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# Nonlinear Model Reduction for Fluid Flows

Samir Sahyoun and Seddik M. Djouadi

***Abstract*— Model Reduction is an essential tool that has been applied in many control applications such as control of fluid flows. Most model reduction algorithms assume linear models and fail when applied to nonlinear high dimensional systems, in particular, fluid flow problems with high Reynolds numbers. For example, proper orthogonal decomposition (POD) fails to capture the nonlinear degrees of freedom in these systems, since it assumes that data belong to a linear space and therefore relies on the Euclidean distance as the metric to minimize. However, snapshots generated by nonlinear partial differential equations (PDEs) belong to manifolds for which the geodesics do not correspond in general to the Euclidean distance. A geodesic is a curve that is locally the shortest path between points. In this paper, we propose a model reduction method which generalizes POD to nonlinear manifolds which have a differentiable structure at each of their points. Moreover, an optimal method in constructing reduced order models for the two-dimensional Burgers' equation subject to boundary control is presented and compared to the POD reduced models.**

## I. Introduction

The control of many processes requires mathematical models that are computationally amenable to control design with the fewest number of states necessary. This is particularly apparent in the control of large scale systems, and systems governed by partial differential equations (PDEs) where the number of states is very high. Some of the applications include flow control, and in particular aerodynamic flow control, where there has been significant interest, see e.g., [1][2][3] [4][7] [8][9][12][14][13][15] [16][5].

Considerable progress has been made in model reduction methods for linear models, especially time-invariant ones, as reported in the excellent monograph [21]. However, most applications of practical interest involve models that are nonlinear. For example, fluid flows are usually governed by Navier-Stokes equations which are highly nonlinear for large Reynolds numbers. These flows arise in many applications such as in control of heat transfer in solid-state circuits, devices, and composite materials, in control, estimation and optimization of energy efficient buildings, in the control of vehicular platoons, microelectromechanical systems (MEMS), smart structures, aerodynamics, combustion, and process control. An important application in combustion control in gas turbines and rockets is flameholder stabilized premixed combustion, where it is crucial to control the flow and flame dynamics to suppress thermoacoustic instabilities [6].

S. Sahyoun and S.M. Djouadi are with the Department of Electrical Engineering & Computer Science, University of Tennessee, Knoxville, TN 37996-2100 ssahyoun@utk.edu, djouadi@eecs.utk.edu

Among the multitude of model reduction techniques, the proper orthogonal decomposition (POD) is arguably the most popular one method used in deriving reduced models for fluid flows governed by nonlinear PDEs for simulation or control purposes. In POD snapshots obtained by conducting experiments or computational fluid dynamic (CFD) simulations are usually used to create an ensemble of solutions with particular open loop control input data. The set is used to construct a set of POD basis modes (see e.g. [15]) and a reduced order model is obtained depending on the energy ratio with the full order model.

Methods borrowed from optimal control theory is used to find the POD basis $\vec{\phi}_k$ which satisfy the desired optimal conditions and the boundary conditions. By solving the system analytically or numerically, the dynamical behavior of the system with different parameters (such as the Reynolds number etc.) can be understood in detail, and under the given optimal conditions, $\vec{\phi}_k$ are the optimal orthogonal bases of the system. In this theory one is allowed to give different optimal conditions for different requirements, and the user can set different approximation demands for the initial condition and the global behavior, respectively. Therefore the optimal truncated low dimensional dynamical systems with particular emphasis on some important temporal-spatial regions can be constructed. From this short introduction, it can be seen that this new theory is quite different from the spatial method, in which the low dimensional dynamical systems is constructed by means of Galerkin projection of the PDE onto a set of predetermined bases.

It is well known that the modes maximize energy in mean square sense, that is, it captures the mean square energy of the snapshot ensemble better than any other basis [17] [11]. With $N$ snapshots in hand the $N \times N$ correlation matrix $L$ defined by

$$L_{i,j} = \langle S_i, S_j \rangle \tag{1}$$

is constructed, where $\langle \, , \, \rangle$ denoted the usual Euclidean inner product.
With $M$ denoting the number of POD modes to be constructed, the first $M$ eigenvalues of largest magnitude, $\{\lambda_i\}_{i=1}^{M}$, of $L$ are found. They are sorted in descending order, and their corresponding eigenvectors $\{v_i\}_{i=1}^{M}$ are calculated. Each eigenvector is normalized so that

$$\|v_i\|^2 = \frac{1}{\lambda_i}. \tag{2}$$

The orthonormal POD basis set $\{\phi_i\}_{i=1}^{M}$ is constructed according to

$$\phi_i = \sum_{j=1}^{N} v_{i,j} S_j, \tag{3}$$

where $v_{i,j}$ is the $j^{th}$ component of $v_i$.

