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Forest Fire Modeling Using Cellular Automata and Percolation Threshold Analysis

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Abstract

Multi-state probabilistic cellular automata are developed for forest fire modeling. We propose a forest fire dynamics model considering intensities of fires as multiple states and having the probability that fire spread depends on the states of the neighboring cells. Furthermore, the idea of percolation threshold is introduced to characterize the strength of the fire propagation. Specifically, we propose a new method to derive the critical probability, below which forest fires do not expand to infinitely large area, using percolation theory and mean-field approximation.

1. Introduction

Forest fires occur in various regions in the world and yield serious harm. In global, hundred thousands of forest fires emerge, and more than 20 million hectares of forests are charred by spreading fires. In order to minimize damages caused by the forest fires, various kinds of researches have been conducted such as forest monitoring or fire discovering by using multiple aerial vehicles, forest fire modeling, and so on.

Accurate forest fire modeling to expect fire spreading is one of the most important tasks to fight forest fires. In these days many models to express forest fire spreading are proposed with several approaches (see [1, 2] as overview). Specifically, in [3, 4] forest fire models using Huygens principle approaches are proposed. By adopting Huygens principle where fires spread in a short time Δt in a circle pattern from each point on the border of fires at time t, the spread of fires are formulated. In [5, 6], forest fire models using reaction-diffusion equations are proposed. In addition, in [7–9], forest fire modeling with cellular automata are proposed. In the cellular automata models, forest is separated into small lattices (celled cells) and each cell takes a state. Fire spread is expressed by giving each cell deterministic or probabilistic dynamics depending on neighboring cells, such that if neighboring cells of a cell A make the cell A ignite.

Some researches propose percolation models for modeling forest fires. In [10–12], the critical probability that fire spread infinitely is discussed when trees exist at each site of the forest with a given probability. Percolation model is used not only for modeling the spread of forest fires but also for water penetration and disease spread. Specifically, characterization of the critical probability that a medium spread infinitely is one of the most important issues.

In this paper, we consider probabilistic cellular automata (PCA) as a forest fire model. Using PCA and regarding the spread of fire as a probabilistic phenomenon, we formulate the model of forest fires. Specifically, as we express intensities of fires with n states that each cell can take, we propose more practical forest fire model. The new model is discussed in Section 2. Next, in regard to the proposed model, we derive the critical probability that fire spreads infinitely by using mean-field approximation and percolation theory. Percolation theory and the derivation of the critical probability are discussed in Sections 3 and 4, respectively. In Section 5, we show numerical results to verify the effectiveness of our method.

2. Problem Setting

2.1. Representation of Probabilistic Cellular Automata

In this section we introduce 2-dimensional 2-state probabilistic cellular automata (PCA), which are used for forest fire modeling. In the 2-dimensional PCA, infinite square lattice is considered and each lattice is called a cell. For forest fire modeling, the square lattice corresponds to forest that sweeps away infinitely.

We consider two states 0 and 1 that each cell takes. For the case of forest fires, state 0 represents green (no fires) and state 1 represents the burning state. Furthermore, the state of each cell changes according to the states of the neighboring cells. In particular, define 4 adjacent cells of the cell A as the neighbors of A. Cells with state 1 make the neighborsf states be 1 with probability p. Concerning forest fire, p is equivalent to the probability that fire spreads.

PCA is used to model systems in which the propagation of media is difficult to analytically represent (e.g., epidemic behaviour or chemical penetration). Specifically, by employing PCA overall state transitions of sys-

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tems with respect to time can be seen easily.

2.2. Multi-State Probabilistic Cellular Automata

In this section we propose 2-dimensional multi-state PCA as a model of forest fire, extending the 2-dimensional 2-state PCA introduced in the previous section. Now, consider the 2-dimensional square lattice and denote the location of each cell as $x \in \mathbb{R}^2$ with the continuous-time index t.

In the 2-dimensional 2-state PCA, the probability p that fire spreads to the neighbor cell is constant. In contrast, the probability that fire spreads in reality differs depending on the fire intensities. Thus, we consider a multi-state cell automaton so that each cell takes one of the values in $S = \{0, 1, 2, \dots, n\}$ depending on the fire intensities.

