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Adaptive Tracking Control of an Underactuated Aerial Vehicle

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Abstract— In this paper, adaptive tracking control of an underactuated quadrotor is addressed. Position and yaw trajectory tracking is designed using state feedback control system and an integrator backstepping approach is applied to this coupled and cascaded dynamic system. The control design is further complicated by considering the parametric uncertainty of the dynamic modeling of the quadrotor aerial-robot vehicle. Projection-based adaptive control schemes are then designed to estimate the unknown parameters. Lyapunov-type stability analysis and numerical simulation results which yields a bounded tracking result are shown to demonstrate the initial validity of the proposed controllers.

I. INTRODUCTION

This paper emphasizes on the control of an underactuated quadrotor to obtain the position tracking about X-, Y-, and Z-axes and also yaw angle tracking along the trajectories in the presence of parametric uncertainty. The translational dynamic model for motion is cascaded and coupled with the rotational dynamics, which causes hard to control the system. To obtain both control objectives simultaneously, a well-known backstepping approach is utilized.

Many researchers (e.g., [2],[10],[14]) have proposed a variety of control solutions for the underactuated quadrotor system. In [15], the authors presented a feedback controller in the underactuated system in the presence of uncertainty. The work in [3] also presented the results of two model-based control techniques applied to an underactued quadrotor for hoveing and vertical takeoff and landing (VTOL) dynamic model. Of particular note, the system dynamics include nonlinearities in the aerodynamic forces. In [15], the authors use feedback linearization to explicitly control the roll, pitch, and yaw angles and the height of a quadrotor vehicle. Of special significance in this work is that the control compensates for wind affects acting on the underactuated quadrotor.

An important property of the quadrotor system is that it can be modeled as coupled and cascaded from the gyroscopic effects which are typically negligible in a hovering model, but can not be neglected during fast maneuvering or for large angular motions. A feasible control solution to a system in this form is the backstepping approach. As general background for this approach, the reader is referred to [12] where the control of cascaded dynamic is addressed. In [9], a backstepping approach to control for a specific model of a quadrotor, the X4 flyer, is presented. This work includes the dynamic complication of the aerodynamic and gyroscopic effects of the rotating blades. The work in [4] presented attitude stabilization of the quadrotor aircraft using the backstepping technique. An emerging alternative to the above sensors is to use vision systems to estimate positions or velocities, which are usually vehicle-based and used to estimate changes in scenery or may be ground-based to monitor a UAV in a fixed area. The work given in [5] and [19] are representatives of the vehicle based vision applications for landing.

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Another issue associated with quadrotor control is that the low-level control objective is often embedded at the center of high-level control objectives such as path planning, target tracking, or coordination with other crafts. The work in [8] presented a trajectory tracking controller for an underactuated small helicopter using a backstepping procedure. In [17], the paper proposed a trajectory tracking control for unmanned air vehicles with constrained velocity and heading rate via a control Lyapunov function. The authors in [1] proposed a solution of the trajectory-tracking and path-following for underactuated autonomous vehicles. The work in [18] presented control strategies for a quadrotor system capable of automatic VTOL, hovering, and obstacle avoidance using simple minimal sensing.

The paper is organized as follows. In Section II a quadrotor-helicopter modeling is presented. The property and assumptions are mentioned. In Section III, the definitions of error signals are developed, a backstepping approach are introduced in the coupled dynamic model of quadrotor. Stability analyses are considered mathematically in Section IV and an adaptive controller for updating the unknown constant parameters is suggested followed by a numerical simulation result in Section V and the concluding remarks in Section VI.

II. SYSTEM MODEL

The dynamic model of the underactuated quadrotor is in the body-fixed reference frame [13] as

$$\begin{bmatrix} mI_3 & O_{3\text{x}3} \\ O_{3\text{x}3} & J \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} -mS(\omega) & O_{3\text{x}3} \\ O_{3\text{x}3} & \text{-}S(\omega)Jw \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} - \begin{bmatrix} N_1(\theta_1, v, |v|) \\ N_2(\theta_2, v, |v|) \end{bmatrix} + \begin{bmatrix} G(R) \\ O_{3\text{x}1} \end{bmatrix} + \begin{bmatrix} B_1, O_{3\text{x}3} \\ O_{3\text{x}1}, B_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{1}$$