With a POD basis in hand, the solution $w$ of the distributed parameter model is approximated as a linear combination of POD modes, i.e.,

$$w \approx \sum_{i=1}^{M} \alpha_i \phi_i \tag{4}$$

This shows that POD find a low dimensional embedding of the snapshots that preserve most of the the energy as measured in a much higher dimensional solution space. However, for nonlinear PDEs the snapshot sets contain nonlinear structures that is not captured by POD, since the latter (numerically) uses the Euclidean distance. Snapshots far apart in the solution manifold may appear deceptively close in the Euclidean distance while they may be far apart as measured by their geodesic.

The paper is organized as follows. In section II we introduce some mathematical background borrowed from differential geometry. In section III solutions of nonlinear PDEs are assumed to belong to a manifold, and a method which generalizes POD to the nonlinear setting is developed. The method essentially deals with manifold data. Section IV gives a method based on optimal control theory to compute the POD basis. In section V we present an application to a nonlinear convective flow that shows the effectiveness of the method. Section VI contains concluding remarks.

## II. Background Theory

Riemannian manifolds, that is manifolds that can be associated with a differentiable structure at each of their points, are considered in this paper. An important property of Riemannian manifolds is that they admit a tangent space of the same dimension as the manifold at each of their points. A Riemannian metric on a manifold $M$ is a smoothly varying inner product $(\ ,\ )$ on the tangent space $T_xM$ at each point $x \in M$. If $v \in T_xM$, then $\|v\| = \sqrt{(v,v)}$. Given a smooth curve segment in $M$, its length is computed by integrating the norm of the tangent vectors along the curve. The Riemannian distance between two points $x, y \in M$, denoted $d(x;y)$, is defined as the minimum length over all possible smooth curves between $x$ and $y$. A geodesic is a curve that locally minimizes the length between points. A manifold is said to be complete if all geodesics extend indefinitely, this means that between any two points there exists a length-minimizing geodesic [19][20].

Given a tangent vector $v \in T_xM$, there exists a unique geodesic, $\gamma_v(t)$, with $v$ as its initial velocity. One can define the exponential map as,

$$\begin{aligned} \mathrm{Exp}_x \ : \quad T_x &\longmapsto M & (5)\\ v &\longmapsto \gamma_v(1) & (6) \end{aligned}$$

The exponential map is a diffeomorphism in a neighborhood of zero, and its inverse in this neighborhood is the Riemannian log map, denoted Logx. Thus for a point y in the domain of $\mathrm{Log}_x$ the geodesic distance between $x$ and $y$ is given by [19]

$$d(x,y) = \|\mathrm{Log}_x(y)\| \tag{7}$$

## III. Nonlinear Proper Orthogonal Decomposition

Consider a set of snapshots $\{S_1, \cdots, S_N\}$, $S_j \in \mathbb{R}^d$. In practical terms the objective of POD is to find an orthonormal basis $\{\phi_1, \cdots, \phi_d\}$ such that

$$\begin{aligned} \phi_1 &= \arg\max_{\|\phi\|=1} \sum_{i=1}^{N} < \phi, S_i > \\ \phi_k &= \arg\max_{\|\phi\|=1} \sum_{i=1}^{N} \sum_{j=1}^{k-1} < \phi_j, S_i >^2 + < \phi, S_i >^2 \end{aligned}$$

The $k$-dimensional subspace $\{\mathrm{span}\{\phi_1, \cdots, \phi_k\}$ maximizes the energy of the snapshots projected to it. To generalize POD for data on manifolds there is a need to extend the concept of a linear subspace to that of a geodesic submanifold, and define the notion of a projection for it.

Recall that a geodesic is a curve that is locally the shortest path between points. A submanifold $S$ of $M$ is said to be geodesic at $x \in S$ if all geodesics of $N$ passing through $x$ are also geodesics of $M$ [19].

