Lyapunov Stability of Periodic Control on a Continuous Stirred Tank Reactor ⋆

Eyan P. Noronha * **Erik B. Ydstie** **

* *Carnegie Mellon University, Pittsburgh, PA 15213 USA (e-mail: enoronha@andrew.cmu.edu).*
** *Carnegie Mellon University, Pittsburgh, PA 15213 USA (e-mail: ydstie@cmu.edu).*

Abstract: Periodic control uses zero mean parametric excitation as a tool to influence the transient behavior of a dynamical system. Unlike conventional methods, based on feedback or feedforward principles, the method of periodic control may not require any measurement of the deviations or disturbances to stabilize an unstable system. The choice of amplitude and frequency in periodic control provides two additional degrees of freedom to stabilize an open-loop system. Stability analysis of periodically forced systems is often limited to linearization methods. This is often not sufficient to assess the global properties of a non-linear system like finding the region of convergence. In this paper, forced oscillations are introduced in the input flow rates of a continuous stirred-tank reactor (CSTR) to operate the process around an unstable point. We use Lyapunov analysis to demonstrate the exponential stability of a CSTR system under the operation of periodic control. For exponentially convergent systems, we derive a theorem to estimate the region of convergence and show that the rate of convergence for a system under weak oscillations is almost the same as that of its averaged system. Numerical simulations of a propylene glycol process are carried out to illustrate the exponential stability of periodic control and verify the results of our analysis.

Keywords: Batch process, CSTR, periodic control, open-loop control, lyapunov stability, exponential stability

1. INTRODUCTION

Traditionally, automatic control of a dynamical system has been accomplished using principles from feedforward and feedback control. In systems where closed-loop control can be difficult to achieve due to lack of critical measurement data or imperfect models, open-loop periodic control is a viable alternative. The stabilizing effect of vibrations was first studied in a pendulum where it was observed that vibrating the suspension point of a pendulum in a vertical line with a reasonably high frequency can stabilize the unstable equilibrium position (Kapitza, 1965). Since then, periodic control has found applications in many areas of electrical systems (Caruntu and Martinez, 2014; Bucolo et al., 2019), mechanical systems (Tahmasian and Katrahmani, 2020; Suttner, 2019), robotics (Yabuno and Kobayashi, 2020; Taha and Kiani, 2019) and also biological systems (Ramírez-Ávila et al., 2019; Ghanaatpishe et al., 2018).

The theory for periodic control was introduced by Meerkov (1973, 1977, 1980), where he discussed the necessary and sufficient conditions for periodic stabilizability and controllability of linear systems. His papers relied on results from averaging theory to study the slow varying dynamics of oscillating systems (Bogoliubov and Mitropolsky, 1961). Bentsman et al. (1989) extended the applications of periodic control to linear systems with time delay and established their robustness conditions using numerical simulations. Lee et al. (1987) and Kabamba et al. (1998) demonstrated the capabilities of using periodic control in reassigning the poles and zeros of a system. Moreau and Aeyels (2004) showed that is is possible to increase the region of stability for linear time-invariant single input single output systems by using a periodic gain. Berg and Wickramasinghe (2015) proposed the use of stability maps to analyse systems using low frequency inputs. Cheng et al. (2018) demonstrates the robustness of periodic control systems and used the method of weak averaging to show input-to-state stability for bounded disturbances.

In the area of non-linear systems, Bellman et al. (1986a,b) developed the periodic control theory for linear multiplicative and vector additive form of oscillations. Bellman et al. (1983) applied periodic control to systems with Arrhenius dynamics, to study the effects of sufficiently fast oscillation in improving stability. Shapiro and Zinn (1997) extended the applicability of periodic control by using nonlinear oscillations, which allowed for a larger class of engineering systems to be open-loop stabilized. Baillieul (1993, 1995) applied periodic control to mechanical systems described by Lagrangian models and Hamiltonian models. To analyze stability of these systems, he used the notion of averaged potential by directly averaging the Hamiltonian function. Bullo (2002) provides sufficient conditions, under

⋆ This research is supported by partial funding from Center for Advanced Process Decision-making (CAPD) at Carnegie Mellon University

which periodic control of a mechanical system can be described using an averaged potential. Cheng et al. (2018) used a Lyapunov based sampling method to determine upper bounds on the disturbances for nonlinear systems stabilized using periodic control.

