Nonlinear state feedback control design for port-Hamiltonian systems with unstructured component ⋆

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Abstract: This paper considers the problem of state feedback controller design to stabilize generalized Hamiltonian systems with an unstructured component. This class of models enable one to exploit the structure of port-Hamiltonian systems for feedback control design while relaxing the constraint of deriving an exact structured port-Hamiltonian representation. For a given stabilizable nonlinear system, and with some assumptions on the unstructured part of the dynamics, a stabilizing control law is designed and asymptotic stability of a desired equilibrium of the system is demonstrated. A numerical illustration of the proposed approach is presented to demonstrate the design method.

Keywords: Dynamic feedback control, Stabilization, port-Hamiltonian systems, Lyapunove stability,

1. INTRODUCTION

The port-Hamiltonian (pH) structure is practical to describe a wide range of dynamical system including chemical processes. This structure gives a clear relation between the dynamics and the energy of the system. For mechanical systems, the Hamiltonian function in pH systems is the total energy and can be used as a Lyapunov function for the system (van der Schaft, 2017). One issue when applying this approach for chemical process systems is to represent mass and energy dynamics as generalized Hamiltonian systems (see for example Ramirez et al. (2013)). Another problem is to design a suitable controller via different techniques for this class of systems, a problem investigated by many researchers (Ortega et al., 2008; Dörfler et al., 2009; Ramírez et al., 2009; Donaire and Junco, 2009; Castaños et al., 2009). Finding a suitable feedback law to shape the Hamiltonian dynamics of a given system requires the solution of a system of matching Partial Differential Equations (PDEs). For static state feedback controller design, proposed techniques from the literature solving the matching and non-matching PDE equations are generally based on control via interconnection (Ortega et al., 2008; Castaños et al., 2009), power shaping (Liu et al., 2010; Dirksz and Scherpen, 2012), and energy shaping through Interconnection-Damping Assignment (IDA-PBC) (Dörfler et al., 2009; Ramírez et al., 2009). Generalized canonical transformation is a way to change the pH system without changing the inherent property of the system. This approach has been shown to simplify the feedback controller design but still needs the solution of PDEs in the transformation process (Fujimoto et al., 2012).

⋆ This research was supported by The Natural Sciences and Engineering Research Council of Canada (NSERC) via the Discovery Grant program.

The controller design problem was further developed by extending the system dynamics in a closed loop that is conceptually equal to the PI controllers. This helps to stabilize the system more efficiently, especially in the presence of disturbance or uncertainties. Some researchers demonstrated that by having the solution to the matching equations, integral action can improve the control performance to reject disturbance while preserving the Hamiltonian structure (Donaire and Junco, 2009; Ryalat and Laila, 2018). The dynamic extension approach has also been applied to the power shaping control of pH systems (Liu et al., 2010). In (Nguyen et al., 2019), some results are obtained to stabilize the system at the desired equilibrium by explicitly defining a reference dynamics that is different from the actual dynamics and then designing an stabilizing dynamic controller to derive the errors to the origin. It was even illustrated that the PDE solution could be relaxed by abandoning the objective of preservation in the closed-loop of the pH structure, which is the condition that gives rise to the PDEs (Donaire et al., 2016; Borja et al., 2016). However, in most cases, a clear pH representation of the system is required. An approach to the approximate representation problem for the stabilization of control affine systems is presented in (Guay and Hudon, 2016) without using port-Hamiltonian systems theory for control design.

In the present contribution, we are interested in representing a control affine system as the combination of a structured dynamical component (a port-Hamiltonian system) and an unstructured dynamical component. Under mild assumptions on the unstructured part, the objective is to design a controller, using the structured part of the dynamics, stabilizing the overall system at a desired equilibrium. The noticeable advancement here, contrary to the conventional approaches, is that we avoid solving PDEs in designing the controller, while using information on the structured part of the dynamics to find a simpler controller. Results show that the proposed controller has a good performance and is powerful enough to stabilize the trajectories at desired equilibrium points. This approach is valuable for systems derived from balance laws, as a port-Hamiltonian representation for this class of systems is usually difficult to obtain in general.

The paper is organized as follows. In section 2, background material is briefly reviewed and the stabilizing control problem for port-Hamiltonian systems is formulated. In Section 3, a stabilizing control for the system is designed and stability of the systems in closed-loop with the proposed controller is proved. In Section 4, the van de Vusse reacting system is studied and numerical simulations are given to illustrate the application of the proposed approach. Conclusions and areas for further investigations are discussed in Section 5.

2. BACKGROUND AND PROBLEM FORMULATION

2.1 Port-Hamiltonian Systems

We consider control affine systems of the form

$$\dot{x} = F(x) + G(x)u, \tag{1}$$

where $x \in \mathcal{D} \subset \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Throughout the present note, we assume the following.