Following [20] the projection of a point $m \in M$ onto a geodesic submanifold $S$ of $M$ is defined as the point on $S$ that is nearest to $m$ in Riemannian distance. Therefore, the projection operator $P_N : M \longmapsto S$ is defined as [20]

$$P_S(m) = \arg\min_{s \in S} d(m,s)^2 \tag{8}$$

Since projection is defined by a minimization, there is no guarantee that the projection of a point exists or that it is unique. However, by restricting to small enough neighborhoods the projection can be guaranteed to exist and is unique. Using (7) the projection operator can be written as

$$P_S(m) = \arg\min_{s \in S} \|\mathrm{Log}_m(s)\|^2 \tag{9}$$

Let $\mu$ be the mean vector of the snapshots $\{S_1, \cdots, S_N\}$, i.e.,

$$\begin{aligned} \mu &= \arg\min_{x \in M} \sum_{i=1}^{N} d(x, S_i)^2 \\ &= \arg\min_{x \in M} \sum_{i=1}^{N} \|\mathrm{Log}_{S_i}(x)\|^2 \end{aligned} \tag{10}$$

Nonlinear POD is defined by first constructing an orthonormal basis of tangent vectors $\phi_1, \cdots, \phi_d \in T_\mu M$ that span $T_\mu M$. Letting $V_k = \mathrm{span}(\{\phi_1, \cdots, \phi_d\})$ the corresponding nonlinear POD submanifolds are $\mathcal{S}_k := \mathrm{Exp}_\mu(V_k)$. The basis vectors for $T_\mu M$ can be computed by [20]

$$\phi_1 = \arg\max_{\|\phi\|=1} \sum_{i=1}^{N} \|\mathrm{Log}_\mu(P_{\mathcal{S}}(S_i))\|^2 \tag{11}$$

where $\mathcal{S} = \mathrm{Exp}_\mu(\mathrm{span}(\phi))$, and

$$\phi_k = \arg\max_{\|\phi\|=1} \sum_{i=1}^{N} \|\mathrm{Log}_\mu(P_{\mathcal{S}}(S_i))\|^2 \tag{12}$$

where $\mathcal{S} = \mathrm{Exp}_\mu(\mathrm{span}(\phi_1, \cdots, \phi_{k-1}, \phi))$.
The concepts discussed above rely on determining the distance $d(\cdot\ ,\ \cdot)$, or the exponential and logarithm maps. This is not possible in general, in particular, when using only snapshots. A more tractable problem is to define a distance on $M$ is to embed it in a Euclidean space and use the Euclidean distance between points. This notion of distance is extrinsic to $M$, that is, it depends on the ambient space and the choice of embedding. One such embedding is the so-called locally linear embedding (LLE) [18]. Each point in the snapshot ensemble with sufficiently close neighbors to lie on or near a locally linear patch of the manifold. The geometry of these patches can be characterized by linear coefficients that reconstruct each data point from its neighbors. The neighborhoods can be constructed as in the first method. The error is measured by the cost function

$$E(W) = \sum_i \left|S_i - \sum_j W_{ij} S_j\right|^2 \tag{13}$$

where $S_j$ denotes snapshot $j$, $W_{ij}$ represents the contribution of $j$th snapshot to the $i$th reconstruction. The weights $W_{ij}$ are computed by minimizing the cost function $E(W)$ w.r.t. $W$ subject to the constraint that $W_{ij} = 0$ if $S_j$ is not a neighbor of $S_i$, and $\sum_i W_{ij} = 1$ to enforce symmetry and invariance w.r.t. translations, rotations, and rescalings. Each snapshot $S_i$ is mapped in a low dimensional vector $Y_i$ representing global internal coordinate on the manifold, and obtained as the minimizer of the embedding cost function

$$\varepsilon(Y) = \sum_i \left|Y_i - \sum_j W_{ij} Y_j\right|^2 \tag{14}$$

where $W_{ij}$'s are fixed. The cost (14) is quadratic in $Y_i$ and can be minimized by solving a sparse eigenvalue problem.