where $v(t) \in \mathbf{R}^3$ denotes the linear velocity and $\omega(t) \in \mathbf{R}^3$ represents the angular velocity. $N_1(\theta_1, v, |v|)$ are aerodynamic forces on the rigid-body and $N_2(\theta_2, v, |v|)$ denotes aerodynamic induced moments where $|v|$ is the norm of linear velocity $v(t)$. A gravity vector is denoted as $G(R) = mgR^T(\Theta)E_z \in \mathbf{R}^3$where and $E_z = [0, 0, 1]^T$ denotes the Z-axis unit vector in the coordinates of the inertial frame and $G(R)$ is represented in the body-fixed frame by the pre-multiplication of the direction cosine matrix $R^T(\Theta)$. The input $u_1(t) \in \mathbf{R}^1$ provides lifting force in the z-direction and $u_2(t) \in \mathbf{R}^3$ creates rotational torque along the roll, pitch, and yaw directions. $m \in \mathbf{R}^1$ is the mass of the quad-rotor, $J \in \mathbf{R}^{3\text{x}3}$ denotes a positive definite diagonal inertia matrix and $g \in \mathbf{R}^1$ denotes the gravitational acceleration due to the gravity. These are all assumed to be unknown constants including the coefficients of the aerodynamic terms. The specific form of the quad-rotor links the inputs to the dynamics via $B_1 = [0, 0, 1]^T \in \mathbf{R}^3$ and $B_2 = I_3 \in \mathbf{R}^{3\text{x}3}$ which is a 3×3 identity matrix. Additionally, $O_{3\text{x}1} \in \mathbf{R}^3$ represents a 3×1 zero vector and $O_{3\text{x}3} \in \mathbf{R}^{3\text{x}3}$ represents a 3×3 zero matrix, and $S(\cdot) \in \mathbf{R}^{3\text{x}3}$ is a general form of skew-symmetric matrix. The translational and rotational kinematic equations in the body-fixed reference frame are given by

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} R^T(\Theta) & O_{3\text{x}3} \\ O_{3\text{x}3} & T^{-1}(\Theta) \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{\Theta} \end{bmatrix} \triangleq D(R,T)\dot{x} \in \mathbf{R}^6 \tag{2}$$

where $D(R,T) \triangleq \left[R^\intercal(\Theta), O_{3\text{x}3}; O_{3\text{x}3}, T^{-1}(\Theta)\right] \in \mathbf{R}^{6\text{x}6}$ and $x \triangleq [p^\intercal, \Theta^\intercal]^\intercal \in \mathbf{R}^6$ were defined, $p(t) \in \mathbf{R}^3$ contains the position of the body-fixed reference frame relative to the inertial frame, and its derivative $\dot{p}(t) \in \mathbf{R}^3$ represents the translational velocity in the inertial frame. The Euler based rotation matrix $R(\Theta) = R_{z,\psi} \cdot R_{y,\theta} \cdot R_{x,\phi} \in SO(3)$ that translates the quantity from a body-fixed frame into inertial coordinates is calculated from the following form

$$R(\Theta) = \begin{bmatrix} c\psi c\theta & c\psi s\theta s\phi - s\psi c\phi & s\psi s\phi + c\psi c\phi s\theta \\ s\psi c\theta & c\psi c\phi + s\phi s\theta s\psi & s\theta s\psi c\phi - c\psi s\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix} \tag{3}$$

where $c\cdot = \cos(\cdot)$ and $s\cdot = \sin(\cdot)$ are used and the matrix is represented by rotating first yaw, pitch, and then roll directions when transforming from the inertial to the body frame. The body-fixed angular velocities are transformed by the matrix $T(\Theta) \in \mathbf{R}^{3\text{x}3}$ to the time derivative vector $\dot{\Theta}(t) = [\dot{\phi}, \dot{\theta}, \dot{\psi}]^T \in \mathbf{R}^3$ of Euler angles $\Theta(t) = [\phi, \theta, \psi]^T \in \mathbf{R}^3$, describing the orientation of the body-fixed frame relative to the inertial frame, denoted as [7]

$$T(\Theta) = \begin{bmatrix} T_x(\Theta) \\ T_y(\Theta) \\ T_z(\Theta) \end{bmatrix} = \begin{bmatrix} 1 & s\phi t\theta & c\phi t\theta \\ 0 & c\phi & -s\phi \\ 0 & s\phi / c\theta & c\phi / c\theta \end{bmatrix} \tag{4}$$

where $\dot{\Theta}(t) = T(\Theta)\omega$. The dynamics in (1) can be compacted and transformed into the inertial frame for subsequent control development to yield the dynamic model by substituting from (2) for $[v^\intercal, \omega^\intercal]^\intercal$ and differentiating (2) to substitute for $[\dot{v}^\intercal, \dot{\omega}^\intercal]^\intercal$ in (1) yielding

$$\bar{M}\ddot{x} = \bar{C}\dot{x} - \bar{N} - \bar{G} + \bar{B}U \tag{5}$$

where the system matrices $\bar{M}(\cdot)$, $\bar{C}(\cdot)$,$\bar{N}(\cdot)$, $\bar{G}(\cdot)$, and $\bar{B}(\cdot)$ have uncertain constant parameters, $\theta_i (i = 1, ..., p)$, and $\bar{M} \triangleq MD \in \mathbf{R}^{6\text{x}6}$ denotes the inertia matrix, $\bar{N} \triangleq [N_1^T,\ N_2^T]^T \in \mathbf{R}^6$ is an aerodynamic damping term, $\bar{G}(R) \triangleq [G_1^T,\ O_{1\text{x}3}]^T \in \mathbf{R}^6$ is a gravity term, and $\bar{B} \in \mathbf{R}^{6\text{x}4}$ represents the input matrix, $U \in \mathbf{R}^4$ denotes apparently the underactuated system where the horizontal motion is mainly due to the orientation of the total thrust, and the time derivative of $D(R,T)$ is denoted as

$$\tfrac{d}{dt}\left(D(R,T)\right) \triangleq \bar{D}(R,T,\dot{x}) \in \mathbf{R}^{6\text{x}6} \tag{6}$$