Assumption 1. The control affine system (1) is stabilizable at a point x^*, *i.e.,* the following expression holds:

$$\text{span}\{f(x), ad_f G(x), \cdots, ad_f^k G(x) \mid \forall x \in \mathcal{D}\backslash\{x^*\}, k \in \mathbb{Z}_+\} = \mathbb{R}^n. \tag{2}$$

Following (van der Schaft, 2017), it is possible to express the system (1) as a standard port-Hamiltonian system with dissipation

$$\dot{x} = [J(x) - R(x)]\nabla H(x) + G(x)u, \tag{3}$$

where $x \in \mathcal{D} \subset \mathbb{R}^n$ expressing the states and $u \in \mathbb{R}^m$ is the input signal. The function $H : \mathbb{R}^n \to \mathbb{R}$ is the Hamiltonian function, $J(x)$ is a skew-symmetric structure matrix, and $R(x)$ is a positive definite symmetric matrix. The notation $\nabla H(x)$ denotes the vector $\frac{\partial H}{\partial x} = [\frac{\partial H}{\partial x_1} \cdots \frac{\partial H}{\partial x_n}]^T$.

An interesting property of this class of systems is that the Hamiltonian function $H(x)$ can play the role of Lyapunov function for stability analysis (Ortega et al., 2002), *i.e.*, if $H(x)$ admits a strict minimum at $x^* = 0$, then x^* is a stable equilibrium of the unforced systems

$$\dot{x} = [J(x) - R(x)]\nabla H(x). \tag{4}$$

Usually, the control input can be used to shape the Hamiltonian function $H(x) \to H_d(x)$ (and eventually to inject additional damping $R(x) \to R_d(x)$) such that under a proper feedback $u = \alpha(x)$, the closed loop system has the Hamiltonian structure, *i.e.*,

$$[J(x) - R(x)]\nabla H(x) + G(x)u = [J(x) - R_d(x)]\nabla H_d(x) \tag{5}$$

and $H_d(x)$ admits a strict minimum at a desired equilibrium x^*. This is the problem that has been widely addressed in the literature and many applications are discussed (Ramírez et al., 2009; Dörfler et al., 2009) but requires the solution of a set of PDE equations, the matching equations, to obtain an exact port-Hamiltonian system in closed-loop.

In the present note, we depart from this approach and consider control affine systems expressed as approximated port-Hamiltonian systems, *i.e.*, we consider that the control affine system (1) can be written as

$$\dot{x} = [J(x) - R(x)]\nabla H(x) + \Psi(x) + G(x)u, \tag{6}$$

where the properties of the unstructured part of the dynamics $\Psi(x)$ are to be established below.

2.2 Problem formulation

We consider control affine systems (1) and assume that the system is stabilizable following Assumption 1. The original system is re-expressed in the form (6). This can be achieved in practice since the unstructured part of the dynamics $\Psi(x)$ can be fixed as desired by the user. In general, the structured part of dynamics can be made as desired. This is formalized in the following assumption.

Assumption 2. The Hamiltonian function in system (6) is assumed to be a quadratic function of the state variables with the unique minimum at the origin, *i.e.*, $H(x) = \frac{1}{2}x^T x$.

The approximate representation of control affine systems as port-Hamiltonian systems is obviously not unique. In the sequel, we demonstrate that the proposed representation can still be exploited in the derivation of a suitable state feedback stabilizing control law, under mild assumptions on the unstructured part of the dynamics $\Psi(x)$.

As the analysis shows in the next Section, the key assumption is that the system (6) (equivalently system (1)) is locally stabilizable. Moreover, we make the following assumption of the unstructured component of the dynamics $\Psi(x)$.

Assumption 3. The unstructured vector $\Psi(x)$ has Lipschitz property in x and meets the inequality

$$\|\Psi(x)\| \leq q(x)\|R(x)\nabla H(x)\| \quad \forall x \in \mathcal{D}\backslash\{x^*\}, \tag{7}$$

where $\mathcal{D} \subset \mathbb{R}^n$ is a domain of interest centered at the origin and the state-dependent bound is bounded locally by a constant, *i.e.*, $q(x) \leq q$ on $\mathcal{D}$.

In other words, the contribution of the unstructured part of the dynamics must be bounded by the natural dissipation of the system, encoded in the structured part of the dynamics. As stabilization analysis of (exact) port-Hamiltonian systems is based on the dissipation of the system, a natural extension in our context is to relate the unstructured part of the dynamics to the dissipation rate of the structured part of the dynamics.

3. STABILIZING CONTROLLER DESIGN

3.1 Stabilizing at nonzero steady state

We design a control law to move the trajectories to desired equilibrium x^*. For completeness of the analysis we consider the more general case in which the system is not fully actuated. Hence, we split the system state variables x into actuated (x_a) and non-actuated (x_{na}) state variables. Subsequently, all the operators would be regrouped in two sets. For the actuated and non-actuated parts, the operators and vector functions are sub-scripted by 1 and 2, respectively. Actuated and non-actuated matrix elements are denoted by 11 and 22, respectively. We also denote the state vector as $\boldsymbol{x} = [x_a, x_{na}]^T$. Using this notation, we can rewrite the dynamics (6) as

$$\begin{bmatrix} \dot{x}_a \\ \dot{x}_{na} \end{bmatrix} = \begin{bmatrix} J_{11}(\boldsymbol{x}) - R_{11}(\boldsymbol{x}) & J_{12}(\boldsymbol{x}) \\ -J_{21}^T(\boldsymbol{x}) & J_{22}(\boldsymbol{x}) - R_{22}(\boldsymbol{x}) \end{bmatrix} \begin{bmatrix} \nabla H(x_a) \\ \nabla H(x_{na}) \end{bmatrix} + \begin{bmatrix} \Psi_1(\boldsymbol{x}) \\ \Psi_2(\boldsymbol{x}) \end{bmatrix} + \begin{bmatrix} g(\boldsymbol{x}) \\ 0 \end{bmatrix} u, \tag{8}$$