Cells with state 0 are assumed yet to be burning, and the states with state n are assumed to be completely burnt out, otherwise burning. That is, cells with state $i \in S_{\text{burn}} \triangleq \{1, 2, \dots, n-1\}$) are burning cells. Here we describe the state of the cell x at time t as $s(x,t) \in S$.

Next, we focus on the probability that fire spreads from the cell with $s(x,t) = i$ into unburned cells in a short time dt. Since the probability that fire spreads into unburned cells depends on the fire intensity, we define the probability that fire moves from the cell x into its neighbors at time t as

$$p(s(x,t)) = \begin{cases} 0, & \text{if } s(x,t) = 0 \text{ or } n, \\ p_i \mathrm{d}t, & \text{if } s(x,t) = i,\ i \in S_{\text{burn}}, \end{cases} \qquad (1)$$

where $p_i \in [0,1]$ is a constant.

While the cell with $s(x,t) = 0$ is unburned, the probability that fire spreads is also 0. Equivalently, since the cell x with $s(x,t) = n$ is completely burnt, the probability that fire spreads is also 0. Furthermore, the cell with $s(x,t) = 0$ becomes $s(x, t + \mathrm{d}t) = 1$ when fire is transferred. That is, $s(x,t) = 1$ is assumed to be the early stage of burning states. Thus, it follows that

$$P\{s(x,t+dt) = 1 | s(x,t) = 0\} \;=\; \sum_{i=0}^{n} p(i) n_i(x,t), \quad (2)$$

$$P\{s(x,t+dt) = i | s(x,t) = 0\} \;=\; 0, \quad i \in S\{0,1\}, \quad (3)$$

where $P\{s(x,t+dt) = j | s(x,t) = i\}$ represents the conditional probability that the state of the cell with $s(x,t) = i$ changes to j in a short time dt and $n_i(x,t)$ represents the number of the neighbors of the cell x, the states of which are i at time t. The cells with $s(x,t) = i$ have effects only on the cells with the state of 0. Furthermore, once a cell begins to burn, the fire intensity of the cell fluctuates without any effects from the neighbor cells. Thus, we describe the probability that the state of the cell with state $i \in S_{\text{burn}}$) changes to j $(\neq i)$ in a short time dt as

$$P\{s(x,t+dt) = j | s(x,t) = i\} \;=\; a_{ij} \mathrm{d}t, \qquad i \in S_{\text{burn}}, j \in S \backslash \{0, i\}, \quad (4)$$

where $a_{ij} \in [0,1]$ is a constant.

Finally, as the state of the cell with s(x, t) = n would be unchanged anymore, the following equation holds

$$P\{s(x,t+dt) = i | s(x,t) = n\} = 0, \quad i \in S \backslash \{n\}. \quad (5)$$

In Sections 3 and 4 below, the critical probability of the proposed multi-state PCA that fire spreads infinitely is discussed.

3. Percolation Theory

Percolation theory is the theory that determines if a medium that emerges at a certain point, such as water or disease, spreads infinitely or not [13]. Percolation is classified broadly into two types: the site percolation where intermediates exist at each site of a given area with probability q and the bond percolation where each couple of neighboring sites connects with probability q. Figures 3.1, 3.2 show the 2-dimensional square lattice site percolation and bond percolation, respectively, where sites (small circles in these figures) correspond to cells of PCA. Furthermore, the connected set of sites is called a cluster, which corresponds to the set of sites surrounded by the dotted lines in the figures.

In percolation theory, the critical probability that a medium spreads infinitely is one of the main focuses and is defined according to the following. First, we consider the cluster C_0 that contains the origin. Then, for a given probability q we define $\theta(q)$ as

$$\theta(q) = P_q(\|C_0\| = \infty), \quad (6)$$

where $\|C_0\|$ denotes the number of the sites contained in C_0. The function $\theta(q)$ given by (6) describes the probability that the cluster having infinite number of sites exists. In other words, $\theta(q)$ represents the probability that a medium, such as water or disease, spreads infinitely. Furthermore, by using $\theta(q)$ we define the critical probability that a medium spreads infinitely as

$$q_{\text{c}} = \inf\{q \in [0,1] : \theta(q) > 0\}. \quad (7)$$

The critical probability q_{c} represents the probability q above which there is a possibility of existence of the cluster having infinite amount of sites. That is, a medium does not spread infinitely with probability q that satisfies $q < q_{\text{c}}$.