where $\frac{d}{dt}(R^\intercal) = \dot{R}^\intercal = -S(\omega)R^\intercal$, and $\frac{d}{dt}(T^{-1}(\Theta)) = \frac{\partial}{\partial\Theta}(T^{-1}(\Theta))\dot{\Theta} \in \mathbf{R}^{3\text{x}3}$ where $\frac{\partial}{\partial\Theta}(T^{-1}(\Theta)) \in \mathbf{R}^{3\text{x}3\text{x}3}$ is a tensor. Finally, $\bar{C} \triangleq CD - M\bar{D} \in \mathbf{R}^{6\text{x}6}$ is a Coriolis-centrifugal force matrix as

$$\bar{C} = \begin{bmatrix} O_{3\text{x}3} & O_{3\text{x}3} \\ O_{3\text{x}3} & -S(\omega)JT^{-1} - J\frac{\partial}{\partial\Theta}(T^{-1})\dot{\Theta} \end{bmatrix}. \tag{7}$$

The dynamic system given in (5) satisfies the following property.

P1: The unknown system parameters are upper and lower bounded to satisfy the following inequalities

$$\underline{\theta}_{ij} \leq \theta_{ij} \leq \overline{\theta}_{ij} \tag{8}$$

where θ_{ij} is the jth parameter of the ith parameter vector $\theta_i(\cdot)$.

P2: The aerodynamic forces and moments in (1) are being linearly parameterized form

$$\begin{aligned} N_1(\theta_1, v, |v|) &\equiv Y_1(v, |v|)\theta_1, \\ N_2(\theta_2, v, |v|) &\equiv Y_2(v, |v|)\theta_2 \end{aligned} \tag{9}$$

where $Y_1(v, |v|) \in \mathbf{R}^{3\text{x}3}$ and $Y_2(v, |v|) \in \mathbf{R}^{3\text{x}3}$ are known regression matrices and $\theta_1 \in \mathbf{R}^3$ and $\theta_2 \in \mathbf{R}^3$ are unknown constant parameters vectors.

The following assumption is made regarding specific components of the dynamic model.

A1: The pitch angle (θ) and the roll angle (ϕ) in $\Theta(t)$ do not close to $\pm\frac{\pi}{2}$ so that $T^{-1}(\Theta)$ in $D(R,T)$ is invertible and B_b^{-1}, $\frac{d}{dt}(B_b^{-1})$ exist (see in (42)), respectively.

A2: The desired trajectories and up to their third derivatives are all bounded; i.e., $p_d(t)$, $\dot{p}_d(t)$, $\ddot{p}_d(t)$, and $\dddot{p}_d\ (t) \in \mathcal{L}_\infty$ and $\psi_d(t)$, $\dot{\psi}_d(t)$, and $\ddot{\psi}_d(t) \in \mathcal{L}_\infty$.

III. FEEDBACK TRACKING CONTROL

The quad-rotor aerial vehicle is under-actuated, and hence, the translational position, $p(t) \in \mathbf{R}^3$, along with yaw, $\psi(t) \in \mathbf{R}^1$ are chosen to be controlled.

A. Error System Development

The position tracking error, denoted as $e_p(t)$, is defined in the body-fixed frame as the transformed difference between the inertial-frame based position, $p(t)$, and the inertial-frame based desired position, denoted as $p_d(t) \in \mathbf{R}^3$, in the manner

$$e_p \triangleq R^T(p - p_d) \in \mathbf{R}^3. \quad (10)$$

The position tracking error rate, $\dot{e}_p\ (t) \in \mathbf{R}^3$, is obtained by taking the time derivative of (10), defined as

$$\dot{e}_p = -S(\omega)e_p + v - R^T\dot{p}_d \quad (11)$$

where the definition of $e_p(t)$ in (10), $v(t) = R^\top \dot{p}$ from (2), and $\dot{R}^T = -S(\omega)R^T$ were utilized. Note that the last two terms in (11) denotes the velocity error. For subsequent adaptive control development, adding and subtracting $\frac{1}{m}R^T\dot{p}_d(t)$ yields

$$\dot{e}_p = -S(\omega)e_p + \tfrac{1}{m}e_v + \tfrac{1}{m}R^T\dot{p}_d - R^T\dot{p}_d \quad (12)$$

where the virtual translational velocity tracking error, denoted by $e_v(t) \in \mathbf{R}^3$, in (12) is defined as

$$e_v \triangleq mv - R^T\dot{p}_d. \quad (13)$$

Note that this is not real velocity error but manipulated for the control development, making the design with no estimated term especially, important in uncertain system. In addition, this term is not used in the proof of stability analysis. The final form of the position tracking error is obtained from (12) and (13) as follows

$$\dot{e}_p = -S(\omega)e_p + \tfrac{1}{m}e_v + (\tfrac{1}{m} - 1)R^T\dot{p}_d. \quad (14)$$