In many chemical reactor systems, the use of periodic control has been known to improve performance by increasing process yields and stability (Silveston and Hudgins, 2012). Cinar et al. (1986, 1987, 1988) conducted experiments and carried out theoretical analysis of a simplified continuous stirred-tank reactor (CSTR) model to test the effects of periodic inputs. Sterman and Ydstie (1990a,b, 1991) used averaging and second order perturbation methods to analyze periodically operated CSTRs for multi-input and reversible processes. Heidarinejad et al. (2012) found periodic operation to be the best practical choice in maximizing the average production rate. Zuyev et al. (2017) came to a similar conclusion for the optimal control design of a particular family of reaction systems in a CSTR. Nikolić et al. (2020) used nonlinear frequency response method for evaluating the time-average performance of a CSTR subjected to two periodic inputs.

To the best of our knowledge, the stability of periodically forced CSTR systems has only been carried out locally through linearization. This may serve as a limitation, since the knowledge on the size of the stability regions is often required for practical applications. Moreover, the time-varying output of a periodically forced process can be difficult track. Therefore, knowledge on the convergence and behaviour of the output under periodic control can be useful. In this paper, we provide global stability analysis of a non-isothermal CSTR system by choosing the squared average error function as a candidate for the Lyapunov function. A Lyapunov transformation is used as a map between the state and the slow varying variable. This allows us to track the periodic output of the CSTR process from the trajectory of the averaged equation. Additionally, Lyapunov analysis can be used to prove exponential convergence of the CSTR to a periodic orbit and provide an upper bound on its rate of convergence.

The paper will be divided into 3 parts. Section 2 presents the problem setup and uses variation of constants to come up with a Lyapunov transformation. Section 2.2 recalls the theorem of averaging and provides a theorem to estimate stability regions and convergence rates of weakly periodic systems. Section 3 applies the Lyapunov transformation and the theory of averaging to a propylene glycol production process. Section 3.1 analyzes the stability of the averaged CSTR system using Lyapunov analysis and provides numerical results on the exponential convergence of a CSTR under the action of periodic control.

2. MATHEMATICAL THEORY

Consider a nonlinear equation of the form

$$\frac{dx}{dt} = X(x),$$

where t is time, x is a state, $x = \{x_1, ..., x_n\}$ and X is a vector-valued function, $X = \{X_1, ..., X_n\}$. Introduce in this equation periodic forcing of the form

$$\frac{dx}{dt} = X(x) + B\left(\frac{t}{\epsilon}\right)x, \tag{1}$$

where $0 < \epsilon \ll 1$ and $B(t/\epsilon)$ is a periodic zero-mean matrix. The theorem of averaging can be directly applied to (1) but requires the inclusion of higher order terms for a notable approximation (Taha et al., 2015). Instead we transform x into a slow varying variable using the variation of constants.

2.1 Variation of constants

Using the method of variation of constants, we seek a solution to (1) depending on a 'fast' time scale variable $\tau = t/\epsilon$ and a 'slow' time scale variable t:

$$x = x(t, \tau).$$

Then $x(t, T)$ satisfies the equation

$$\frac{\partial x}{\partial t} + \frac{1}{\epsilon}\frac{\partial x}{\partial \tau} = X(x) + B(\tau)x. \tag{2}$$

Isolating the variables involving fast time scales we arrive at:

$$\frac{1}{\epsilon}\frac{\partial x}{\partial \tau} = B(\tau)x, \tag{3}$$

$$x(t, \tau) = \Phi(\tau, \epsilon)y(t), \tag{4}$$

where $\Phi(T, \epsilon)$ is the principle fundamental matrix of (3) and $y(t)$ is a slow varying transformation of $x(t, \tau)$. Since $\Phi(T, \epsilon)$ is periodic and bounded, $y(t)$ is a Lyapunov (stabilility preserving) transformation of $x(t)$. Substituting equation (4) into (2):

$$\frac{dy}{dt} = \Phi(\tau, \epsilon)^{-1}X\Big(\Phi(\tau, \epsilon)y\Big) = Y(y, t, \epsilon). \tag{5}$$

2.2 Theorem of Averaging

In this section we present some mathematical results related to the theory of averaging. These results have been put together from Bogoliubov and Mitropolsky (1961), and Guckenheimer and Holmes (2013) with some modest modifications to determine the convergence rate of periodic control to a stationary solution and to provide the region of convergence.

Lemma 1. Given a system of ordinary differential equations of the form (5), we shall seek a change of variables of the form

$$y = z + \epsilon u(z, t, \epsilon),$$

such that (5) becomes

$$\frac{dz}{dt} = Y_0(z) + \epsilon Y_1(z, t, \epsilon), \tag{6}$$

where Y_0 is slow varying part of Y and Y_1 is a periodic functions with same frequency as $B(\tau)$. Additionally, the system of first approximation:

$$\frac{dz}{dt} = Y_0(z), \tag{7}$$

for the same set of initial conditions is related to (5) by $\|y(t) - z(t)\| = O(\epsilon), \forall t \in [0, \infty)$ when (7) is asymptotically stable or on the time scale $1/\epsilon$ otherwise.