such that $J_{12}(\boldsymbol{x}) \neq 0$. Note that based on Assumption 2, the Hamiltonian function $H(\boldsymbol{x}) = H_1(x_a) + H_2(x_{na}) = \frac{1}{2}(x_a^T x_a + x_{na}^T x_{na})$. Furthermore, Assumption 3 can be rewritten as follows:

$$\left\| \begin{bmatrix} \Psi_1(\boldsymbol{x}) \\ \Psi_2(\boldsymbol{x}) \end{bmatrix} \right\| < q(\boldsymbol{x}) \left\| \begin{bmatrix} R_{11}(\boldsymbol{x}) \nabla H_1(x_a) \\ R_{22}(\boldsymbol{x}) \nabla H_2(x_{na}) \end{bmatrix} \right\| \tag{9}$$

We do not use the unstructured component in development of the controller but we use this information for stability analysis. We consider the Hamiltonian function $H(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{x}$, hence, $\nabla H(\boldsymbol{x}) = \boldsymbol{x}$. Since the equilibrium can be a nonzero state, we shift the Hamiltonian function such that it has a local minimum at $\boldsymbol{x}^*$. Hence, the Hamiltonian function is re-expressed as

$$H(\boldsymbol{x} - \boldsymbol{x}^*) = \tfrac{1}{2}(\boldsymbol{x} - \boldsymbol{x}^*)^T(\boldsymbol{x} - \boldsymbol{x}^*) \tag{10}$$

In the sequel, we design a dynamic controller of the form

$$\begin{aligned} u(\boldsymbol{x}, \boldsymbol{x}^*, \xi) &= \alpha(\boldsymbol{x}, \boldsymbol{x}^*) + K_I(\boldsymbol{x})\xi \\ \dot{\xi} &= -K_I(\boldsymbol{x})^T g^T(\boldsymbol{x})(x_a - x_a^*), \end{aligned} \tag{11}$$

where K_I can be a constant or state dependent positive matrix. The extended dynamic variables $\xi \in \mathbb{R}^m$. The following Proposition states and proves the asymptotic stability of the system (6) at desired equilibrium $\boldsymbol{x}^*$ when we add the integral action to the system.

Proposition 4. Consider the port-Hamiltonian system with unstructured component (6) interconnected in closed-loop with controller (**??**), where

$$\begin{aligned} \alpha(\boldsymbol{x}, \boldsymbol{x}^*) = [g(\boldsymbol{x}^*)^T g(\boldsymbol{x})]^{-1}[-g(\boldsymbol{x}^*)^T(A + K_p R_{11} x_a) \\ + g(\boldsymbol{x}^*)^T g(\boldsymbol{x}^*) u^*], \end{aligned}$$

where K_p is positive square constant gain matrix in $\mathbb{R}^{n\times n}$, $A = [\Delta J_{11} - \Delta R_{11}]x_a + [J_{12}(\boldsymbol{x}) - J_{12}(\boldsymbol{x}^*)]x_{na}$, $\Delta J_{11} = J_{11}(\boldsymbol{x}) - J_{11}(\boldsymbol{x}^*)$, and $\Delta R_{11} = R_{11}(\boldsymbol{x}) - R_{11}(\boldsymbol{x}^*)$.

Under Assumption 1, 2, and 3, the desired equilibrium $\boldsymbol{x}^*$ is asymptotically stable.

Proof. We consider the Lyapunov function

$$V(\boldsymbol{x}, \boldsymbol{x}^*, \xi) = \tfrac{1}{2}(\boldsymbol{x} - \boldsymbol{x}^*)^T(\boldsymbol{x} - \boldsymbol{x}^*) + \tfrac{1}{2}\xi^T\xi.$$

Taking the time derivative, we have

$$\dot{V}(\boldsymbol{x}, \boldsymbol{x}^*, \xi) = (\boldsymbol{x} - \boldsymbol{x}^*)^T(\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}^*) + \xi^T\dot{\xi}$$

We denote $\delta x = x - x^*$ for each actuated and non-actuated states. Considering quadratic property of Hamiltonian function, we have