In the 2-dimensional square lattice bond percolation case, the critical probability is proven to be $q_{\text{c}} = 0.5$ analytically. Moreover, in the case of site percolation,

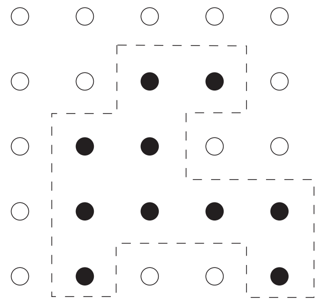

Figure 3.1: Square lattice site percolation

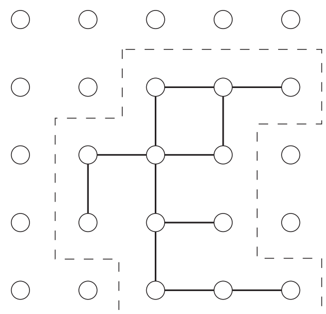

Figure 3.2: Square lattice bond percolation

q_c is known to be 0.592746 approximately [14]. In the next section, we derive the critical probability of the multi-state PCA that we propose.

4. Critical Probability for Spreading the Fire

In comparison to percolation theory that deals with problems which have static properties, [15] applied percolation theory to dynamical systems. In [15] the dynamical epidemic problem, where people having diseases recover after a certain period of time, is replaced by a 3-dimensional percolation where a temporal dimension is added to two spatial dimensions. Since the percolation that [15] discussed has a directivity with respect to temporal axis, it is called directed percolation.

Although some methods to derive the critical probability of PCA replacing to the 3-dimensional directed percolation are proposed, two problems would emerge if we considered to apply those methods to the multi-state PCA. First, it is known to be difficult to derive the critical probability of directed percolation analytically. Second, due to the consideration of multiple states it is impossible that we convert our proposed PCA into the 3-dimensional percolation. In this paper, we derive the critical probability using mean-field approximation and the 2-dimensional percolation theory.

4.1. Mean-Field Approximation to PCA

In this section, we introduce the notion of mean-field approximation that is used in this paper. Mean-field approximation is used to derive the dynamics of the proportion of cells with a specific state, such as burning or disease, with respect to the systems expressed by the PCA such as forest fire or epidemic propagation. For example, in [16,17] useful methods regarding to cure to protect epidemic propagation is proposed using mean-field approximation.

Applying procedure of mean-field approximation to PCA and getting the dynamics of the proportion of cells with each state, it is possible to derive time responses with any given initial conditions.

Now, we define a binary variable $\delta_i(x,t)$ as

$$\delta_i(x,t) = \begin{cases} 1, & \text{if } s(x,t) = i, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

where $i \in S$. And we also define $\langle f(x) \rangle$ as the average value of the function $f(x)$ of the overall cells x. Then, the proportion of the cells with state i is described by $\langle \delta_i(x,t) \rangle$. In other words, the number of cells with state i included in k cells chosen arbitrarily is written by $k\langle \delta_i(x,t) \rangle$. Equivalently, the proportion of couples of the cells with state i and the cell with state j is described by $\langle \delta_i(x,t) \rangle \langle \delta_j(x_k^{\mathrm{nb}},t) \rangle$ where x_k^{nb}, $k = 1,2,3,4$, represent the 4 neighbor cells of the cell x. Moreover, as we suppose that the states of arbitrary two cells are independent of each other, it follows that

$$\sum_{k=1}^{4} \langle \delta_0(x,t) \delta_i(x_k^{\mathrm{nb}},t) \rangle \simeq 4 \langle \delta_0(x,t) \rangle \langle \delta_i(x_k,t) \rangle \tag{9}$$

where the coefficient 4 of the right-hand side stems from the fact that the number of the neighbors of the cell x is equal to 4. The above approximation is called mean-field approximation [16].