## IV. POD Based on Optimal Control

Consider the general nonlinear initial-boundary value problem [22]

$$\frac{\partial \vec{w}}{\partial t} + N(\vec{w}) = 0, \vec{\mathbf{x}} \in \Omega, t > 0 \tag{15}$$
$$\vec{w}(\vec{\mathbf{x}}, 0) = \vec{w}_0(\vec{\mathbf{x}}), \vec{\mathbf{x}} \in \Omega, \tag{16}$$
$$\vec{w}(\vec{\mathbf{x}}, t)|_{\partial\Omega} = \vec{g}(\vec{\mathbf{x}}, t), t > 0,$$

where for $t > 0, N : H \subset V \to F$, is a nonlinear operator, where $H, V$ and $F$ are the Hilbert spaces. $H$ is the linear subspace of $V$. The inner products on $H, F$ and $V$ are $(.,.)_H, (.,.)_F, (.,.)_V$, respectively. The corresponding norm defined on $V$ is $\|\vec{w}\|_v = (\vec{w}, \vec{w})_v^{1/2}, \forall \vec{w} \subset V$.
Define a space

$$\mathcal{B}_N = \{\phi = \vec{\phi}_1(\vec{\mathbf{x}}), \cdots, \vec{\phi}_N(\vec{\mathbf{x}})^T | \vec{\phi}_i \in H, (\vec{\phi}_i, \vec{\phi}_j) = \delta_{ij}\}$$

$\vec{w}(\vec{\bar{x}}, t)$ is decomposed as:

$$\vec{w}(\vec{\bar{x}}, t) = \vec{w}_N(\vec{\mathbf{x}}, t) + \vec{w}_R(\vec{\mathbf{x}}, t) \tag{17}$$
$$\approx \sum_{k=1}^{N} a_k(t) \vec{\phi}_k(\vec{\mathbf{x}}). \tag{18}$$

Then, we project the PDE onto the unknown bases $\vec{\phi}_k$ to get the Galerkin model

$$\dot{a}_k(t) = G_k(a_k; \vec{\phi}_1, \cdots, \vec{\phi}_N; \nabla\vec{\phi}_1, \cdots, \nabla\vec{\phi}_N, \cdots) \tag{19}$$
$$a_k(0) = (\vec{w}_0(\vec{\mathbf{x}}), \vec{\phi}_k).$$

Now the optimization function to be solved is as follows: Find $\phi_{opt} \in \mathcal{B}_N$ such that:

$$J(\phi_{opt}) = \min_{\phi \in \mathcal{B}_N} J(\phi) \tag{20}$$
$$where \quad J(\phi) = \int_0^T \|\vec{w}_R\|_v^2 dt$$

## V. Application to the 2D Burgers' Equation

The specific problem geometry considered is shown in Figure 1. The idea and methods presented here could be modified to apply to a different geometry or obstacle shape.

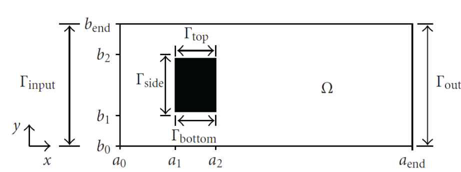


Fig. 1. Problem Geometry

The problem statement with its corresponding boundary conditions and governing equations was taken from [13]. A realistic example of this geometry in an aerodynamic application would be a payload hatch open during flight with actuator control only on the boundary. Let $\Omega_{gap}$ be the region defined by $[a_1, a_2] \times [b_1, b_2]$. Let $\Omega_{full}$ be the region defined by $(a_0, a_{end}) \times (b_0, b_{end})$. Then the problem domain is given by $\Omega = \Omega_{full}/\Omega_{gap}$. In this problem setup, $\Omega_{gap}$ is an obstacle.
The system dynamics that act within the problem domain are described by the two-dimensional ($2D$) Burgers' equation [16]

$$\frac{\partial}{\partial t} w(t, x, y) + \nabla.F(w)$$
$$= \frac{1}{R_e}\left(\frac{\partial^2}{\partial x^2} w(t, x, y) + \frac{\partial^2}{\partial y^2} w(t, x, y)\right), \tag{21}$$

where $R_e$ is the Reynolds number and the form of $F(w)$ is

$$F(w) = \left[c_1 \frac{w^2(t, x, y)}{2} \qquad c_2 \frac{w^2(t, x, y)}{2}\right]^T. \tag{22}$$