After taking the time derivative of $e_v(t)$ in (13), substituting the first row in (1) for $m\dot{v}(t)$, $\dot{R}^T = -S(\omega)R^T$, and then applying the definition of $e_v(t)$ in (13), we get the velocity error rate as

$$\dot{e}_v = -S(\omega)e_v + G(R) - Y_1(v,|v|)\theta_1 - R^T\ \ddot{p}_d + B_1u_1 \quad (15)$$

where P2 was used to replace $N_1(\theta_1, v, |v|)$. The yaw angle tracking error, $e_\psi(t) \in \mathbf{R}^1$, is defined as

$$e_\psi \triangleq \psi - \psi_d. \quad (16)$$

The goal in the control development will be to ensure that $e_\psi(t)$ and $e_p(t)$ are driven to small values. The yaw angle rate error is derived by taking the time derivative of (16) as follows

$$\dot{e}_\psi = \dot{\psi} - \dot{\psi}_d = T_z(\Theta)\omega - \dot{\psi}_d \in \mathbf{R}^1 \quad (17)$$

where $T_z(\Theta) \in \mathbf{R}^{1\text{x}3}$ is the third row vector of $T(\Theta)$ from (4). Note that $T_z(\Theta)\omega(t) = \dot{\psi}(t)$ in $\dot{\Theta}(t)$ where $\dot{\psi}_d(t)$ is the desired yaw anglular velocity in the body-fixed frame. In order to further develop the control design, the filtered position tracking error signal $r_p(t) \in \mathbf{R}^3$ is defined in the following manner [5]

$$r_p \triangleq e_v + \alpha e_p + \delta \quad (18)$$

where $\alpha \in \mathbf{R}^1$ is a positive constant and $\delta = [0, 0, \delta_3]^T \in \mathbf{R}^3$ is a constant design vector in which $\delta_3 \in \mathbf{R}^1$ is a scalar constant. The filtered position tracking error can be combined with the yaw tracking error to create a composite tracking error $r(t) \in \mathbf{R}^4$ in the manner

$$r = [r_p^T,\ e_\psi]^T. \quad (19)$$

The filtered tracking error dynamics can be found by first differentiating (19) to yield

$$\dot{r} = [\dot{r}_p^T,\ \ \dot{e}_\psi]^T = [\dot{e}_v^T + \alpha\ \dot{e}_p^T,\ \ \dot{e}_\psi]^T \in \mathbf{R}^4. \quad (20)$$

The filtered position tracking error rate, $\dot{r}_p(t)$, is obtained by substituting (14) and (15), and the term $S(\omega)\delta$ has been added and subtracted to facilitate introduction of $\dot{r}_p(t) \in \mathbf{R}^3$ on the right-hand side as

$$\begin{aligned}\dot{r}_p \ = \ & \alpha v - S(\omega)r_p - \alpha R^T\dot{p}_d - R^T\ \ddot{p}_d - \tfrac{e_p}{m} + \\ & [G(R) - Y_1\theta_1 + \tfrac{e_p}{m}] + [S(\omega)\delta + B_1u_1]\end{aligned} \quad (21)$$

where $\frac{e_p}{m}$ is subtracted and added for the subsequent stability analysis and $\alpha v - \alpha R^T\dot{p}_d = \frac{\alpha}{m}e_v + (\frac{\alpha}{m} - \alpha)R^T\dot{p}_d$ was used for differentiating the measurable and unknown terms. It is now a straightforward matter to substitute from (17) and (21) into (20) to yield the open-loop filtered tracking error dynamics in the following form

$$\begin{aligned}\dot{r} \ = \ & \begin{bmatrix} \alpha v - S(\omega)r_p - \alpha R^T\dot{p}_d - R^T\ \ddot{p}_d + W_1\Theta_1 - \frac{e_p}{m} \\ -\dot{\psi}_d \end{bmatrix} \\ & + \begin{bmatrix} -S(\delta) & B_1 \\ T_z(\Theta) & 0 \end{bmatrix} \begin{bmatrix} \omega \\ u_1 \end{bmatrix}\end{aligned} \quad (22)$$

where the cross product on vectors, $\omega(t)$ and δ, was used; $S(\omega)\delta = -S(\delta)\omega$. Note that δ is a bounding constant vector which is utilized to incorporate the coupled dynamics between translational and rotational dynamics via the matrix $S(\delta)\omega(t)$ in the following backstepping approach and $\dot{r}(t)$ is derived from the position error rate $\dot{e}_p(t)$, yaw angle error rate $\dot{e}_\psi\ (t)$, and the translational dynamics $m\dot{v}(t)$ from (15) where $v(t)$ is coupled with the angular velocity $\omega(t)$. In (22) the following is developed for parameter terms:

P3: The combined term, $W_1\Theta_1 \in \mathbf{R}^3$, which is

$$W_1\Theta_1 = G(R) - Y_1(v,|v|)\theta_1 + \tfrac{e_p}{m}, \quad (23)$$

satisfies a linear parameterization where $W_1 \in \mathbf{R}^{3\times 5}$ is a regression matrix and $\Theta_1 \in \mathbf{R}^5$ is a constant parameter vector.