Now, we derive the difference equation with respect to the proportion $\langle \delta_i(x,t) \rangle$ of the cells in state i in a short time dt. The difference equation of $\langle \delta_i(x,t) \rangle$ is given by

$$\begin{aligned} &\langle \delta_i(x,t+\mathrm{d}t) \rangle - \langle \delta_i(x,t) \rangle \\ &= \sum_{j=0, j\neq i}^{n} \langle \delta_j(x,t) P\{s(x,t+\mathrm{d}t) = i | s(x,t) = j\} \rangle \\ &\quad - \sum_{j=0, j\neq i}^{n} \langle \delta_i(x,t) P\{s(x,t+\mathrm{d}t) = j | s(x,t) = i\} \rangle, \end{aligned} \tag{10}$$

where the first term of right-hand side denotes the increasing rate of the cells with state i and the second term denotes the decreasing rate of the cells with state

i. Substituting (2)–(5) into (10), it follows that

$$\begin{aligned}
&\langle \delta_1(x, t+\mathrm{d}t)\rangle - \langle \delta_1(x,t)\rangle \\
&\quad = \langle \delta_0(x,t) \sum_{i=0}^{n} p_i n_i(x,t)\rangle \\
&\qquad + \sum_{j=2}^{n} (a_{j1}\langle \delta_j(x,t) - a_{1j}\langle \delta_1(x,t)\rangle)\mathrm{d}t, \qquad (11)\\
&\langle \delta_i(x, t+\mathrm{d}t)\rangle - \langle \delta_i(x,t)\rangle \\
&\quad = \sum_{j=1, j\neq i}^{n} (a_{ji}\langle \delta_j(x,t) - a_{ij}\langle \delta_i(x,t)\rangle)\mathrm{d}t, \\
&\qquad\qquad i \in S_{\mathrm{burn}}\backslash\{1\}, \qquad (12)\\
&\langle \delta_n(x, t+\mathrm{d}t)\rangle - \langle \delta_n(x,t)\rangle \\
&\quad = \sum_{j=1}^{n-1} a_{jn}\langle \delta_j(x,t)\rangle \mathrm{d}t, \qquad (13)
\end{aligned}$$

We omit the difference equation with respect to $\langle \delta_0(x,t)\rangle$ since $\langle \delta_0(x,t)\rangle$ is determined to satisfy the condition given by

$$\sum_{i\in S} \langle \delta_i(x,t)\rangle = 1. \qquad (14)$$

While the function $n_i(x,t)$ contained in the first term of the right-hand side of (11) represents the number of the neighbors with state i of the cell x, the function is written as

$$n_i(x,k) = \sum_{k=1}^{4} \delta_i(x_k^{\mathrm{nb}}, t), \qquad (15)$$

and the first term of right-hand side of (11) is given

$$\begin{aligned}
&\sum_{i=0}^{n} p_i \langle \delta_0(x,t) n_i(x,t)\rangle \mathrm{d}t \\
&\quad = \sum_{i=0}^{n} p_i \langle \delta_0(x,t) \sum_{k=1}^{4} \delta_i(x_k^{\mathrm{nb}}, t)\rangle \mathrm{d}t \\
&\quad = \sum_{i=0}^{n} \sum_{k=1}^{4} p_i \langle \delta_0(x,t) \delta_i(x_k^{\mathrm{nb}}, t)\rangle \mathrm{d}t. \qquad (16)
\end{aligned}$$

Furthermore, it follows from (9) and (16) that (11) becomes

$$\begin{aligned}
&\langle \delta_1(x, t+\mathrm{d}t)\rangle - \langle \delta_1(x,t)\rangle \\
&\quad = \sum_{i=0}^{n} 4p_i \langle \delta_0(x,t)\rangle \langle \delta_i(x,t)\rangle \mathrm{d}t \\
&\qquad + \sum_{j=2}^{n} (a_{j1}\langle \delta_j(x,t) - a_{1j}\langle \delta_1(x,t)\rangle)\mathrm{d}t. \qquad (17)
\end{aligned}$$

Now we define $\langle \delta_i(x,t)\rangle = y_i(t)$ since $\langle \delta_i(x,t)\rangle$ is a function only of t. In this case, it follows from (12), (13), and (17) that

$$\frac{\mathrm{d}y_1(t)}{\mathrm{d}t} = \sum_{i=0}^{n} 4p_i y_0(t) y_i(t) + \sum_{j=2}^{n} (a_{j1} y_j(t) - a_{1j} y_1(t)), \qquad (18)$$

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} = \sum_{j=1, j\neq i}^{n} (a_{ji} y_j(t) - a_{ij} y_i(t))\mathrm{d}t, \quad i \in S_{\mathrm{burn}}\backslash\{1\}, \qquad (19)$$

$$\frac{\mathrm{d}y_n(t)}{\mathrm{d}t} = \sum_{j=1}^{n-1} a_{jn} y_j(t). \qquad (20)$$

The equations (18)–(20) describe the dynamics of the proportions of cells with each state.