Proof. The proof for this lemma is outlined in Theorem 4.1.1 of Guckenheimer and Holmes (2013).

When periodic control is applied, the stability of the system is often analyzed using the averaged equation. Therefore, it is desirable to know if the asymptotic stability of the averaged autonomous system (7) implies the existence of a stationary solution to the periodic equation (5) which is also asymptotically stable. This stems from the intuition that for small ϵ the dynamics of the system is dominantly influenced by Y_0, which is asymptotically stable. In the next theorem, we provide a proof to determine the region of convergence when an asymptotically stable fixed point is perturbed using weak non-linear oscillations.

Theorem 2. If z_s is an exponentially stable equilibrium point of (7) in U_ρ, where U_ρ is the region of convergence, then for any $\delta < \rho$, there exists an ϵ' such that for any $\epsilon < \epsilon'$, a unique stationary solution $y^*(t)$ to (6) exists and any trajectory $y(t) \in U_\delta$ exponentially converges to $y^*(t)$ with the same rate of convergence.

Proof. The Proof for this theorem is outlined in Appendix A.

3. SPECIFIC EXAMPLE: CSTR

Consider a non-isothermal CSTR used to carry out the production of propylene glycol by the hydrolysis of propylene oxide with sulphuric acid as a catalyst. Water is supplied in excess, so the rate of reaction is first-order in propylene oxide concentration (C_A). The reactor mole and energy balance, assuming perfect mixing, constant volume, and neglecting changes in the kinetic and potential energy is

$$\frac{dC_A}{dt'} = \frac{F}{V}(C_{Af} - C_A) - k_0 \exp(-E_a/RT)C_A \tag{8}$$

$$\frac{dT}{dt'} = \frac{F}{V}(T_f - T) + (-\Delta H)k_0 \exp(-E_a/RT)C_A - h(T - T_j) \tag{9}$$

This system has the following parameter values (Bequette, 2003):

$E_a = 32,400$ Btu/lbmol	$k_0 = 16.96 \times 10^{12}$ hr^{-1}
$h = 0.9$ hr^{-1}	$T_f = 60°F = 519.67R$
$V = 500$ ft^3	$C_{af} = 0.132$ lbmol/ft^3
$T_j = 68.378°F = 528.048R$	$\bar{F} = 2000$ft^3/hr
$-\Delta H = 732.4$ °F ft^3/lbmol	$R = 1.987$ Btu/lbmol°F

The reactor is designed with an operating flow rate of 2000 ft^3/hr, and a desired conversion of 50% (C_A = 0.066 lbmol/ft^3), resulting in a reactor temperature of 101.1°F. At the designed flow-rate the reactor exhibits multiple steady-state behaviour as shown in Fig. 1. It can be shown that the intermediate temperature, which is our desired operating condition, is unstable. Since the reactor is open-loop unstable, conventional theory would dictate that the reactor can only be operated at 101.1°F using a closed-loop control.

We now investigate the feasibility of stabilizing the unstable operating point using periodic control. The main advantage here being that periodic control does not require any measurements for stabilizability. We oscillate the flowrate about a mean value, $F = F_0(1 + C \sin \omega t')$.

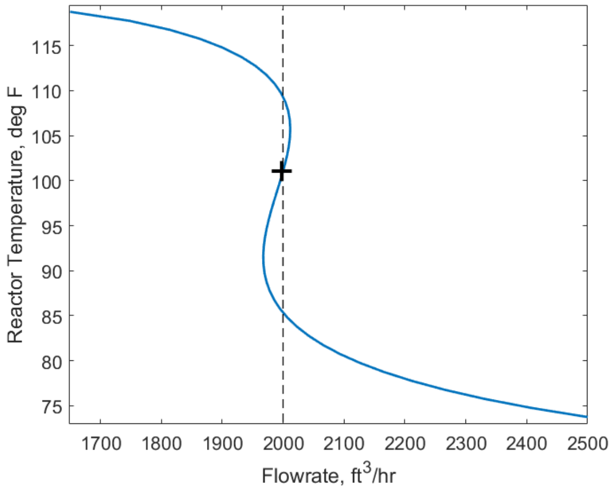


Fig. 1. Steady state characteristic curve of (8)

$$\frac{dC_A}{dt'} = \frac{\bar{F}_0(1 + C \sin \omega t')}{V}(C_{Af} - C_A) - k_0 \exp(-E_a/RT)C_A \tag{10}$$

$$\frac{dT}{dt'} = \frac{\bar{F}_0(1 + C \sin \omega t')}{V}(T_f - T) + (-\Delta H)k_0 \exp(-E_a/RT)C_A - h(T - T_j) \tag{11}$$

Figure 2 demonstrates the behaviour of the CSTR system for large and small amplitudes of $C = 0.5$ and $C = 0.2$. Throughout this study we will compare the stability of these two test cases to verify the application of our theory.

