is available. However, feedback linearization suffers from a lack of robustness if plant-model mismatch exists. The approach presented in this work analyzes the robustness properties of the closed-loop process with specific regard to the controller tuning parameter. Due to this, it is possible to tune the controller such that robustness over the entire operating region is guaranteed even under the assumption of certain types of model mismatch. This method is illustrated with an example and conclusions about its applicability to more general model and controller formulations are presented.

1 Introduction

Nonlinear process control has become increasingly popular in the chemical process industries. This is due to the trend towards speciality products, tighter profit margins, more stringent environmental requirements, as well as advances in nonlinear systems theory and in the numerical implementation of nonlinear controllers (Bequette, 1991).
Feedback linearization is a nonlinear controller design technique that can result in excellent performance if an accurate model of the process is known. However, the closed-loop performance can degrade significantly, even up to the point that the process can become unstable, if the real model contains inaccuracies in the parameters or includes unmodeled dynamics (Henson and Seborg, 1991). There are several possibilities to circumvent this:

- A simpler controller could be used. This approach can increase the robustness but will usually decrease controller performance, especially when the operating region of the process is large.
- A robust nonlinear controller could be designed. However, this will result in controllers that are even more complicated to design, implement, and maintain than regular feedback linearizing controllers.

This paper presents a different approach of dealing with model mismatch for nonlinear controller design. The model and its uncertainties are thoroughly analyzed by performing bifurcation analysis on the closed-loop system. Subsequently, the results from this analysis are used to tune the controller. In particular, it is often possible to tune a feedback linearizing controller such that robust stability is guaranteed.
The procedure is illustrated using an unstable reactor as an example. For this case study, uncertainty in the model parameters will result in an upper bound on the controller tuning parameter, while unmodeled dynamics will result in a lower bound. This upper and lower bound on the tuning parameter correspond to, roughly speaking, a lower and an upper bound on the aggressiveness of the controller. From the bifurcation analysis it can be inferred that the controller will guarantee robust stability over the entire operating region for tuning parameter values between these bounds. In addition to robustness, it is also possible to use these bounds in order to achieve good performance even if there exists mismatch between the real model and the one that is used for designing the controller.

2 Feedback Linearization

Two main categories of designing controllers via feedback linearization can be identified: input-output linearization and state-space linearization. The presentation in this paper will exclusively focus on the former because it is more generally applicable and will result in linear input-output behavior of the system if no model mismatch is present.
Consider a single-input single-output (SISO) nonlinear system with n states of the form

$$\begin{aligned}\dot{x}(t) &= f(x(t)) + g(x(t))u(t)\\ y(t) &= h(x(t)).\end{aligned} \tag{1}$$

If this system has a well-defined relative degree r then it can be transformed into normal form via a diffeomorphism $[\xi^T, \eta^T]^T = \Phi(x)$. The ξ coordinates are defined as

$$\xi_k = \Phi_k(x) = L_f^{k-1}h(x), \quad 1 \leq k \leq r \tag{2}$$

and the $\eta_j = \Phi_{r+j}(x), 1 \leq j \leq n-r$ (Isidori, 1995; Kravaris and Kantor, 1990) correspond to the internal dynamics of the closed-loop process. The normal form of the system is then given by

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_r &= L_f^r h(x) + L_g L_f^{r-1} h(x) u \\ \dot{\eta} &= q(\xi, \eta) \\ y &= \xi_1 \end{aligned} \tag{3}$$

The map between the input and the output can be linearized by choosing a static state feedback control law

$$u = \frac{v - L_f^r h(x)}{L_g L_f^{r-1} h(x)} \tag{4}$$

such that the r-th equation of (3) becomes $\dot{\xi}_r = v$. It is possible to place the poles of the closed-loop transfer function for the linearized subsystem ξ in the complex plane by choosing an appropriate feedback v. For the purpose of this paper only one tuning parameter, ϵ, which represents the time constant of the closed-loop system is used to shape the closed-loop response. A feedback linearizing controller in terms of the original states is then given by

$$u = \frac{-L_f^r h(x) - \frac{\binom{r}{r-1}}{\epsilon} L_f^{r-1} h(x) - \ldots - \frac{\binom{r}{1}}{\epsilon^{r-1}} L_f h(x)}{L_g L_f^{r-1} h(x)} + \frac{\frac{1}{\epsilon^r}(y_{sp} - h(x))}{L_g L_f^{r-1} h(x)} \tag{5}$$