4.2. Derivation of Critical Probability

In this section, we derive the critical probability of the 3-state PCA with $n = 3$ as the simplest case. By using (14) and (18)–(20), the state equations in regards to 3-state PCA are given by

$$\begin{aligned}
y_1(t) &= 4p_1 y_1(t)(1 - y_1(t) - y_2(t)) - a_{12} y_1(t), \qquad (21)\\
y_2(t) &= a_{12} y_1(t), \qquad (22)\\
y_0(t) &+ y_1(t) + y_2(t) = 1. \qquad (23)
\end{aligned}$$

The state variables $y_0(t)$, $y_1(t)$, and $y_2(t)$ represent the proportions of the unburned cells, the burning cells, and the burnt cells, respectively, at time t. The critical probability of 3-state PCA is defined as the probability that fire spreads infinitely for each a_{12}, which corresponds to the probability of the state transition from the burning cells to the burnt cells.

Now, let $y(t) \triangleq [y_1(t), y_2(t)]^{\mathrm{T}}$. Figure 4.1 shows the trajectories of the system given by (21)–(23) with the initial condition $y(0) = [0.0001, 0]^{\mathrm{T}}$ on y_1-y_2 space, where $a_{12} = 0.5$ and $p_1 = 0.2, 0.4, 0.8$, respectively. As shown in the figure, the trajectories started from the neighborhood of $[y_1, y_2]^{\mathrm{T}} = [0, 0]^{\mathrm{T}}$ converge to points on y_2 axis. That is, the number of burning cells eventually goes to 0.

The equilibrium point y^* of the system (21)–(23) is derived as

$$y^* = \begin{bmatrix} 1-c \\ 0 \\ c \end{bmatrix}, \qquad (24)$$

where $c \in [0, 1]$ is a constant depending on the initial condition. Now, let $\tilde{y} \triangleq y - [y_1^*, y_2^*]$ and approximate (21) and (22) linearly around the equilibrium point, then it follows that

$$\dot{\tilde{y}}(t) = \begin{bmatrix} 4p_1(1-c) - a_{12} & 0 \\ a_{12} & 0 \end{bmatrix} \tilde{y}(t) \triangleq \tilde{A}\tilde{y}(t). \qquad (25)$$

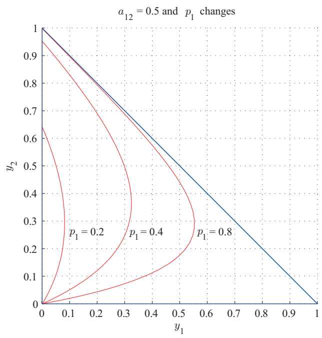

Figure 4.1: Several state trajectories on y_1-y_2 space

Note that the eigenvalues of $\tilde{A}$ are $\lambda_1 = 0$ and $\lambda_2 = 4p_1(1-c) - a_{12}$. Since one of the eigenvalues of $\tilde{A}$ is on the imaginary axis, stability of the system (21)–(23) is analyzed by the center manifold theory [18]. If λ_2 is negative, that is

$$c > 1 - \frac{a_{12}}{4p_1}, \tag{26}$$

the hyperplane given as the center manifold exists in the neighborhood of y^*, and the system trajectories (21)–(23) with any initial states near the equilibrium point converge to the center manifold. Consequently, the equilibrium point that satisfies (26) is proven to be stable.

As shown in Figure 4.1, the trajectories starting from the neighborhood of $[0,0]^{\mathrm{T}}$, which is the unstable equilibrium point, reach the stable equilibrium points given by (26). Thus, the system trajectories starting from the neighborhood of $[0,0]^{\mathrm{T}}$ converge to a stable equilibrium point on y_2-axis if $[0,0]^{\mathrm{T}}$ is the unstable equilibrium point.