Equations (10) and (11) can be reduced to a dimensionless form as:

$$\frac{dx_1}{dt} = -(1 + C \sin \omega t_r t)x_1 + \text{Da}(1 - x_1) \exp\{\gamma x_2/(1 + x_2)\},$$

$$\frac{dx_2}{dt} = -(1 + C \sin \omega t_r t)x_2 + \text{B Da}(1 - x_1) \exp\{\gamma x_2/(1 + x_2)\} - \beta(x_2 - \delta_j),$$

where

$$x_1 = \frac{(C_{Af} - C_A)}{C_{Af}} \quad x_2 = \frac{(T - T_f)}{T_f} \quad t_r = V/\bar{F}_0$$

$$t = t'/t_r \quad \gamma = \frac{E}{RT_f} \quad Da = k_0 e^{-\gamma} t_r$$

$$\text{B} = \frac{(-\Delta H)C_{Af}}{T_f} \quad \beta = ht_r \quad \delta_j = \frac{T_j - T_f}{T_f}.$$

This can be rearranged to a standard linear multiplicative form:

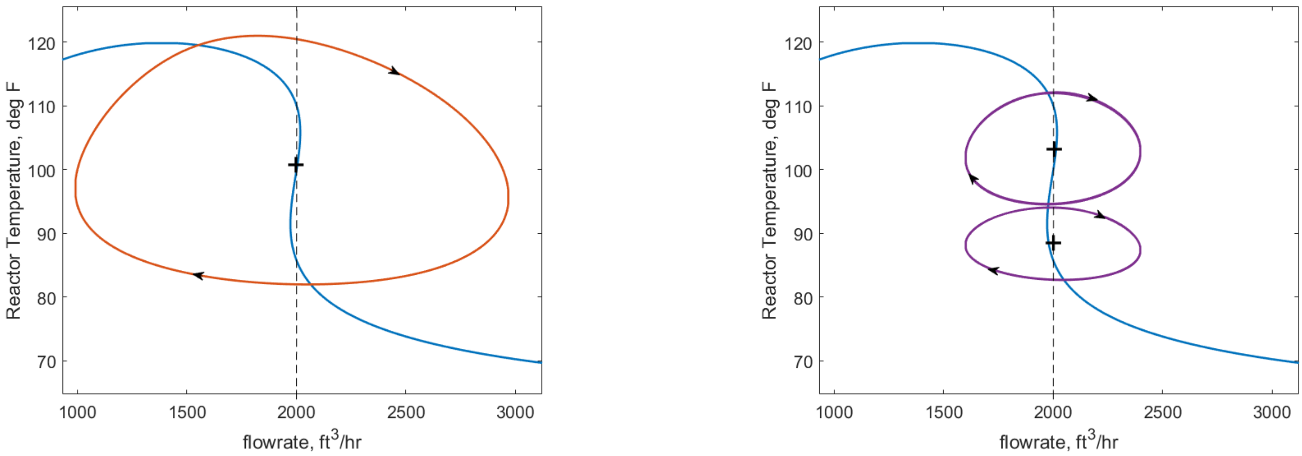
$$\frac{dx}{dt} = X(x) + B(\tau)x, \tag{12}$$

where $x = \{x_1, x_2\}$, $\epsilon = (\omega t_r)^{-1}$, $\tau = t/\epsilon$ and

$$X(x) = \begin{bmatrix} -x_1 + \text{Da}(1 - x_1) \exp\{\gamma x_2/(1 + x_2)\} \\ -x_2 + \text{B Da}(1 - x_1) \exp\{\gamma x_2/(1 + x_2)\} - \beta(x_2 - \delta_j) \end{bmatrix},$$

$$B(\tau) = \begin{bmatrix} -C \sin(\tau) & 0 \\ 0 & -C \sin(\tau) \end{bmatrix}.$$

The principle fundamental matrix for such a system is calculated as:



(a) forced oscillations at large amplitude of C=0.5

(b) forced oscillation at small amplitude of C=0.2

Fig. 2. Response of a CSTR system to small and large amplitude periodic input

$$\Phi(\tau,\alpha) = \begin{bmatrix} e^{\alpha\cos(\tau)} & 0 \\ 0 & e^{\alpha\cos(\tau)} \end{bmatrix}, \tag{13}$$

where $\alpha = \epsilon C$ is taken to be an independent parameter, regulated by controlling the amplitude of oscillations. Using $\Phi(\tau,\alpha)$ as the choice of a Lyapunov transformation, we transform (12) into a slow varying system

$$\frac{dy}{dt} = \Phi(\tau,\alpha)^{-1} X\Big(\Phi(\tau,\alpha) y\Big) \tag{14}$$