B. Integrator Backstepping

An integrator backstepping approach [12] is applied to the system in the following manner. The equation (22) can be described a general error dynamic form as

$$\dot{r} = f_1(r) + g_1\mu \quad (24)$$

where the first row vector is substituted into $f_1(r) \in \mathbf{R}^4$, the last matrix and vector are substituted into $g_1 \in \mathbf{R}^{4\text{x}4}$ and $\mu = \begin{bmatrix} \omega^T & u_1 \end{bmatrix}^T \in \mathbf{R}^4$, respectively. Modifying the $\mu(t)$ to change the variable [11] by adding and subtracting a new control signal $\bar{u}_1(t)$ into (24) yields

$$\dot{r} = f_1(r) + g_1\bar{u}_1 + g_1(\mu - I_4\bar{u}_1). \quad (25)$$

Then, manipulating the last parenthesis term in (25) yields

$$\mu - I_4\bar{u}_1 = \begin{bmatrix} \omega - B_z\bar{u}_1 \\ u_1 - B_o\bar{u}_1 \end{bmatrix} = \begin{bmatrix} \omega - B_z\bar{u}_1 \\ 0 \end{bmatrix} \quad (26)$$

where $B_z = [I_3, O_{3x1}] \in \mathbf{R}^{3x4}$ and the actual translational control input $u_1(t)$ is designed by

$$u_1 = B_o\bar{u}_1 \text{ and } B_o = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbf{R}^{1\times4}. \quad (27)$$

Introducing an auxiliary signal $z(t) \in \mathbf{R}^3$ in order to inject the ficticious control signal $\bar{u}_1(t)$ into the rotational dynamics with $\omega(t)$ from the translational dynamics is defined as

$$z = \omega - B_z\bar{u}_1 \quad (28)$$

Thus, the open-loop error signals are obtained

$$\dot{r} = \begin{bmatrix} \alpha v - S(\omega)r_p - \alpha R^T\dot{p}_d - R^T\ddot{p}_d + W_1\Theta_1 - \frac{e_p}{m} \\ -\dot{\psi}_d \end{bmatrix} + B_b\bar{u}_1 + B_b\begin{bmatrix} z \\ 0 \end{bmatrix} \quad (29)$$

where $B_b(\cdot) \in \mathbf{R}^{4x4}$, defined as g_1 in (25), is given by

$$B_b = \begin{bmatrix} -S(\delta) & B_1 \\ T_z(\Theta) & 0 \end{bmatrix}. \quad (30)$$

Taking the time derivative of $z(t)$ in (28) and multiplying by the inertia matrix, J, yields

$$J\dot{z} = J\dot{\omega} - JB_z\dot{\bar{u}}_1. \quad (31)$$

Substituting the second equation of (1) for $J\dot{\omega}(t)$ into (31), grouping terms, and invoking P2 for the linear parameterization of $N_2(\theta_2, v, |v|)$ produces

$$J\dot{z} = -S(\omega)J\omega - Y_2(v,|v|)\theta_2 - JB_z\dot{\bar{u}}_1 + B_2u_2 \quad (32)$$

where the control input $u_2(t)$ is finally appeared and it will be desinged later to derive the closed-loop controller form while satisfying the system stability. Therefore, this error dynamics can be viewed as backstepping - $\bar{u}_1(t)$ through the integrator. Before designing the control input $u_2(t)$ in (32), the following is made for the other terms in a similar manner in P2:

P4: A linear parameterization has been developed

$$W_3\Theta_3 = -S(\omega)J\omega - Y_2(v,|v|)\theta_2 - JB_z\dot{\bar{u}}_1 \quad (33)$$

where $W_3(\cdot) \in \mathbf{R}^{3\times q}$ is a known regression matrix and $\Theta_3 \in \mathbf{R}^q$ is a constant parameter vector, in which q is the number of uncertain parameters.

Using P4, (32) is rewritten as

$$J\dot{z} = W_3(\cdot)\Theta_3 + B_2u_2. \quad (34)$$

C. Controller Formulation

The controller development in this section is based on the assumption that all the states are measurable but the parameters are unknown.

1) Translational Input Design: Based on the error signals given in (29), the control input $\bar{u}_1(t)$ can be designed based on Lyapunov-type stability anlysis [6] (see (49)) as

$$\bar{u}_1 = B_b^{-1}\left(\begin{bmatrix} \alpha R^T\dot{p}_d - \alpha v + R^T\ddot{p}_d - W_1\hat{\Theta}_1 \\ \dot{\psi}_d \end{bmatrix} - \frac{\rho_1^2 r}{\varepsilon_1} - k_r r\right) \quad (35)$$

where the right side of (35) can be substituted into U, i.e., $\bar{u}_1 \triangleq B_b^{-1}U$, the measurable signals such as desired trajectories can be directly canceled, the linearly parameterized term, $W_1\hat{\Theta}_1 \in \mathbf{R}^3$, is defined corresponding to (23) as

$$W_1\hat{\Theta}_1 = \underbrace{\left[gR^T(\Theta)E_z,\ Y_1(v,|v|),\ e_p\right]}_{W_1}\underbrace{\begin{bmatrix} \hat{m} & \hat{\theta}_1^T & \frac{1}{\hat{m}} \end{bmatrix}^T}_{\hat{\Theta}_1}, \quad (36)$$