When this control law is applied to the process, the closed-loop transfer function between the system output y and the set point y_{sp} becomes

$$\frac{y}{y_{sp}} = \frac{1}{(\epsilon s + 1)^r} \tag{6}$$

under the assumption that $y(0) = y_{sp}(0)$.
It is also possible to include integral action in the controller in order to compensate for possible inaccuracies in the model. The feedback linearizing controller with integral action is given by

$$u = \frac{-L_f^r h(x) - \frac{\binom{r+1}{r}}{\epsilon} L_f^{r-1} h(x) - \ldots - \frac{\binom{r+1}{2}}{\epsilon^{r-1}} L_f h(x)}{L_g L_f^{r-1} h(x)} + \frac{\frac{\binom{r+1}{1}}{\epsilon^r}(y_{sp} - h(x)) + \frac{1}{\epsilon^{r+1}} \int_0^t (y_{sp} - h(x)) d\tau}{L_g L_f^{r-1} h(x)} \tag{7}$$

resulting in the closed-loop transfer function

$$\frac{y}{y_{sp}} = \frac{(r+1)\epsilon s + 1}{(\epsilon s + 1)^{r+1}} \tag{8}$$

under the condition that $y(0) = y_{sp}(0)$.
The transfer functions shown in equations (6) and (8) only represent the closed-loop system behavior if the matching conditions are satisfied up to a degree of at least r. Since this investigation specifically focuses on controller tuning under the influence of model mismatch, the controller implementation shown in equation (7) will be used. Due to the model mismatch it will not be possible to exactly achieve the closed-loop response shown in equation (8). However, the integrating term in the controller will ensure that the desired set point can be reached and appropriate tuning of the controller can still result in good performance for many cases.
For the implementation of feedback linearizing controllers it is usually postulated that the internal dynamics of the process is stable and that the values of the states are exactly known. While this investigation also uses the latter assumption, it will be shown that the validity of the former assumption can easily be analyzed as part of the proposed tuning method.

3 Bifurcation Analysis

Bifurcation theory allows to systematically identify critical points on the steady state manifold of a parametrized ODE or DAE system. The term critical point refers to a point at which the dynamic behavior of the system changes qualitatively. For example, at Hopf and saddle-node bifurcations, stable and unstable steady states meet. Therefore, stability boundaries in the process parameter space can be investigated by locating these critical points for the system of interest. The use of bifurcation theory in conjunction with parameter continuation is well established. As one-parametric curves of steady states are calculated by parameter continuation, critical points can be detected by monitoring sign changes of appropriate test functions (Beyn *et al.*, 2002). Once a bifurcation point has been detected, a curve of bifurcation points can be calculated by continuation from this point, just as a curve of steady states was calculated starting from a known steady state in the first step. The sequence of continuation, detection of critical points, and subsequent continuation of a critical point can be repeated for critical points of higher order, e.g. for a cusp point found on a curve of saddle-node bifurcations. While many higher order critical points are related to exotic dynamic behavior, some reveal information which can be exploited for engineering purposes. Most notably in the present context, sets of cusp points bound regions of the process parameter space in which no saddle-node bifurcations occur. Similarly, sets of a particular type of degenerate Hopf point bound regions in which no Hopf points occur. Since Hopf and saddle-node points mark the stability boundary, knowledge about the location of degenerate Hopf and cusp points will be exploited in section 4 to identify regions in which no loss of stability can occur for any value of the parameters. Remarkably, bifurcation analysis has rarely been applied to closed-loop processes to the authors' knowledge (Cibrario and Lévine, 1991; Littleboy and Smith, 1998). The relation of the present paper to bifurcation theory-based design methods will be briefly discussed in section 6.

4 Robust Controller Tuning

During controller tuning a trade-off is always associated with performance and robustness requirements. This is due to the fact that good performance leads to aggressive controllers which will usually result in a decrease in the robustness of the closed-loop process. A balance between these objectives has to be found. Methods for tuning linear controllers are well established (Skogestad and Postlethwaite, 1996) but this is not the case for nonlinear controllers, where performance and robustness cannot easily be quantified.