To obtain an autonomous first order approximation of (14), we expand the matrix exponentials of (14) and collect the means of each term upto an accuracy of $O(\alpha^4)$

$$\frac{dz}{dt} = X(z) + \sigma^2 Y(\mathbf{z}), \tag{15}$$

where, $\sigma^2 = C^2/4\omega^2$ and $Y(\mathbf{z}) = [Y_1(\mathbf{z}), Y_2(\mathbf{z})]^T$, with

$$Y_1(\mathbf{z}) = \frac{\mathrm{Da}\exp\{\gamma z_2/(1+z_2)\}}{(1+z_2)^4} \times \Big[1 + (4-\gamma-\gamma z_1)z_2 + (6 - 4\gamma + \gamma^2 - \gamma^2 z_1)z_2^2 + (4 - 3\gamma + \gamma z_1)z_2^3 + z_2^4\Big]$$

$$Y_2(\mathbf{z}) = \beta\delta_j + \mathrm{B}\, Y_1(\mathbf{z})$$

Figure 3 shows a qualitative agreement of the average system with the periodic behaviour of the CSTR. The colored curves denote the forced oscillating solutions of (12). The solid black curve represents the solution of the averaged equation (15). Given the dynamics of the average equation, it is possible to estimate the behaviour of the oscillating solution by performing the Lyapunov transformation on the averaged solution. This estimate is illustrated as a dotted black curve and is calculated using

$$\hat{x}(t) = \Phi(\tau,\alpha) z(t).$$

Since $\Phi(\tau,\alpha)$ is bounded and $z(t)$ converges to a stable equilibrium, $|\hat{x}(t) - x(t)| = O(\epsilon)$, $\forall t \in [0,\infty)$.

3.1 Averaged Lyapunov function

Consider the following candidate for a Lyapunov function:

$$W(z) = \overline{z}^T \overline{z}.$$

For the sake of convenience, we use an overline to denote the difference operator. Therefore, $\overline{f(z)} = f(z) - f(z_s)$, where z_s is the required operating point. The time derivative of this function gives us,

$$\begin{aligned}\dot{W}(z) &= \overline{z}^T \left(\overline{\frac{dz}{dt}}\right) + \left(\overline{\frac{dz}{dt}}\right)^T \overline{z} \\ &= \overline{z}^T (\overline{X(z)} + \sigma^2 \overline{Y(z)}) + (\overline{X(z)} + \sigma^2 \overline{Y(z)})^T \overline{z}\end{aligned}$$

Now, $\overline{X(z)} = P(z_2)\overline{z}$, where $P(z_2)$ is defined as

$$\begin{bmatrix} -1 - Da\, \overline{e^{\gamma z_2/(1+z_2)}} & \dfrac{Da}{\overline{z_2}}(1 - z_{1s})\overline{e^{(\gamma z_2/(1+z_2)}} \\ -BDa\, \overline{e^{\gamma z_2/(1+z_2)}} & -1 - \beta + \dfrac{Da}{\overline{z_2}}(1 - z_{1s})\overline{e^{(\gamma z_2/(1+z_2)}} \end{bmatrix}$$