In this case, the value for $c_1$ is equal to 1 and $c_2$ is equal to 0. The value used is 300, a small Reynolds number, but it still allows for the nonlinearity to show in the problem.
Dirichlet boundary conditions located on the obstacle top and bottom are denoted by $\Gamma_{top}$ and $\Gamma_{bottom}$. A Dirichlet boundary condition is a first-type boundary condition that specifies the values of the solution defined by $f(x)$ on a domain boundary [17]. The form of the boundary condition is

$$w(t,x,y) = f(t,x,y) \qquad \forall (x,y) \in \partial\Omega \tag{23}$$

The boundary conditions on the top and bottom are described by the following:

$$\begin{aligned} w(t,\Gamma_{bottom}) &= u_{bottom}(t)\Psi_{bottom}(x), \\ w(t,\Gamma_{top}) &= u_{top}(t)\Psi_{top}(x), \end{aligned} \tag{24}$$

Here $u_{top}(t)$ and $u_{bottom}(t)$ are the control inputs on the top and bottom boundaries, respectively; the spatial functions $\Psi_{top}(x)$ and $\Psi_{bottom}(x)$ describe the spatial effect that the controls have on the top and bottom boundaries.
The boundary condition on the airflow intake side is

$$w(t,\Gamma_{in}) = f(y) \tag{25}$$

and it is parabolic in nature. The airflow outtake side has a Neumann boundary condition that has the form [23].

$$\frac{\partial}{\partial x} w(t,\Gamma_{out}) = 0 \tag{26}$$

On all of the remaining boundaries of $\Omega, w(t,x,y)$ is set equal to 0 for all values of $t$. Finally, the initial conditions for the interior are given by

$$w(0,x,y) = w_0(x,y) \in L^2(\Omega). \tag{27}$$

A numerical solution was found by simulation using Comsol . The resulting system model contains a little more than 4000 states. The velocity field may be represented as:

$$\begin{aligned} \vec{w} &= \vec{W} + \vec{w_N} + \vec{w_R} \\ &= \vec{W} + \sum_{k=1}^{N} a_k(t)\vec{\phi}_k(\vec{\mathbf{x}}) + \vec{w}_R \end{aligned} \tag{28}$$

where $\vec{W} = W\vec{i} + V\vec{j}$ is the mean velocity, $\vec{w} = w\vec{i} + v\vec{j}$, $\vec{\mathbf{x}} = x\vec{i} + y\vec{j}$, $\vec{w}_R$ is the remainder and $\vec{\phi}_k = \phi_k\vec{i} + \varphi_k\vec{j}$ is the optimal basis that satisfies $\int_\Omega \phi_k\varphi_l d\Omega = \delta_{kl}$ Using POD in [15], projection of Burgers' equations onto the space of the optimal basis $\vec{\phi}_k$ gives the reduced order model.
In the simulation, we will use the first 3 POD modes as initial modes for the optimal reduction process. Our optimal functional condition will be [22]:

$$\begin{aligned} J(\phi) &= \int_0^T \|\vec{w}_R\|^2 dt \\ &= \int_0^T \big(w - \sum_{k=1}^{N} a_k(t)\phi_k(\vec{\mathbf{x}}), w - \sum_{k=1}^{N} a_k(t)\phi_k(\vec{\mathbf{x}})\big)dt \end{aligned} \tag{29}$$

and the generalized optimal functional condition will be:

$$\begin{aligned} J^g(\phi) = \int_0^T \big(\sum_{k=1}^{N}[a_k^2 - 2a_k(w,\phi_k)] + \vec{\lambda}^T(\vec{G} - \dot{\vec{a}})\big)dt \\ + \sum_{\substack{k,l=1 \\ k \le l}}^{N} [(\phi_k,\phi_l) - \delta_{kl}] \end{aligned} \tag{30}$$

where $\vec{\lambda}(t)$ is $N$ dimensional vector of Lagrangian multipliers. The optimization problem is solved by conjugate gradients algorithm. We use

$$a_k(0) = \int_\Omega (\vec{w}_0 - \vec{W}).\vec{\phi}_k d\Omega$$

as initial values to the optimization problem.
In order to nd the optimal bases $\vec{\phi}_k$, let the variation of $J^g$ equals to zero, i.e., $\delta J^g = 0$. Using the basic theorem of variation method, from the terms involving $\delta a_k$, we get the ODEs of $\lambda_k$ as