In the following a methodology for determining upper and lower bounds for tuning parameters for feedback linearizing controllers is presented. This approach is based upon bifurcation analysis of the closed-loop system. The method will be illustrated by an example and a generalization of this tuning method is discussed in the next section.

Consider a continuous stirred tank reactor (CSTR) for an exothermic, irreversible reaction, $A \rightarrow B$ (Uppal *et al.*, 1974). Assuming constant liquid volume, the following dynamic model can be derived based upon a component and an energy balance:

$$\dot{C}_A = \frac{q}{V}\left(C_{Af} - C_A\right) - k_0 \exp\left(-\frac{E}{RT}\right) C_A \qquad (9)$$

$$\dot{T} = \frac{q}{V}\left(T_f - T\right) - \frac{\Delta H}{\rho C_p} k_0 \exp\left(-\frac{E}{RT}\right) C_A + \frac{UA}{V\rho C_p}\left(T_c - T\right) \qquad (10)$$

The values of the parameters and the nominal operating conditions for this process are shown in Table 1. The temperature of the cooling fluid, T_c, can be manipulated and the reactor temperature, T, is measured. This results in a system consisting of two states with a single input and a single output. The bifurcation diagram of the open-loop system is shown in Figure 1. The equilibria of the system consist of two stable branches and one unstable branch connecting the two

Variable	Value	Variable	Value
q	$100\frac{L}{min}$	$\frac{E}{R}$	$8750K$
C_{Af}	$1\frac{mol}{L}$	k_0	$7.2 \cdot 10^{10}\frac{1}{min}$
T_f	$350K$	UA	$5 \cdot 10^4\frac{J}{minK}$
V	$100L$	T_c	$300K$
ρ	$1000\frac{g}{L}$	C_A	$0.5\frac{mol}{L}$
C_p	$0.239\frac{J}{gK}$	T	$350K$
ΔH	$-5 \cdot 10^4\frac{J}{mol}$		

Table 1: Parameters for the CSTR

stable ones. The system also has two limits points and one Hopf point. The nominal operating point shown in Table 1 lies on the open-loop unstable branch. When a

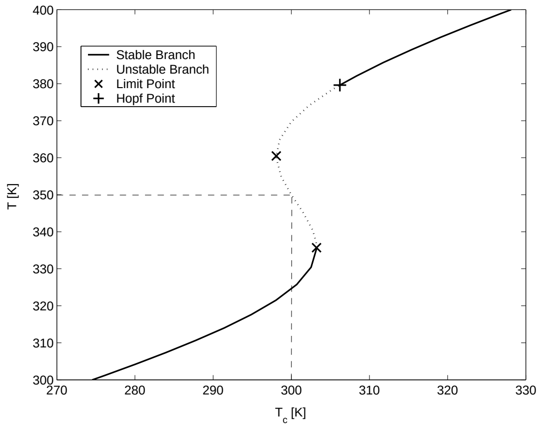

Figure 1: Bifurcation diagram of the open-loop system.

controller like the one given by equation (7) is designed for this process it results in the following feedback control law

$$u = \frac{-\frac{q}{V}\left(T_f - T\right) + \frac{\Delta H}{\rho C_p} k_0 \exp\left(-\frac{E}{RT}\right) C_A + \frac{UA}{V\rho C_p} T}{\frac{UA}{V\rho C_p}} + \frac{\frac{2}{\epsilon}\left(T_{sp} - T\right) + \frac{1}{\epsilon^2}\int_0^t \left(T_{sp} - T\right) d\tau}{\frac{UA}{V\rho C_p}} \qquad (11)$$

$$T_c = u \qquad (12)$$

Assuming that there is no mismatch between the plant and the model, the controller of the form of equation (11) results in a system that has a stable input-output behavior as well as a stable internal dynamics for any value of ϵ and any set point, T_{sp}, within the operating region. This can easily be verified by applying bifurcation analysis to the closed-loop system. For this nominal case the value of ϵ can be made arbitrarily small, resulting in a very fast response. However, in an on-line application there is always some mismatch between the plant and the model which can also lead to restrictions for the controller tuning.