and $\overline{Y(z)} = Da\, Q(z_2)\overline{z}$ where $Q(z_2)$ is given by

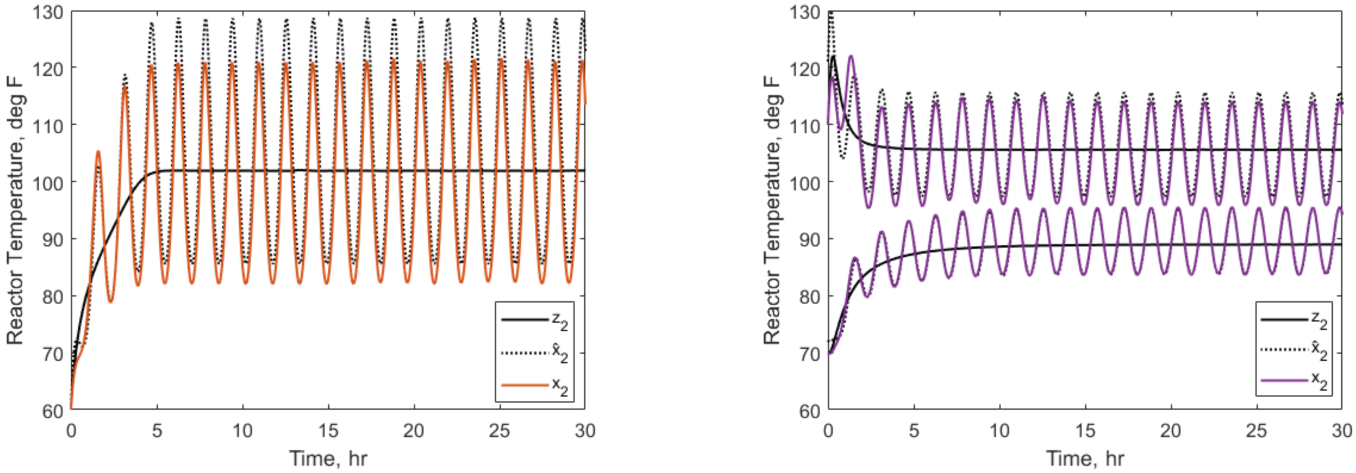
$$\begin{bmatrix} \overline{\gamma z_2 e^{(\gamma z_2/(1+z_2)} \dfrac{(z_2^2 - \gamma z_2 - 1)}{(1+z_2)^4}} & \dfrac{(1 - z_{1s})}{\overline{z_2}} \overline{e^{(\gamma z_2/(1+z_2)}} \\ \overline{\mathrm{B}\gamma z_2 e^{(\gamma z_2/(1+z_2)} \dfrac{(z_2^2 - \gamma z_2 - 1)}{(1+z_2)^4}} & \mathrm{B}\dfrac{(1 - z_{1s})}{\overline{z_2}} \overline{e^{(\gamma z_2/(1+z_2)}} \end{bmatrix}$$

Therefore,

$$\begin{aligned}\dot{W}(z) &= \overline{z}^T\ [P(z_2) + P(z_2)^T + Da\,\sigma^2(Q(z_2) + Q(z_2)^T)]\overline{z} \\ &\le \lambda_{max}(z_2) W(z) \\ &\le \lambda W(z),\end{aligned}$$

where $\lambda_{max}(z_2)$ is the combined maximum eigenvalue of the symmetric matrices inside the square brackets and $\lambda = \Big(\sup_{z_2 \in U_\rho} \lambda_{max}(z_2)\Big)$. If $\lambda_{max}(z_2)$ is negative for all values of z_2 inside a region of convergence then from Theorem 2, we can conclude that there exists a stationary periodic orbit $x^*(t)$ inside this region. In addition, we also conclude that any solution $x(t)$ to (12) starting within the region of convergence exponentially converges to this stationary orbit as $\|x(t) - x^*(t)\| \le Ke^{-\lambda t}$.

Figure 4 plots the largest eigenvalues for the two test cases. This plot allows us to see the region of convergence by looking at when the largest eigenvalue of the system is negative. We see that for small amplitude oscillations the region around the operating point of $T = 101.1°$F has positive eigenvalue and is therefore unstable.



(a) Temperature variation plot for large amplitude forcing of C=0.5

(b) Temperature variation plot for small amplitude forcing of C=0.2

Fig. 3. The orange and purple curves denote the time varying response of the CSTR system to large and small amplitude inputs, respectively. Solid black line denotes the solution to the averaged equation. The dotted black curve is an estimate of the time varying response calculated using the average solution.

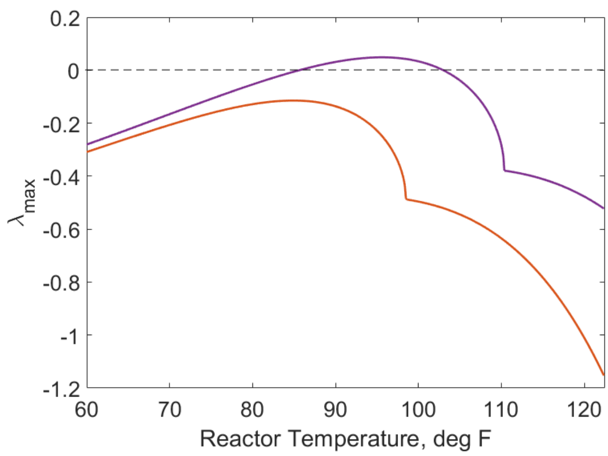


Fig. 4. Orange curve denotes the largest eigenvalue for C=0.5 and purple for C=0.2 respectively

REFERENCES

Baillieul, J. (1995). Energy methods for stability of bilinear systems with oscillatory inputs. *International Journal of Robust and Nonlinear Control*, 5(4), 285–301. doi:10.1002/rnc.4590050405.

Baillieul, J. (1993). Stable average motions of mechanical systems subject to periodic forcing. *Dynamics and Control of Mechanical Systems: The falling cat and related problems*, 1, 1–23.