$$\begin{aligned} \dot{\lambda}_k = &\sum_{l,m=1}^{N} \lambda_l \int_\Omega \big(\vec{\phi}_k.(\vec{\phi}_m.\nabla\vec{\phi}_l) + \vec{\phi}_k.(\vec{\phi}_l.\nabla\vec{\phi}_m)\big)d\Omega a_m \\ &+ \sum_{l=1}^{N} \lambda_l \int_\Omega [\vec{\phi}_k.(\vec{\phi}_l\nabla\vec{W} + \vec{W}.\nabla\vec{\phi}_l) + \frac{1}{R_e}\nabla\vec{\phi}_k.\nabla\vec{\phi}_l]d\Omega \\ &- 2(a_k - \int_\Omega \vec{w}.\vec{\phi}_k d\Omega), \qquad \lambda_k(T) = 0. \end{aligned} \tag{31}$$

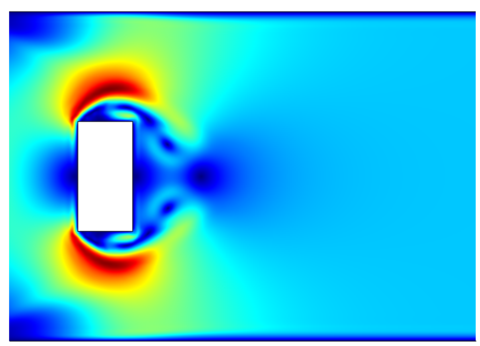

Fig. 2. Full Solution

Conjugate gradients algorithm is used to solve the problem where we alternate in $x$ and $y$ directions to find the optimal bases $\phi_k$ and $\varphi_k$. The POD bases which are the bases obtained with the method in [16], are used as the initial bases for the iterations. Figures 2,3 and 4 show the full soulution, POD reduced model and Optimal bases POD respectively.

## VI. Conclusion

In this paper, a generalization of POD to manifold data generated by nonlinear PDEs is discussed. The manifolds are assumed to be Riemmanian, so that nonlinear POD relies on computing geodesic submanifolds. Then, based on optimal

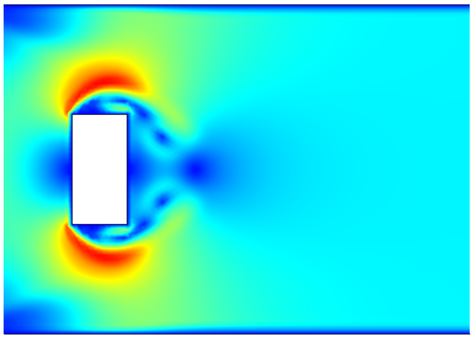

Fig. 3. POD reduced model

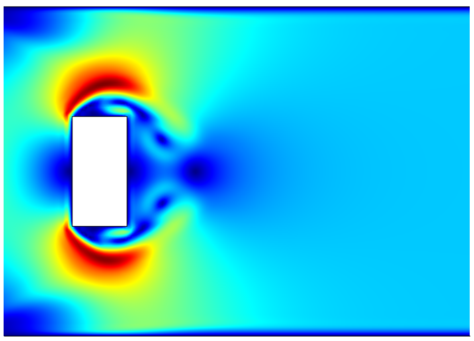

Fig. 4. Optimal basis POD reduced model

control theory and using snapshots from a numerical solution of the 2D Burgers' equation, optimal bases are derived. The first three POD modes were used as initials for the optimization problem. The result shows that this approach gives better results compared with standard POD.

## REFERENCES


[1] K. Willcox and J. Peraire, " Balanced Model Reduction via the Proper Orthogonal Decomposition," *AIAA JOURNAL* Vol. 40, No. 11, November 2002, pp. 2323-2330

[2] Caraballo, E., Samimy, M., and DeBonis, J., "Low Dimensional Modeling of Flow for Closed-Loop Flow Control," AIAA Paper 2003-0059, January 2003.

[3] Carlson, H., Glauser, M., Higuchi, H., and Young, M., "POD Based Experimental Flow Control on a NACA-4412 Airfoil," AIAA Paper 2004-0575, January 2004.