$$\begin{aligned} \begin{bmatrix} \dot{x}_a - \dot{x}_a^* \\ \dot{x}_{na} - \dot{x}_{na}^* \end{bmatrix} &= \begin{bmatrix} J_{11}(\boldsymbol{x}) - R_{11}(\boldsymbol{x}) & J_{12}(\boldsymbol{x}) \\ -J_{21}^T(\boldsymbol{x}) & J_{22}(\boldsymbol{x}) - R_{22}(\boldsymbol{x}) \end{bmatrix} \begin{bmatrix} x_a \\ x_{na} \end{bmatrix} \\ &+ \begin{bmatrix} \Psi_1(\boldsymbol{x}) \\ \Psi_2(\boldsymbol{x}) \end{bmatrix} + \begin{bmatrix} g(\boldsymbol{x}) \\ \boldsymbol{0} \end{bmatrix} u \\ &- \begin{bmatrix} J_{11}(\boldsymbol{x}^*) - R_{11}(\boldsymbol{x}^*) & J_{12}(\boldsymbol{x}^*) \\ -J_{21}^T(\boldsymbol{x}^*) & J_{22}(\boldsymbol{x}^*) - R_{22}(\boldsymbol{x}^*) \end{bmatrix} \begin{bmatrix} x_a^* \\ x_{na}^* \end{bmatrix} \\ &- \begin{bmatrix} \Psi_1(\boldsymbol{x}^*) \\ \Psi_2(\boldsymbol{x}^*) \end{bmatrix} - \begin{bmatrix} g(\boldsymbol{x}^*) \\ \boldsymbol{0} \end{bmatrix} u^* \\ &= \begin{bmatrix} J_{11}(\boldsymbol{x}^*) - R_{11}(\boldsymbol{x}^*) & J_{12}(\boldsymbol{x}^*) \\ -J_{21}^T(\boldsymbol{x}^*) & J_{22}(\boldsymbol{x}^*) - R_{22}(\boldsymbol{x}^*) \end{bmatrix} \begin{bmatrix} \delta x_a \\ \delta x_{na} \end{bmatrix} \\ &+ \begin{bmatrix} \Delta J_{11} - \Delta R_{11} & J_{12}(\boldsymbol{x}) - J_{12}(\boldsymbol{x}^*) \\ -[J_{21}^T(\boldsymbol{x}) - J_{21}^T(\boldsymbol{x}^*)] & \Delta J_{22} - \Delta R_{22} \end{bmatrix} \begin{bmatrix} x_a \\ x_{na} \end{bmatrix} \\ &+ \begin{bmatrix} \Psi_1(\boldsymbol{x}) - \Psi_1(\boldsymbol{x}^*) \\ \Psi_2(\boldsymbol{x}) - \Psi_2(\boldsymbol{x}^*) \end{bmatrix} + \begin{bmatrix} g(\boldsymbol{x}) \\ \boldsymbol{0} \end{bmatrix} u - \begin{bmatrix} g(\boldsymbol{x}^*) \\ \boldsymbol{0} \end{bmatrix} u^* \end{aligned}$$

Interconnection of this equation with dynamic controller (11) gives

$$\begin{aligned} &\begin{bmatrix} \dot{x}_a - \dot{x}_a^* \\ \dot{x}_{na} - \dot{x}_{na}^* \\ \dot{\xi} \end{bmatrix} = \\ &\begin{bmatrix} J_{11}(\boldsymbol{x}^*) - R_{11}(\boldsymbol{x}^*) & J_{12}(\boldsymbol{x}^*) & g(\boldsymbol{x})K_I \\ -J_{21}^T(\boldsymbol{x}^*) & J_{22}(\boldsymbol{x}^*) - R_{22}(\boldsymbol{x}^*) & 0 \\ -K_I^T g^T(\boldsymbol{x}) & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \delta x_a \\ \delta x_{na} \\ \xi \end{bmatrix} \\ &+ \begin{bmatrix} \Delta J_{11} - \Delta R_{11} & J_{12}(\boldsymbol{x}) - J_{12}(\boldsymbol{x}^*) & \boldsymbol{0} \\ -[J_{21}^T(\boldsymbol{x}) - J_{21}^T(\boldsymbol{x}^*)] & \Delta J_{22} - \Delta R_{22} & 0 \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} x_a \\ x_{na} \\ \boldsymbol{0} \end{bmatrix} \\ &+ \begin{bmatrix} \Phi_1(\boldsymbol{x}, \boldsymbol{x}^*) \\ \Phi_2(\boldsymbol{x}, \boldsymbol{x}^*) \\ \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} g(\boldsymbol{x}) \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \alpha(\boldsymbol{x}, \boldsymbol{x}^*) - \begin{bmatrix} g(\boldsymbol{x}^*) \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} u^*. \end{aligned} \tag{12}$$

where $\Phi_i(\boldsymbol{x}, \boldsymbol{x}^*) = \Psi_i(\boldsymbol{x}) - \Psi_i(\boldsymbol{x}^*)$ is the new unstructured dynamic. We have

$$\begin{aligned} \dot{V}(\boldsymbol{x}, \boldsymbol{x}^*, \xi) &= \delta x_a^T[J_{11}(\boldsymbol{x}^*) - R_{11}(\boldsymbol{x}^*)]\delta x_a \\ &+ \delta x_a^T J_{12}(\boldsymbol{x}^*)\delta x_{na} - \delta x_{na}^T J_{12}(\boldsymbol{x}^*)\delta x_a \\ &+ \delta x_{na}^T[J_{22}(\boldsymbol{x}^*) - R_{22}(\boldsymbol{x}^*)]\delta x_{na} \\ &+ \delta x_a^T[\Delta J_{11} - \Delta R_{11}]x_a + \delta x_a^T[J_{12}(\boldsymbol{x}) - J_{12}(\boldsymbol{x}^*)]x_{na} \\ &- \delta x_{na}^T[J_{21}^T(\boldsymbol{x}) - J_{21}^T(\boldsymbol{x}^*)]x_a + \delta x_{na}^T[\Delta J_{22} - \Delta R_{22}]x_{na} \\ &+ \delta x_a^T\Phi_1(\boldsymbol{x}, \boldsymbol{x}^*) + \delta x_{na}^T\Phi_2(\boldsymbol{x}, \boldsymbol{x}^*) \\ &+ \delta x_a^T g(\boldsymbol{x})\alpha(\boldsymbol{x}, \boldsymbol{x}^*) \\ &+ \delta x_a^T g(\boldsymbol{x})K_I(\boldsymbol{x})\xi - \xi^T g^T(\boldsymbol{x})K_I(\boldsymbol{x})^T\delta x_a - \delta x_a^T g(\boldsymbol{x}^*)u^*. \end{aligned}$$