Bellman, R., Bentsman, J., and Meerkov, S.M. (1986a). Vibrational control of nonlinear systems: Vibrational stabilizability. *IEEE Transactions on Automatic Control*, 31(8), 710–716. doi:10.1109/TAC.1986.1104384.

Bellman, R., Bentsman, J., and Meerkov, S.M. (1983). Vibrational control of systems with Arrhenius dynamics. *Journal of Mathematical Analysis and Applications*, 91(1), 152–191. doi:10.1016/0022-247X(83)90099-9.

Bellman, R.E., Bentsman, J., and Meerkov, S.M. (1986b). Vibrational Control of Nonlinear Systems: Vibrational Controllability and Transient Behavior. *IEEE Transactions on Automatic Control*, 31(8), 717–724. doi:10.1109/TAC.1986.1104383.

Bentsman, J., Fakhfakh, J., and Lehman, R.B. (1989). Vibrational stabilization of linear time delay systems and its robustness with respect to the delay size. *Systems & control letters*, 12(3), 267–272.

Bequette, B.W. (2003). *Process control: modeling, design, and simulation*. Prentice Hall Professional.

Berg, J.M. and Wickramasinghe, I.M. (2015). Vibrational control without averaging. *Automatica*, 58, 72–81.

Bogoliubov, N.N. and Mitropolsky, Y.A. (1961). *Asymptotic methods in the theory of non-linear oscillations*, volume 10. CRC Press.

Bucolo, M., Buscarino, A., Famoso, C., Fortuna, L., and Frasca, M. (2019). Control of imperfect dynamical systems. *Nonlinear Dynamics*, 98(4), 2989–2999.

Bullo, F. (2002). Averaging and vibrational control of mechanical systems. *SIAM Journal on Control and Optimization*, 41(2), 542–562.

Caruntu, D.I. and Martinez, I. (2014). Reduced order model of parametric resonance of electrostatically actuated mems cantilever resonators. *International Journal of Non-Linear Mechanics*, 66, 28–32.

Cheng, X., Tan, Y., and Mareels, I. (2018). On robustness analysis of linear vibrational control systems. *Automatica*, 87, 202–209.

Cinar, A., Rigopoulos, K., Meerkov, S.M., and Shu, X. (1986). Vibrational Control of an Exothermic CSTR. *1986 American Control Conference*, 33(3), 353–365.

Cinar, A., Rigopoulos, K., Shu, X., and Meerkov, S. (1988). Vibrational control of chemical reactors: Stabilization and conversion improvement in an exothermic cstr. In *Dynamics and Control of Chemical Reactors and Distillation Columns*, 153–157. Elsevier.

Cinar, A., Deng, J., Meerkov, S., and Shu, X. (1987). Vibrational control of an exothermic reaction in a cstr: Theory and experiments. *AIChE journal*, 33(3), 353–365.

Ghanaatpishe, M., Bayiz, Y.E., Cheng, B., and Fathy, H.K. (2018). Rotary versus flapping flight: An application study for optimal periodic control theory. In *Dynamic Systems and Control Conference*, volume 51890, V001T11A003. American Society of Mechanical Engineers.

Guckenheimer, J. and Holmes, P. (2013). *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, volume 42. Springer Science & Business Media.

Heidarinejad, M., Liu, J., and Christofides, P.D. (2012). Economic model predictive control of nonlinear process systems using lyapunov techniques. *AIChE Journal*, 58(3), 855–870.

Kabamba, P., Meerkov, S., and Poh, E.K. (1998). Pole placement capabilities of vibrational control. *IEEE transactions on automatic control*, 43(9), 1256–1261.

Kapitza, P.L. (1965). Dynamical stability of a pendulum when its point of suspension vibrates, and pendulum with a vibrating suspension. *Collected papers of PL Kapitza*, 2, 714–737.

Lee, S., Meerkov, S., and Runolfsson, T. (1987). Vibrational feedback control: Zeros placement capabilities. *IEEE Transactions on Automatic Control*, 32(7), 604–611.

Meerkov, S.M. (1973). On averaging trajectories of slow dynamical systems. *Uspekhi Mat. Nauk*, 9(9), 1609–1617. doi:10.18287/0134-2452-2015-39-4-453-458.

Meerkov, S.M. (1977). Vibrational control theory. *Journal of the Franklin Institute*, 303(2), 117–128. doi: 10.1016/0016-0032(77)90040-0.

Meerkov, S.M. (1980). Principle of Vibrational Control: Theory and Applications. *IEEE Transactions on Automatic Control*, 25(4), 755–762. doi: 10.1109/TAC.1980.1102426.

Moreau, L. and Aeyels, D. (2004). Periodic output feedback stabilization of single-input single-output continuous-time systems with odd relative degree. *Systems & control letters*, 51(5), 395–406.