in which $W_1 \in \mathbf{R}^{3\times5}$ is a known regression matrix and $\hat{\Theta}_1 = \in \mathbf{R}^5$ is the estimated constant parameter vector. The $\rho_1(\cdot)$ term is a non-decreasing function which will be shown later (see (51)). Finally, the closed-loop filtered tracking error dynamics for $\dot{r}(t)$ is formed by substituting (35) into (29) to yield

$$\dot{r} = -k_r r + \begin{bmatrix} W_1\tilde{\Theta}_1 - S(\omega)r_p - \frac{e_p}{m} \\ 0 \end{bmatrix} - \frac{\rho_1^2 r}{\varepsilon_1} + B_b\begin{bmatrix} z \\ 0 \end{bmatrix} \quad (37)$$

where the parameter mismatch term $\tilde{\Theta}_1(\cdot) \in \mathbf{R}^5$ is introduced as follows

$$\tilde{\Theta}_1 = \Theta_1 - \hat{\Theta}_1 \text{ and } \tilde{\Theta}_1 = \begin{bmatrix} \tilde{m}_1 & \tilde{\theta}_1^T & \tilde{m}_2 \end{bmatrix}^T, \quad (38)$$

in which $\tilde{m}_1 = m - \hat{m}$ and $\tilde{m}_2 = \frac{1}{m} - \frac{1}{\hat{m}}$.

2) Torque Input Design: The control input $u_2(t) \in \mathbf{R}^3$ is now formulated from (34), making use of (33), in the following form

$$u_2 = B_2^{-1}\left(-k_z z - \bar{B}_b^T r - W_3(p,R,v,\omega)\hat{\Theta}_3\right) \quad (39)$$

where the feedback term $z(t)$ is designed to stabilize the $z(t)$-dynamics, the transposition of $B_b(\cdot)$ is formed as

$$B_b^T = [-S(\delta)^T, T_z^T(\Theta)] = [S(\delta), T_z^T(\Theta)] \in \mathbf{R}^{3x4}, \quad (40)$$

the linear parameterization was developed as

$$W_3\hat{\Theta}_3 = -S(\omega)\hat{J}\omega - Y_2(v,|v|)\hat{\theta}_2 - \hat{J}B_z\dot{\bar{u}}_1 \in \mathbf{R}^3 \quad (41)$$

where $W_3(\cdot) \in \mathbf{R}^{3\times41}$ is the regression matrix and the estimated paramter $\hat{\Theta}_3(\cdot) \in \mathbf{R}^{41}$ is given in order to compendate Θ_3. It is clear that both $S(\omega)\hat{J}\omega(t)$ and $Y_2(v,|v|)\hat{\theta}_2$ terms can be easily linearly parameterized but the model for $\dot{\bar{u}}_1(t)$ is developedas follows: the time derivative of $\bar{u}_1(t)$ can be computed using the definition in (35) and can be represented as

$$\dot{\bar{u}}_1 \triangleq (B_b^{-1})\frac{d}{dt}U + \frac{d}{dt}(B_b^{-1})U \quad (42)$$

where $U(t)$ is the parenthetical terms on the right equation in (35) and the time derivative of $U(t)$ is calculated as

$$\frac{d}{dt}U = \begin{bmatrix} -\alpha\ \dot{v} - \dot{W}_1(v)\hat{\Theta}_1 - W_1\ \dot{\hat{\Theta}}_1 \\ 0 \end{bmatrix} - \frac{d(\rho_1^2 r)}{\varepsilon_1 dt} - k_r\ \dot{r} + \begin{bmatrix} \alpha R^T\ \ddot{p}_d - \alpha S(\omega)R^T\dot{p}_d - S(\omega)R^T\ \ddot{p}_d + R^T\ \dddot{p}_d \\ \ddot{\psi}_d \end{bmatrix} \quad (43)$$

where $\dot{r}\ (t)$ in (37) will be used, $\dot{v}\ (t)$ will be substituted from (1) including estimated parameters as $\dot{v} = -S(\omega)v + \frac{1}{m}(W_1\hat{\Theta}_1 + B_1 u_1)$, the time derivative of $W_1(\cdot)$ in (43) is defined as

$$\dot{W}_1\hat{\Theta}_1 = \dot{G}(\hat{m}, g) - \dot{Y}_1(v)\hat{\theta}_1 + \frac{\dot{e}_p}{\hat{m}}, \quad (44)$$

$\dot{\hat{\Theta}}_1$ will be given in (52), and the time derivative of $\frac{\rho_1^2 r}{\varepsilon_1}$ in (43) yields

$$\frac{d}{dt}\left(\frac{\rho_1^2 r}{\varepsilon_1}\right) = \frac{1}{\varepsilon_1}\left[2\rho_1 \cdot \frac{d(\rho_1)}{dt} \cdot \left(\frac{v_d^T \dot{v}_d}{\|v_d\|_s}\right) r + \rho_1^2 \dot{r}\right]. \quad (45)$$