Assume that the heat transfer coefficient, UA, of the real plant represented by equation (10) is not identical to the one for the model shown in equation (11) due to uncertainty in this parameter. For the plant a value of UA equal to $5 \cdot 10^4 \frac{J}{minK}$ is used whereas for the model a value of $5.5 \cdot 10^4 \frac{J}{minK}$ is assumed. When bifurcation analysis of this closed-loop system is performed, it is found that the system can become unstable for high values of ϵ because there is a Hopf point along the equilibrium curve for some values of T_{sp}. Starting from this Hopf point, a Hopf curve can be computed where both ϵ and T_{sp} are varied. The curve shown in Figure 2 results from this where the shaded region is unstable and the region outside of the Hopf curve corresponds to stable steady states. It can be concluded that the system will always be stable if ϵ is smaller than a certain value corresponding to the peak of the Hopf curve.

Next, it was investigated how this peak moves with variations in the parameter uncertainty. This way it can be established how much the controller can be detuned without losing stability for a specific uncertainty in the model parameter. The corresponding curve is shown in Figure 3. This curve provides information about how large ϵ can be chosen for a certain mismatch in the model parameter in order to guarantee robust stability. If the model mismatch is less than what was assumed for the controller design then the closed-loop system will also be stable. It should be pointed out that values of ϵ that are close to the critical ϵ will usually result in low performance of the closed-loop system. When the model parameter UA is chosen to be less than the real value of the plant ($UA = 5 \cdot 10^4 \frac{J}{minK}$) then this model mismatch has a stabilizing effect. For such a case any value of ϵ will result in a stable closed-loop system.

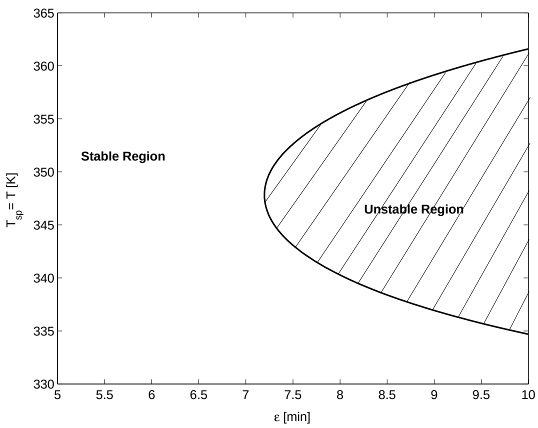

Figure 2: Hopf curve of the closed-loop system for model mismatch in UA of 10%.

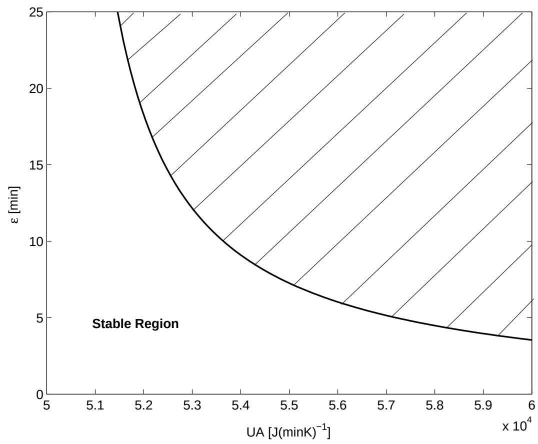

Figure 3: ϵ vs. UA along the peak of the Hopf curve.

Similar investigations have been performed for mismatch in other model parameters (k_0, $\frac{E}{R}$). All of these lead to similar conclusions that there exists an upper bound for the value of ϵ for some form of model mismatch. Tuning the controller more aggressively by choosing a smaller value of ϵ than this bound will guarantee robustness against parameter uncertainty for the investigated case over the entire operating region.

Another form of model mismatch that needs to be considered is unmodeled dynamics. Since no model can describe every detail of a process with perfect precision, it is commonly assumed that the fast dynamics of the process can be neglected. While this is generally a good assumption, it does lead to a bound on the achievable closed-loop performance. For linear controllers the uncertainty can be described in the frequency domain and unmodeled dynamics will result in a large uncertainty weight at high frequencies (Skogestad and Postlethwaite, 1996). Unfortunately, such a characterization of the uncertainty is not possible for nonlinear systems. However, bifurcation analysis can be performed on a system that contains the most important part of the fast dynamics while a controller that has no knowledge about this dynamics is used to control it. For this case study, the plant model is augmented by the following two equations that describe the actuator dynamics as an overdamped second order process

$$\begin{aligned} \epsilon_v \dot{T}_c &= -T_c + z \\ \epsilon_v \dot{z} &= -z + u \end{aligned} \tag{13}$$

where ϵ_v corresponds to the time constant of the cooling system. The equations in (13) replace the original equation (12) for the following investigation. The goal is to tune the controller such that robust stability is guaranteed for this form of model mismatch.