Canceling some terms we obtain,

$$\begin{aligned} \dot{V}(\boldsymbol{x}, \boldsymbol{x}^*, \xi) &= -\delta x_a^T R_{11}(\boldsymbol{x}^*)\delta x_a - \delta x_{na}^T R_{22}(\boldsymbol{x}^*)\delta x_{na} \\ &+ \delta x_a^T[\Delta J_{11} - \Delta R_{11}]x_a + \delta x_a^T[J_{12}(\boldsymbol{x}) - J_{12}(\boldsymbol{x}^*)]x_{na} \\ &- \delta x_{na}^T[J_{21}^T(\boldsymbol{x}) - J_{21}^T(\boldsymbol{x}^*)]x_a + \delta x_{na}^T[\Delta J_{22} - \Delta R_{22}]x_{na} \\ &+ \delta x_a^T\Phi_1(\boldsymbol{x}, \boldsymbol{x}^*) + \delta x_{na}^T\Phi_2(\boldsymbol{x}, \boldsymbol{x}^*) \\ &+ \delta x_a^T g(\boldsymbol{x})\alpha(\boldsymbol{x}, \boldsymbol{x}^*) - \delta x_a^T g(\boldsymbol{x}^*)u^*. \end{aligned}$$

Since the system is stabilizable (Assumption 1), stability of non-actuated dynamics at the desired equilibrium x_{na}^* is guaranteed without control action. Hence, the non-actuated error terms δx_{na} vanish. Thus,

$$\begin{aligned} \dot{V}(\boldsymbol{x}, \boldsymbol{x}^*, \xi) &= -\delta x_a^T R_{11}(\boldsymbol{x}^*)\delta x_a + \delta x_a^T[\Delta J_{11} - \Delta R_{11}]x_a \\ &+ \delta x_a^T[J_{12}(\boldsymbol{x}) - J_{12}(\boldsymbol{x}^*)]x_{na} \\ &+ \delta x_a^T\Phi_1(\boldsymbol{x}, \boldsymbol{x}^*) + \delta x_a^T g(\boldsymbol{x})\alpha(\boldsymbol{x}, \boldsymbol{x}^*) - \delta x_a^T g(\boldsymbol{x}^*)u^*. \end{aligned} \tag{13}$$

Note that assuming stabilizability of the system the unstructured part of the non-actuated dynamics vanish as well. In this equation, the first term is always negative, hence, to guarantee that $\dot{V}(\boldsymbol{x} - \boldsymbol{x}^*) \leq 0$, we should find $\alpha(\boldsymbol{x}, \boldsymbol{x}^*)$ such that

$$\delta x_a^T[(\Delta J_{11}-\Delta R_{11})x_a+(J_{12}(\boldsymbol{x})-J_{12}(\boldsymbol{x}^*))x_{na}] \\ +\delta x_a^T\Phi_1(\boldsymbol{x},\boldsymbol{x}^*)+\delta x_a^T g(\boldsymbol{x})\alpha(\boldsymbol{x},\boldsymbol{x}^*)-\delta x_a^T g(\boldsymbol{x}^*)u^*=0. \tag{14}$$

Using Assumption 3, we replace Φ_1 by $K_pR_{11}\nabla H(x_a)$ or $K_pR_{11}\delta x_a$ where K_p is a positive diagonal gain matrix with all entries equal or larger than q. By having this term we can adjust the controller such that the effect of unstructured component is taken in effect and hence, we can ensure the strict negativity of Lyapunov derivative when trajectories are far from the desired equilibrium point $(\boldsymbol{x}^*,\xi^*)$. Therefore, equation (14) becomes:

$$\delta x_a^T[(\Delta J_{11}-\Delta R_{11})x_a+(J_{12}(\boldsymbol{x})-J_{12}(\boldsymbol{x}^*))x_{na}] \\ +\delta x_a^T K_pR_{11}\delta x_a+\delta x_a^T g(\boldsymbol{x})\alpha(\boldsymbol{x},\boldsymbol{x}^*)-\delta x_a^T g(\boldsymbol{x}^*)u^*=0.$$

Finally, for

$$A+g(\boldsymbol{x})\alpha(\boldsymbol{x},\boldsymbol{x}^*)+K_pR_{11}\delta x_a-g(\boldsymbol{x}^*)u^*=0,$$

with $A=[\Delta J_{11}-\Delta R_{11}]x_a+[J_{12}(\boldsymbol{x})-J_{12}(\boldsymbol{x}^*)]x_{na}$ and selecting

$$\alpha(\boldsymbol{x},\boldsymbol{x}^*)=[g(\boldsymbol{x}^*)^Tg(\boldsymbol{x})]^{-1}[-g(\boldsymbol{x}^*)^T(A+K_pR_{11}x_a) \\ +g(\boldsymbol{x}^*)^Tg(\boldsymbol{x}^*)u^*],$$

ensures that $\dot V(\boldsymbol{x}-\boldsymbol{x}^*)\le 0$. By design of the Hamiltonian function under Assumption 2, the origin of the dynamic system is the largest invariant set, and by invoking Lasalle's invariance principle, we conclude that desired equilibrium is asymptotically stable in closed-loop Khalil (2002). ■

3.2 Stabilizing at the origin

This section presents the specific case where the system is to be stabilized at the origin *i.e.,* $\boldsymbol{x}^*=0$. Followed by the general idea of closed-loop Lyapunov stability, we end up in a simpler dynamic controller that guarantees the stability of the system at the origin. Next Proposition states the main result of this section.