Finally, (42) can now be implemented using (11), (37), $\dot{v}\ (t)$, $\dot{W}_1\hat{\Theta}_1$ with $\dot{G} = -mgS(\omega)R^T E_z$, and $\frac{d}{dt}(\frac{\rho_1^2 r}{\varepsilon_1})$ to produce parameterization as

$$\ddot{u}_1 = \underbrace{[(B_\mu^{-1})\phi_a + \frac{d(B_\mu^{-1})}{dt}\phi_b]}_{\Phi_1} + \underbrace{[(B_\mu^{-1})\phi_c + \frac{d(B_\mu^{-1})}{dt}\phi_d]}_{\Phi_2\hat{\Theta}_2} \quad (46)$$

where $\Phi_1(\cdot) \in \mathbf{R}^4$ is a signal matrix without having parameters and $\Phi_2 \in \mathbf{R}^{3\times l}$ is the regression term and $\hat{\Theta}_2 \in \mathbf{R}^l$ is a unknown parameter vector. After substituting (39) into (34) and rearranging the equation, we have then the final form for the closed-loop system as

$$J\dot{z} = -k_z z + W_3\tilde{\Theta}_3 - \bar{B}_b^T r \quad (47)$$

where $W_3\tilde{\Theta}_3$ is the regression estimation error, $\tilde{\Theta}_3(\cdot)$, is defined as $\tilde{\Theta}_3 = \Theta_3 - \hat{\Theta}_3 \in \mathbf{R}^{41}$.

IV. STABILITY ANALYSIS

Theorem 1: The control law of (35) and (39) ensure that the tracking error is semi-globally asymptotically bounded as

$$\|\eta(t)\| \to 0 \text{ as } t \to \infty.$$

provided $\alpha > \frac{m}{2\lambda_1}$ and $\varepsilon_1 < \frac{2||r||^2}{\lambda_1}$.

Proof: The non-negative functions $V(t)$ is defined as $V = \frac{1}{2}e_p^T e_p + \frac{1}{2}r^T r + \frac{1}{2}z^T J z + \frac{1}{2}\tilde{\Theta}_1^T\Gamma_1^{-1}\tilde{\Theta}_1 + \frac{1}{2}\tilde{\Theta}_3^T\Gamma_3^{-1}\tilde{\Theta}_3$. The time derivative of $V(t)$ yields

$$\dot{V} = e_p^T\dot{e}_p + r^T\dot{r} + z^T J\dot{z} - \tilde{\Theta}_1^T\Gamma_1^{-1}\ \dot{\hat{\Theta}}_1 - \tilde{\Theta}_3^T\Gamma_3^{-1}\ \dot{\hat{\Theta}}_3 \quad (48)$$

where $\dot{\tilde{\Theta}}_1\ (t) = -\ \dot{\hat{\Theta}}_1$ and $\dot{\tilde{\Theta}}_3\ (t) = -\ \dot{\hat{\Theta}}_3$ in which the parameter estimation errors are utilized by taking the time derivative of (38). After substituting (14), (29), and (34) into (48) and using $B_b[z^T, 0]^T = \bar{B}_b z$ produces

$$\dot{V} = e_p^T\left(-S(\omega)e_p + \frac{1}{m}(r_p - \alpha e_p - \delta) + (\frac{1}{m} - 1)R^T\dot{p}_d\right) + r^T\left(-k_r r + \begin{bmatrix} -S(\omega)r_p - \frac{1}{m}e_p \\ 0 \end{bmatrix} - \frac{\rho_1^2}{\varepsilon_1}r\right) + r_p^T\bar{B}_b z - z^T(k_z z + \bar{B}_b^T r) + \tilde{\Theta}_1^T\left[W_1^T r_p - \dot{\hat{\Theta}}_1\right] + \tilde{\Theta}_3^T\left[W_3^T z - \dot{\hat{\Theta}}_3\right] \quad (49)$$

where the controller inputs given in (35) and (39) were designed and then substituted into (29) and (34), respectively which result in (37) and (47), and the definition of $r_p(t)$ was used for $e_v(t)$ in the above $\dot{e}_p(t)$. Then, the following scaler terms are canceled each other $\frac{1}{m}e_p^T r_p$ and $\frac{1}{m}r_p^T e_p$, $r_p^T\bar{B}_b z$ and $z^T\bar{B}_b^T r_p$,and skew-symmetric terms are removed as $e_p^T S(\omega)e_p$ and $r_p^T S(\omega)r_p$. After rearranging, the equation (49) yields

$$\dot{V} = e_p^T[(\tfrac{1}{m} - 1)v_d - \tfrac{\delta}{m}] - \frac{\alpha e_p^T e_p}{m} - k_r r^T r - \frac{\rho_1^2 r^T r}{\varepsilon_1} - k_z z^T z + \tilde{\Theta}_1^T\left[W_1^T r_p - \dot{\hat{\Theta}}_1\right] + \tilde{\Theta}_3^T\left[W_3^T z - \dot{\hat{\Theta}}_3\right] \quad (50)$$

where $v_d = R^T\dot{p}_d$ was defined and (52) was used for the last two bracketed terms.

A3: The right bracketed term in (50) is upper bounded in the following manner

$$\tfrac{1}{m}[(1-m)v_d - \delta] \leq \rho_1(\|v_d\|_s) \quad (51)$$

where $\rho_1(\|v_d\|_s)$ denotes a positive bounded non-decreasing function in $\|v_d\|$.