Investigation of the closed-loop system under the assumption of unmodeled dynamics given by equation (13) ($\epsilon_v = 0.02\,\text{min}$) but no parametric uncertainty shows that the system exhibits a Hopf point when T_{sp} is held constant and ϵ is varied. Computing the Hopf curve by starting from the Hopf point and varying both T_{sp} and ϵ results in the curve shown in Figure 4. This figure reveals that the unmodeled dynamics results in a lower bound on the controller tuning parameter, ϵ, for any fixed value of T_{sp}. Since the curve shown in Figure

4 has a peak at about $\epsilon = 0.055\,\mathrm{min}$, robust stability can be guaranteed for any T_{sp} by setting $\epsilon > 0.055\,\mathrm{min}$. Figure 5 shows how the peak on the Hopf curve moves with a variation of the time constant of the unmodeled dynamics, ϵ_v. If ϵ is chosen to be greater than a certain

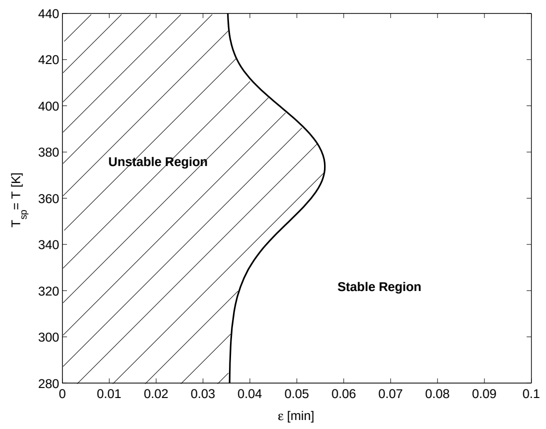

Figure 4: Hopf curve of the closed-loop system for unmodeled dynamics with $\epsilon_v = 0.02min$.

value for a specific ϵ_v, then the system will be stable over the entire operating region. It can also be concluded that the closed-loop system will be stable for any value of ϵ_v that is smaller than the one that was used for the design. In summary, there are upper and

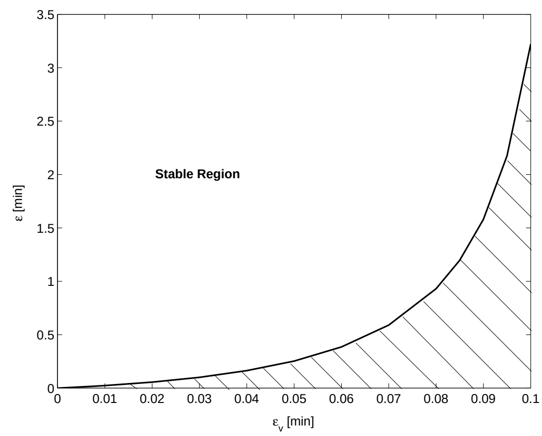

Figure 5: ϵ vs. ϵ_v along the peak of the Hopf curve.

lower bounds for the controller tuning parameter ϵ for this case study. The upper bound results from uncertainty in the model parameters while the lower bound is caused by unmodeled dynamics.

In a final step, the effect of parameter uncertainties on the location of the lower bound for ϵ and the effect of unmodeled dynamics on the location of the upper bound for ϵ are investigated. The existence of unmodeled dynamics has a very mild stabilizing effect on the system for high values of ϵ because the unstable region shown in Figure 2 is moved further to the right when the value of ϵ_v is increased. A similar effect is taking place for the lower bound of ϵ when UA for the model is chosen to be larger than the parameter in the model because this will move the unstable region in Figure 4 further to the left. However, if the value of UA in the model underestimates the real value of the parameter then the unstable region in Figure 4 will move slightly to the right.