By using the value of the critical probability q_{c} of the square lattice site percolation and comparing it to $y_2(t)$ as the proportion of burnt cells, we can determine whether the probability that fire spreads infinitely exists. Furthermore, in the case of square lattice site percolation, the critical probability q_{c} approximately equals to 0.592746. Thus, if the converging point with respect to a given initial condition

$$\lim_{t\to\infty} y(t) = y^{\mathrm{inf}} = \left[\ y_1^{\mathrm{inf}}, y_2^{\mathrm{inf}}\ \right], \tag{27}$$

is precisely known, then the critical probability can be derived to be $y_2^{\mathrm{inf}} = 0.592746$.

In what follows, we calculate the value of y^{inf}. By using (21) and (22), it follows that

$$\frac{\mathrm{d}y_1}{\mathrm{d}y_2} = \frac{4p_1}{a_{12}}(1 - y_1 - y_2) - 1, \tag{28}$$

so that

$$\alpha y_1 + \frac{\mathrm{d}y_1}{\mathrm{d}y_2} = -\alpha y_2 + (\alpha - 1), \tag{29}$$

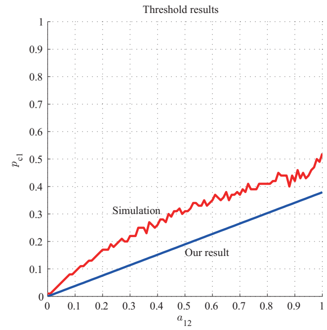

Figure 5.1: Comparison of the analytical and numerical results

where $\alpha \triangleq 4p_1/a_{12}$. Note that (29) is an inhomogeneous linear differential equation so that the solution with the initial condition $y = [0,0]^{\mathrm{T}}$ is given by

$$y_1 = -\exp(-\alpha y_2) - y_2 + 1. \tag{30}$$

With $y_1 = 0$, it follows that

$$y_2 = \frac{1}{\alpha}\mathrm{lambertw}(-\alpha e^{-\alpha}) + 1, \tag{31}$$

where $w = \mathrm{lambertw}(x)$ represents the solution of the equation $we^w = x$, is the solution for y_2. Note that letting $\beta \triangleq \mathrm{lambertw}(-\alpha e^{-\alpha})$ in (31), there are two solutions for β; the trivial solution $\beta = -\alpha$ and the other nontrivial solution describing the value of the converging point with respect to y_2 of the trajectory which starts from the neighborhood of $y = [0,0]^{\mathrm{T}}$, that is, y_2^{inf}. If the value of y_2^{inf} is below the critical probability of square lattice site percolation $q_{\mathrm{c}} = 0.592746$, then the probability that fire spreads is 0. Therefore, by denoting $\alpha_{\mathrm{c}} = p_{\mathrm{c}1}/a_{12}$ where $p_{\mathrm{c}1}$ is the critical probability of the 3-state PCA, α_{c} satisfies

$$\frac{1}{\alpha_{\mathrm{c}}}\mathrm{lambertw}(-\alpha_{\mathrm{c}} e^{-\alpha_{\mathrm{c}}}) = 0.592746. \tag{32}$$

By solving (32), $\alpha_{\mathrm{c}} = 1.51552$ is obtained so that consequently

$$p_{\mathrm{c}1} = 0.37888 a_{12}, \tag{33}$$

is the critical probability for given a_{12}.

5. Numerical Results

In this section, we compare the values of the critical probability that we derived to the numerical results. In the simulation, we suppose 100×100 square lattice and the initial condition where the center cell of the lattice is burning (state 1). Figure 5.1 shows the critical probabilities $p_{\mathrm{c}1}$ as a_{12} moves from 0 to 1, where the horizontal axis represents a_{12} and the vertical axis represents $p_{\mathrm{c}1}$.

Comparing our analytical result to the numerical result, the value of $p_{\mathrm{c}1}$ seems to be similar when a_{12} is below 0.1. In contrast, when a_{12} is above 0.1, there is an discrepancy by around 0.1 between the values of p_{c1}. Furthermore, as shown in the figure the analytical result is more conservative. These differences may be caused by the approximation process of the model using mean-field approximation.