Proposition 5. Consider the port-Hamiltonian system with unstructured component defined in (6). Assume that the system is locally stabilizable and that the unstructured component $\Psi(\boldsymbol{x})$ meets the conditions of Assumption 3 for a representation with Hamiltonian meeting Assumption 2. Then, the state feedback controller given by

$$u(\boldsymbol{x},\xi)=[g^T(\boldsymbol{x})g(\boldsymbol{x})]^{-1}g^T(\boldsymbol{x}) \\ \times[-K_pR_{11}(\boldsymbol{x})\nabla H_1(x_a)+K_I(\boldsymbol{x})\xi] \tag{15} \\ \dot\xi=-K_I^T(\boldsymbol{x})\nabla H_1(x_a),$$

where K_p is positive square constant gain matrix, $K_I(\boldsymbol{x})$ is a positive state dependent integral gain matrix and $\xi\in\mathbb{R}^m$ locally stabilizes the system at the origin .

Proof. We consider the Lyapunov function candidate

$$V(x,\xi)=\tfrac{1}{2}(\boldsymbol{x}^T\boldsymbol{x}+\xi^T\xi). \tag{16}$$

To ensure the closed-loop stability, we should find a controller such that the Lyapunov function derivative is negative definite except at the origin.

The closed loop augmented system is given as

$$\begin{bmatrix}\dot x_a\\ \dot x_{na}\\ \dot\xi\end{bmatrix}=\begin{bmatrix}\tilde J_{11}(\boldsymbol{x}) & J_{12}(\boldsymbol{x}) & K_I(\boldsymbol{x})\\ -J_{21}^T(\boldsymbol{x}) & \tilde J_{22}(\boldsymbol{x}) & \mathbf{0}\\ K_I^T(\boldsymbol{x}) & \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}x_a\\ x_{na}\\ \xi\end{bmatrix}+\begin{bmatrix}\Psi_1(\boldsymbol{x})\\ \Psi_2(\boldsymbol{x})\\ \mathbf{0}\end{bmatrix}, \tag{17}$$

where

$$\tilde J_{11}(\boldsymbol{x})=J_{11}(\boldsymbol{x})-(K_p+I)R_{11}(\boldsymbol{x}) \\ \tilde J_{22}(\boldsymbol{x})=J_{22}(\boldsymbol{x})-R_{22}(\boldsymbol{x}).$$

Taking the time derivative of the Lyapunov function, we have

$$\begin{aligned}\dot V=&\boldsymbol{x}^T\dot{\boldsymbol{x}}+\xi^T\dot\xi\\ =&[x_a^T\ x_{na}^T]\begin{bmatrix}\dot x_a\\ \dot x_{na}\end{bmatrix}+\xi^T\dot\xi\\ =&-(K_p+I)x_a^TR_{11}(\boldsymbol{x})x_a+x_a^TJ_{12}(\boldsymbol{x})x_{na}+x_a^TK_I(\boldsymbol{x})\xi\\ &+x_a^T\Psi_1(\boldsymbol{x})-x_{na}^TJ_{21}^T(\boldsymbol{x})x_a-x_{na}^TR_{22}(\boldsymbol{x})x_{na}\\ &+x_{na}^T\Psi_2(\boldsymbol{x})-\xi^TK_I^T(\boldsymbol{x})x_a\\ =&-(K_p+I)x_a^TR_{11}(\boldsymbol{x})x_a-x_{na}R_{22}(\boldsymbol{x})x_{na}+x_a^T\Psi_1(\boldsymbol{x})\\ &+x_{na}^T\Psi_2(\boldsymbol{x}).\end{aligned} \tag{18}$$

Using Assumption 3, we re-express the contribution of the unstructured terms. We also note that matrices $R_{11}(\boldsymbol{x})$ and $R_{22}(\boldsymbol{x})$ are positive for all $\boldsymbol{x}$ from the representation of the system. We denote their respective maximal eigenvalues by $r_{11}(\boldsymbol{x})$, and $r_{22}(\boldsymbol{x})$. We obtain the following inequality

$$\begin{aligned}\dot V\le&-(K_p+1)r_{11}\|x_a\|^2-r_{22}(\boldsymbol{x})\|x_{na}\|^2\\ &+q(\boldsymbol{x})r_{11}(\boldsymbol{x})\|x_a\|^2+q(\boldsymbol{x})r_{22}(\boldsymbol{x})\|x_{na}\|^2.\end{aligned} \tag{19}$$

Under Assumption 3, there exists a positive bound, locally, q for $q(\boldsymbol{x})$ and, since the system is stabilizable, it is possible to find controller gains to ensure that

$$\dot V\le-(K_p+1-q\cdot r_{11})\|x_a\|^2-r_{22}(1-q)\|x_{na}\|^2<0 \tag{20}$$

for all $x\in\mathcal{D}\setminus\{0\}$. The origin of the closed-loop system is thus asymptotically stable in closed-loop. This concludes the proof. ■

Remark 6. By tuning the controller gains, we can obtain an acceptable performance against model uncertainy and disturbance rejection. For nominal port-Hamiltonian system, separate analysis has been carried to show robustness (Ryalat and Laila, 2018) and disturbance rejection (Donaire and Junco, 2009; Ferguson et al., 2017) by designing dynamic controllers.