Summarizing, it can be stated that the value of the controller tuning parameter ϵ for the worst case scenario of 1) uncertainty in the parameter UA of up to $\pm 10\%$ 2) unmodeled dynamics of the form of equation (13) with $\epsilon_v \leq 0.02\,\mathrm{min}$ can be determined from the diagram shown in Figure 6. Any value of ϵ between the peak values of $0.059424\,\mathrm{min}$ and $7.1969\,\mathrm{min}$ will result in robust stability of the closed-loop system over the entire operating region and for any plant model mismatch as described. In order to achieve good per-

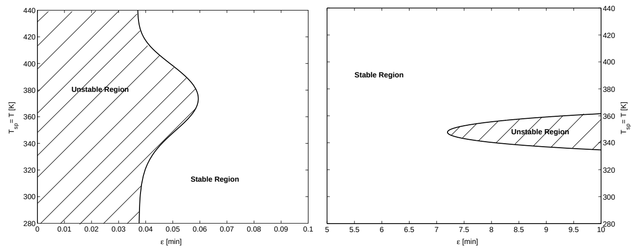

Figure 6: Regions of stability based upon the controller tuning parameter ϵ.

formance in addition to robustness it is recommended to use a value of ϵ that does not lie directly on the stability boundary (e.g. $\epsilon = 0.25\,\mathrm{min}$ for this case). If there is a gap of several orders of magnitude between the smallest and the largest value of the tuning parameter then it is recommended to stay closer to the lower value in order to achieve a faster response. Ultimately, the dynamics of the open-loop process has also to be considered for tuning the controller.

All of the results derived from bifurcation analysis have been confirmed in simulations with the dynamic system.

5 Controller Tuning Procedure

The previous section illustrated the controller tuning method by applying it to a specific example. However, the same tuning method can be applied to processes in general. It contains no restriction about the process to be controlled or the type of controller to be tuned.

For feedback linearizing controllers the following steps should be included in the controller tuning process:

1. Design and implement the controller on the simulated process.
2. Analyze the internal dynamics of the closed-loop

system using bifurcation analysis. This should be performed at the operating point as well as over the entire operating region in order to determine if the internal dynamics will remain stable over this region.

3. Tune the controller to satisfy a nominal performance requirement. This is trivial to do for the nominal case because the input-output behavior of the closed-loop process corresponds to equation (6) or (8) if a controller of the type of equation (5) or (7) is being used.

4. Identify the main sources of parametric uncertainty in the model. Subsequently, analyze the closed-loop system using bifurcation analysis under the assumption of parametric uncertainty. This can result in restrictions on the controller tuning parameters.

5. Perform bifurcation analysis on the closed-loop system under the assumption of unmodeled dynamics. This investigation might place further bounds on the controller tuning parameters.

6. Investigate the region for the controller tuning parameters for which the closed-loop system remains stable under the worst possible combination of parametric uncertainty and unmodeled dynamics over the entire operating region. This is an important point in order to guarantee robust stability of the controller.

7. If the controller tuning parameters that satisfy the nominal performance requirement can also guarantee robust stability then they can be kept. Otherwise, the controller has to be retuned in order to guarantee robust stability. It should be pointed out that it is desirable to use controller tuning parameters that do not lie close to a region of instability of the closed-loop process in order to also achieve good robust performance.

It is stressed that step 6 of the above procedure corresponds to an analysis of cusp and degenerate Hopf points involving more than two parameters. Figure 6, for example, was obtained by locating a degenerate Hopf point from Figure 4 and Figure 2 in three parameters by repeatedly calculating curves in two parameters for a variety of fixed values of the third parameter. Repeated calculation of curves in two parameters will become tedious if extremal points must be found w.r.t. more than three parameters. It is worth noting that bifurcation theory-based design methods (Mönnigmann and Marquardt, 2002) are available which can deal with a larger number of parameters than bifurcation analysis.

Following these steps will result in a controller that is tuned such that it meets nominal stability, nominal performance, as well as robust stability requirements.

6 Conclusions

This paper presented a controller tuning strategy for nonlinear systems. The method is based upon applying bifurcation analysis to the closed-loop system in order to determine regions of stability for the controller tuning parameters. It is often possible to tune the controller such that it meets nominal performance as well as robust stability requirements. This approach was illustrated by tuning a feedback linearizing controller for an unstable nonlinear plant, under the assumption of parametric uncertainty as well as unmodeled dynamics. However, the approach as such can also be applied to different types of controllers, plants with different characteristics, as well as under the assumption of different types of plant-model mismatch.

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