Then, by using Young's inequality, the first bracketed term can be upper bounded by

$$\left\|e_p^T\right\|\rho_1(\xi\|v_d\|) \leq \tfrac{1}{2}\left(\tfrac{1}{\lambda_1}\|e_p\|^2 + \lambda_1\rho_1^2\right)$$

where $\lambda_1 \in \mathbf{R}^1$ is a positive constant. The adaptation laws for the estimated parameter vectors, $\hat{\Theta}_1$ and $\hat{\Theta}_3$, are designed using the projection-based update algorithm in [16] as

$$\dot{\hat{\Theta}}_1 = \text{Proj}\{\Gamma_1 W_1^\top r_p,\ \hat{\Theta}_1\} \text{ and } \dot{\hat{\Theta}}_3 = \text{Proj}\{\Gamma W_3^\top z,\ \hat{\Theta}_3\} \quad (52)$$

where Proj$\{\cdot\}$ is the parameter projection operator, $\hat{\Theta}_1$ and $\hat{\Theta}_3$ are the regression vectors, $\Gamma_1 = \gamma_1 I_5 \in \mathbf{R}^{5\times5}$ and $\Gamma_3 = \gamma_3 I_5 \in \mathbf{R}^{41\times41}$ are constant diagonal gain matrices, in which $\gamma_i (i = 1, 2) \in \mathbf{R}^1$ is a positive constant adaptation gain value. The second and third bracketed terms are compensated each other by using the updated parameter algorithm. Thus, $\dot{V}(t)$ yields

$$\dot{V} \leq -k_r||r||^2 - k_z||z||^2 - (\tfrac{\alpha}{m} - \tfrac{1}{2\lambda_1})||e_p||^2 - \rho_1^2(\tfrac{||r||^2}{\varepsilon_1} - \tfrac{\lambda_1}{2}). \quad (53)$$

Finally, $\dot{V}(t)$ can be upper bound in the following form

$$\dot{V} \leq -k_r||r||^2 - k_z||z||^2 - (\tfrac{\alpha}{m} - \tfrac{1}{2\lambda_1})||e_p||^2 \leq -\lambda_2||\eta||^2, \quad (54)$$

provided

$$\varepsilon_1 < \tfrac{2||r||^2}{\lambda_1} \text{ and } \alpha > \tfrac{m}{2\lambda_1}$$

where $\lambda_2 = \min\left\{k_r, k_z, (\frac{\alpha}{m} - \frac{1}{2\lambda_1})\right\} \in \mathbf{R}^1$ is a positive constant and $\eta(t)$ is defined as $\eta \triangleq [e_p^T,\ r^T,\ z^T]^T$. Utilizing Barbalat's Lemma, the tracking error is locally asymptotically stable; $\|\eta\| \to 0$ as $t \to \infty$ under the given condition. Therefore, we can obtain the result of Thereom 1.

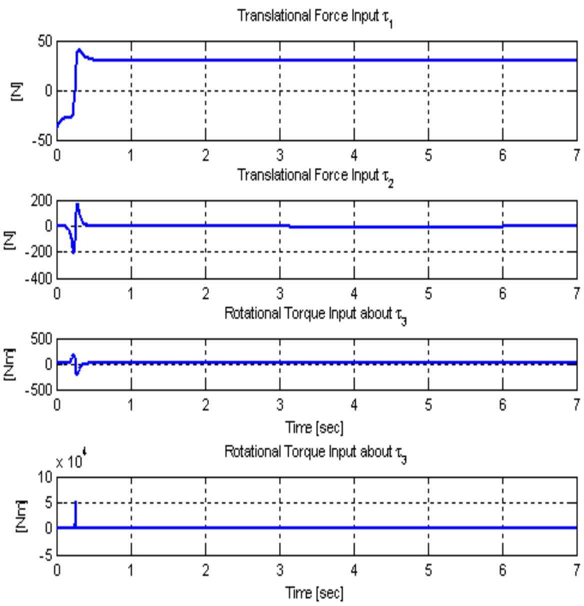

Fig. 1. Force and Torques

V. SIMULATION NOTES

The tracking control in the presence of uncertainty was simulated using a small quad-rotor unmanned aerial vehicle as m=2.9 [kg], g=9.81[ms^2], J=diag(0.4, 0.4, 0.6) [kgm^2]. An aerodynamic coefficients and the constant control parameters for controller and the constant diagonal gains (γ_i) for updating algorithm were given as $C_{d_1} = 0.3$, $C_{d2} = 0.02$, $C_{d3} = 0.3$, $k_{r1} = 10$, $k_{r2} = 10$, and $k_z = 10$, $\alpha = 20$, $\varepsilon_1 = .1$, $\gamma_1 = 30$, $\gamma_3 = 50$, $\delta_3 = -1$. Figure 1 shows the force and torque inputs of the quad-rotor. (More simulation results will be given later)

VI. CONCLUSIONS AND FUTURE WORKS

The goal of designing an state feedback controller for traking control of a quad-rotor UAV system in the presence of parametric uncertainties has been suggested. The estimated parameters are updated by the adaptation laws using the projection algorithm. The controller is designed for position tracking in 3-dimension while yaw tracking which degree-of-freedom is fully used in the given underactuated system. Numerical simulation results are to demonstrate a comprehensive quad-rotor tracking control.

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