Nikolić, D., Seidel-Morgenstern, A., and Petkovska, M. (2020). Nonlinear frequency response analysis of forced periodic operations with simultaneous modulation of two general waveform inputs with applications on adiabatic cstr with square-wave modulations. *Chemical Engineering Science*, 115842.

Ramírez-Ávila, G.M., Kurths, J., Depickere, S., and Deneubourg, J.L. (2019). Modeling fireflies synchronization. In *A Mathematical Modeling Approach from Nonlinear Dynamics to Complex Systems*, 131–156. Springer.

Shapiro, B. and Zinn, B. (1997). High-frequency nonlinear vibrational control. *IEEE Transactions on Automatic Control*, 42(1), 83–90.

Silveston, P.L. and Hudgins, R.R. (2012). *Periodic operation of chemical reactors*. Butterworth-Heinemann.

Sterman, L.E. and Ydstie, B.E. (1990a). The steady-state process with periodic perturbations. *Chemical Engineering Science*, 45(3), 721–736.

Sterman, L.E. and Ydstie, B.E. (1990b). Unsteady-state multivariable analysis of periodically perturbed systems. *Chemical engineering science*, 45(3), 737–749.

Sterman, L.E. and Ydstie, B.E. (1991). Periodic forcing of the cstr: An application of the generalized ii-criterion. *AIChE journal*, 37(7), 986–996.

Suttner, R. (2019). Extremum seeking control for an acceleration controlled unicycle. *IFAC-PapersOnLine*, 52(16), 676–681.

Taha, H.E. and Kiani, M. (2019). A new vibrational control system in nature: Flapping flight. In *AIAA Scitech 2019 Forum*, 1417.

Taha, H.E., Tahmasian, S., Woolsey, C.A., Nayfeh, A.H., and Hajj, M.R. (2015). The need for higher-order averaging in the stability analysis of hovering, flapping-wing flight. *Bioinspiration & biomimetics*, 10(1), 016002.

Tahmasian, S. and Katrahmani, A. (2020). Vibrational control of mechanical systems with piecewise linear damping and high-frequency inputs. *Nonlinear Dynamics*, 99(2), 1403–1413.

Yabuno, H. and Kobayashi, S. (2020). Motion control of a flexible underactuated manipulator using resonance in a flexible active arm. *International Journal of Mechanical Sciences*, 174, 105432.

Zuyev, A., Seidel-Morgenstern, A., and Benner, P. (2017). An isoperimetric optimal control problem for a non-isothermal chemical reactor with periodic inputs. *Chemical Engineering Science*, 161, 206–214.

Appendix A. PROOF OF THEOREM 2

Without loss of generality, assume $z_s = 0$. Consider a flow map ϕ, starting from a point $y_0 \in U_\delta$, describing a unique solution to (7). Since the flow is described inside a region of exponential convergence, it should satisfy

$$|\phi(t-s)| \leq e^{-\lambda(t-s)} \qquad \forall \;\; s \leq t, \; t \in R.$$

We define an integro-differential equation

$$y(t) = \phi(t)y_0 + \int_0^s \epsilon\phi(t-s)Y_1(y,s,\epsilon)ds.$$

Clearly, $y(t)$ is a solution to (6). Furthermore, we assume Y_1 to be Lipschitz in y. In general, Y_1 is C^r when Y is C^r in y. With this we can write

$$|Y_1(y,t,\epsilon) - Y_1(y',t,\epsilon)| \leq \mu(\delta)|y-y'|,$$

where $\mu(\delta)$ is the Lipschitz constant such that $\mu(\delta) \to 0$ when $\delta \to 0$. We can show

$$\begin{aligned} |y-y'| &\leq |\phi(t)|y_0 - y_0'| + \\ &\int_0^s \epsilon|\phi(t-s)|\, |Y_1(y,s,\epsilon) - Y_1(y',s,\epsilon)|\, ds \\ &\leq e^{-\lambda t}|y_0 - y_0'| + \frac{\epsilon\mu(\delta)}{\lambda}|y-y'|_\infty. \end{aligned}$$

From this we see that it is possible find an $\epsilon'(\delta)$ such that for any $\epsilon < \epsilon'$, $\epsilon\mu/\lambda < 1$. Therefore we can find a constant K such that

$$|y-y'| \leq Ke^{-\lambda t}|y_0 - y_0'| \tag{A.1}$$

Since (A.1) describes a contraction, by the contraction mapping theorem, there exists a unique periodic orbit $y^*(t)$ inside U_δ. Additionally, from (A.1) we can show that all trajectories starting within U_δ converge to $y^*(t)$ exponentially at the rate of λ.