4. ILLUSTRATIVE EXAMPLE: VAN DE VUSSE REACTION SYSTEM

To illustrate the proposed construction, we consider the non-isothermal van de Vusse reaction system, considered for example in (Ramírez et al., 2009). The governing equations for the system are given by

$$\begin{aligned}\dot C_A&=-k_1(T)C_A-k_3(T)C_A^2+D(C_{A0}-C_A)\\ \dot C_B&=k_1(T)C_A-k_2(T)C_B-DC_B\\ \dot T&=-\frac{k_1(T)C_A\Delta H_1+k_2(T)C_B\Delta H_2+k_3(T)C_A^2\Delta H_3}{\rho C_p}\\ &\quad+D(T_{in}-T)+\frac{Q}{\rho C_p}.\end{aligned} \tag{21}$$

The control objective is to maximize the concentration of component B by manipulating the dilution rate and heat input to the reactor. In the present study, we fix the dilution rate and use the heat input Q as the sole manipulated variable to design a dynamic controller for stabilizing at the desired equilibrium point. In control affine form, setting $C_A = x_1$, $C_B = x_2$, and $T = x_3$, the system is given by

$$\dot{x} = \begin{bmatrix} -k_1x_1 - k_3x_1^2 + D(C_{A0} - x_1) \\ k_1x_1 - k_2x_2 - Dx_2 \\ -\frac{k_1x_1\Delta H_1 + k_2x_2\Delta H_2 + k_3x_1^2\Delta H_3}{\rho C_p} + D(T_{in} - x_3) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\rho C_p} \end{bmatrix} Q, \tag{22}$$

where $k_i(x_3) = k_{0_i} \exp(\frac{E_i}{Rx_3})$.

The numerical values of the process parameters are given in Table 1. To proceed with the design, we first test if the

Table 1. van de Vusse reaction numerical values

C_{A0}	5 (mol/l)
T_{in}	403.15 K
D	15 (1/h)
C_p	3.01 kJ/(kg K)
ρ	0.9434 (kg/l)
ΔH_1	4.20 (kJ/mol)
ΔH_2	-11.00 (kJ/mol)
ΔH_3	-41.85 (kJ/mol)
k_{10}	1.287×10^{12} l/(mol.h)
k_{20}	1.287×10^{12} l/(mol.h)
k_{30}	9.043×10^{9} l/(mol.h)
E_1/R	-9758.3 K
E_2/R	-9758.3 K
E_3/R	-8560.0 K

system is stabilizable and if the unstructured component is bounded (Assumptions 1 and 3). We can check the stabilizability for the system by obtaining the following matrix

$$S = [f(x), ad_f g(x), ad_f^2 g(x)]. \tag{23}$$

The rank of this matrix is 3 for all x, hence, the system is stabilizable.

4.1 Application of the main results

A possible representation of the van de Vusse system is given using the Hamiltonian function

$$H(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2),$$

the structured matrices

$$J(x) = \begin{bmatrix} 0 & -k_1 & \frac{k_1\Delta H_1}{\rho c_p} \\ k_1 & 0 & \frac{k_2\Delta H_2}{\rho c_p} \\ -\frac{k_1\Delta H_1}{\rho c_p} & -\frac{k_2\Delta H_2}{\rho c_p} & 0 \end{bmatrix}$$

$$R(x) = \begin{bmatrix} k_1 + D & 0 & 0 \\ 0 & k_2 + D & 0 \\ 0 & 0 & D \end{bmatrix}$$

leaving the unstructured part of the dynamics be expressed by

$$\Psi(x) = \begin{bmatrix} -k_3x_1^2 + k_1x_2 - \frac{k_1x_3\Delta H_1}{\rho c_p} + DC_{A0} \\ \frac{-k_2x_3\Delta H_1}{\rho c_p} \\ -\frac{k_3x_1^2\Delta H_3}{\rho c_p} + DT_0 \end{bmatrix}.$$

To test Assumption 3, we need to show that there is a positive q meeting the following inequality on a local domain excluding the equilibrium point:

$$\|\Psi(x)\| < q\|R(x)\nabla H(x)\|. \tag{24}$$

Due to space limitation, detailed calculations are omitted. However, one can show that the above inequality holds with $q = 1$, to be used in the design, for all $x \in \mathcal{D}\backslash\{x^*\}$.