6. Conclusion and Future Extensions

In this paper forest fire was modeled by using PCA. Specifically, multi-state PCA is proposed to express forest fire more accurately, where intensities of fires are expressed by multiple states and probabilities that fire spreads depend on states of cells. Furthermore, we derived the critical probability of PCA. By applying mean-field approximation to the proposed PCA and by using the critical probability analysis for square lattice site percolation $q_c = 0.592746$, we derived the critical probability of 3-state PCA as the easiest case of multi-state PCA. Finally, we compared the critical probability that we derived to the numerical results to verify the effectiveness of our result.

For the future extensions, new field approximation methods are needed to be introduced in order to approximate PCA more accurately and to improve the accuracy of the critical probability. Furthermore, new approaches or extensions of our approach are needed to derive the critical probability of multi-state PCA. Although we derived the critical probability of 3-state PCA in this paper, we have to derive the critical probability of PCA which has n states as general solutions.

References

[1] E. Paster, L. Zarate, E. Planas, and J. Arnaldos, "Mathematical models and calculation systems for the study of wildland fire behaviour," *Progress in Energy and Combustion Science*, vol. 29, pp. 139–153, 2003.

[2] A. L. Sullivan, "A review of wildland fire spread modeling, 1990–present 3: Mathematical analogues and simulation models," *Int. J. Wildland Fire*, vol. 18, pp. 387–403, 2008.

[3] M. A. Finney, "FARSITE: Fire area simulator – Model development and evaluation," U.S. Dep. Agri., Tech. Rep. RMRS-RP-4, 2004.

[4] J. Glasa and L. Halada, "On elliptical model for forest fire spread modeling and simulation," *Mathematics and Computers in Simulation*, vol. 21, pp. 76–88, 2008.

[5] S. D. Watt, A. J. Roberts, and R. O. Weber, "Dimensional reduction of a bushfire model," *Math. Comp. Mod.*, vol. 21, pp. 79–83, 1995.

[6] V. Mendez and J. E. Llebot, "Hyperbolic reaction-diffusion equations for a forest fire model," *Phy. Rev. E*, vol. 56, pp. 6557–6563, 1997.

[7] I. Karafyllidis, "Design of a dedicated parallel processor for the prediction of forest fire spreading using cellular automata and genetic algorithms," *Engineering Applications of Artificial Intelligence*, vol. 17, pp. 19–36, 2004.

[8] A. H. Encinas, L. H. Encinas, S. H. White, A. M. Rey, and G. R. Sanchez, "Simulation of forest fire fronts using cellular automata," *Adv. Eng. Sof.*, vol. 38, pp. 372–378, 2007.

[9] L. H. Encinas, S. H. White, A. M. Rey, and G. R. Sanchez, "Modeling forest fire spread using hexagonal cellular automata," *Applied Mathematical Modeling*, vol. 31, pp. 1213–1227, 2007.

[10] W. Niessen and A. Blumen, "Dynamics of forest fires as a directed percolation model," *J. Phys. A: Math. Gen.*, vol. 19, pp. L289–L293, 1986.

[11] M. J. Kearney, "Compact directed percolation with modified boundary rules: A forest fire model," *J. Phys. A: Math. Gen.*, vol. 35, pp. L421–L425, 2002.

[12] B. Porterie, N. Zekri, J. P. Clerc, and J. C. Loraud, "Modeling forest fire spread and spotting process with small world networks," *Combustion and Flame*, vol. 149, pp. 63–78, 2007.

[13] D. Stauffer and A. Aharony, *Introduction to Percolation Theory.* CRC Preess, 1994.

[14] H. Kesten, "The critical probability of bond percolation on the square lattice equals 1/2," *Commun. Math. Phys.*, vol. 74, pp. 41–59, 1980.

[15] S. R. Broadbent and J. M. Hammersley, "Percolation processes. I. Crystals and mazes," *Proc. Camb. Philos. Soc.*, vol. 53, pp. 629–641, 1957.

[16] J. A. Filipe and G. J. Gibson, "Studying and approximating spatio-temporal models for epidemic spread and control," *Phil. Trans. R. Soc. Lond. B*, vol. 353, pp. 2153–2162, 1998.

[17] A. Messer and H. Hinrichsen, "Crossover from directed percolation to mean field behavior in the diffusive contact process," *J. Stat. Mech.*, vol. P04024, 2008.

[18] H. K. Khalil, *Nonlinear Systems*, 2nd ed. Upper Saddle River, NJ: Prentice-Hall, 1996.