[4] Cohen, K., Siegel, S., McLaughlin, T., and Myatt, J., "Proper Orthogonal Decomposition Modeling of a Controlled Ginzburg-Landau Cylinder Wake Model," AIAA Paper 2003-2405, January 2003.

[5] J. Borggaard, Optimal Reduced-Order Modeling for Nonlinear Distributed Parameter Systems, Proceedings of the American Control Conference, Minneapolis, pp. 1150-1154, 2006.

[6] P.G. Mehta, A. Banaszuk, M.C. Soteriou and I. Mezic, Fuel Control of a Ducted Bluffbody Flame, Proceedings of the 42nd lEEE Conference on Decision and Control Maui, Hawaii USA, pp.2340-2345, December 2003.

[7] C. W. Rowley, Model reduction for fluids using balanced proper orthogonal decomposition, International Journal of Bifurcation and Chaos 15(3) : 9971013, March 2005.

[8] Ilak, M. and Rowley, C. W., "Reduced-Order Modeling of Channel Flow Using Traveling POD and Balanced POD,". *3rd AIAA Fow Control Conference*, San Francisico, June 2006

[9] M. Ilak and C. W. Rowley Modeling of transitional channel flow using balanced proper orthogonal decomposition, Physics of Fluids 20(034103), March 2008.

[10] M. Ilak and C. W. Rowley Reduced-order modeling of channel flow using traveling POD and balanced POD AIAA Paper 2006-3194, 3rd AIAA Flow Control Conference , June 2006.

[11] S.M. Djouadi, On the Optimality of the Proper Orthogonal Decomposition and Balanced Truncation, Proceedings of the 47th IEEE Conference on Decision and Control, Cancun, Mexico, December 2008, pp. 4221- 4226.

[12] Seddik M. Djouadi, R. Chris Camphouse, and James H. Myatt, Empirical Reduced-Order Modeling for Boundary Feedback Flow Control, Journal of Control Science and Engineering Volume 2008 (2008), Article ID 154956, 11 pages.doi:10.1155/2008/154956.

[13] R.C. Camphouse, S.M. Djouadi, and J.H. Myatt, "Feedback Control for Aerodynamics," Proc, of the IEEE CDC, December, 2006

[14] S.M. Djouadi, R.C. Camphouse and J.H. Myatt, Reduced Order Models for Boundary Feedback Flow Control, American Control Conference 2008, Seattle, June 2008, Seattle, Washington, pp. 4005-4010

[15] Camphouse, R.C., "Boundary Feedback Control Using Proper Orthogonal Decomposition Models," *Journal of Guidance, Control, and Dynamics*, Vol. 28, No. 5, September-October 2005, pp 931-938.

[16] Camphouse, R. C. and Myatt, J. H., "Reduced Order Modelling and Boundary Feedback Control of Nonlinear Convection," AIAA Paper 2005-5844, August 2005.

[17] Holmes, P., Lumley, J., and Berkooz, G., *Turbulence, Coherent Structures, Dynamical Systems and Symmetry*, Cambridge University Press, New York, 1996, pp. 86-127.

[18] S.T Roweis and L.. Saul, Nonlinear Dimensionality Reduction by Locally Linear Embeddings, Science, vol. 290, December 2000, pp.2323-2326.

[19] W. M. Boothby, An Introduction to Differentiable Manifolds and Riemannian Geometry, 2nd ed. Academic Press, 1986.

[20] P.T. Flectcher, C. Lu, S.M. Pizer, and S. Joshi, Principal Geodesic Analysis for the Study of Nonlinear Statistics of Shape, IEEE Trans. on Medical Imaging, v. 23, no. 8, Augus 2004, pp.995-1005.

[21] Antoulas, A., Approximation of Large-Scale Dynamical Systems, *Society for Industrial and Applied Mathematics*, Philadelphia, 2005, pp. 2-24.

[22] C. Wu , Large Optimal Truncated Low Dimentional Dynamical Systems , Discrete And Continous Dynamical Systems., vol. 2, No. 4, (1996), 559-583.

[23] T. J. Chung, "Computational Fluid Dynamics, Cambridge University Press", Cambridge, UK, 2002.