In this example, only the temperature dynamic is actuated. Considering the proposed quadratic Hamiltonian function for the system and controller and the reference trajectory to be a static (admissible) equilibrium points x^*, the augmented error dynamic is described by

$$\begin{bmatrix} \dot{x}_1 - \dot{x}_1^* \\ \dot{x}_2 - \dot{x}_2^* \\ \dot{x}_3 - \dot{x}_3^* \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} -k_1^* - D & -k_1^* & \frac{k_1^*\Delta H_1}{\rho c_p} & 0 \\ k_1^* & -k_2^* - D & \frac{k_2^*\Delta H_2}{\rho c_p} & 0 \\ -\frac{k_1^*\Delta H_1}{\rho c_p} & -\frac{k_2^*\Delta H_2}{\rho c_p} & -D & \frac{k_I}{\rho cp} \\ 0 & 0 & -\frac{k_I}{\rho cp} & 0 \end{bmatrix} \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \\ x_3 - x_3^* \\ \xi \end{bmatrix}$$
$$+ \begin{bmatrix} k_1 - k_1^* & k_1^* - k_1 & a & 0 \\ k_1 - k_1^* & k_2 - k_2^* & b & 0 \\ -a & -b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \xi \end{bmatrix}$$
$$+ \begin{bmatrix} \Psi_1(x) - \Psi_1(x^*) \\ \Psi_2(x) - \Psi_2(x^*) \\ \Psi_3(x) - \Psi_3(x^*) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\rho C_p} \\ 0 \end{bmatrix} \alpha(x, x^*) - \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\rho C_p} \\ 0 \end{bmatrix} u^*,$$

where $a = \frac{\Delta H_1}{\rho c_p}(k_1 - k_1^*)$, $b = \frac{\Delta H_2}{\rho c_p}(k_2 - k_2^*)$ and k_i^* is the reaction coefficient evaluated at x_3^*.

We need to find a proper gain for the proportional part of controller. As mentioned in the proof of Proposition 4, this can be done through calculation of $q < q(x)$. Then, we select a gain in the order of calculated q. From equation (11) and Proposition 4, a dynamic controller for this system is defined by

$$\begin{aligned} u(x, x^*, \xi) &= (k_1 - k_1^*)\Delta H_1 x_1 + (k_2 - k_2^*)\Delta H_2 x_2 \\ &\quad - \rho C_p D(x_3 - x_3^*) + u^* + k_I\xi \\ \dot{\xi} &= -\frac{k_I}{\rho C_p}(x_3 - x_3^*). \end{aligned} \tag{25}$$

4.2 Simulation results and discussion

For numerical simulations, we set $K_I = 40$ as integral gain. This value is selected such that the fraction in controller (25) holds a reasonable value. Initially, the reactor is operating at a steady-state of $x^* = (1.18, 0.87, 403.96)$, which corresponds to $u^* = -500KJ/hr$. We seek to reach the optimum steady-state, where C_B is maximized, that is $x^* = (2.02, 1.07, 389)$ which corresponds to $u^* = -1100KJ/hr$ (Nguyen et al., 2018). To show a more illustrative response of the controller, following the presentation in (Ramírez et al., 2009), we present a sequence of step changes before reaching to the desired equilibrium. Fig. 1 shows that the system trajectories converge to the desired steady state. The dynamics of control input is given in Fig. 2.

5. CONCLUSION

We considered the problem of state feedback stabilizing controller design for a class of nonlinear systems described by a structured part, a port-Hamiltonian system, composed with an unstructured component. A state-feedback control design, based on the structured part, is presented. Under mild conditions on the unstructured dynamics related to the natural dissipation of the port-Hamiltonian system, it is shown that the overall closed-loop system renders a desired equilibrium point of the dynamics asymptotically stable. Using the proposed approach, a simpler controller can be designed by exploiting the port-Hamiltonian structure without solving matching equations. The results

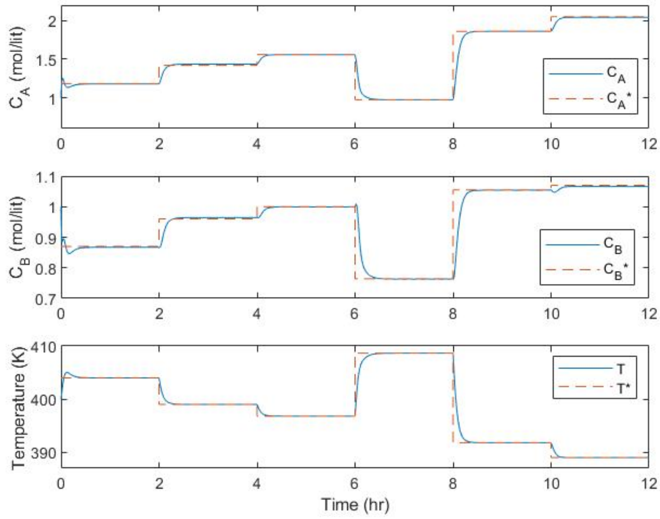

Fig. 1. Closed-loop trajectory response for van de Vusse system

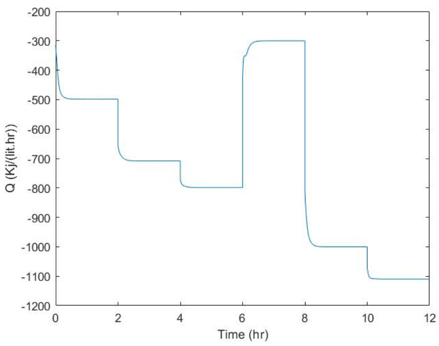

Fig. 2. Dynamic controller output value for van de Vusse system

are validated by simulations of the non-isothermal van de Vusse system. Current investigations focus on the problem of output feedback design for the proposed class of systems.

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