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Can Thermodynamics Be Used to Design Control Systems?

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Abstract— **Thermodynamics is a physical branch of science that governs the thermal behavior of dynamical systems. The laws of thermodynamics involving conservation of energy and nonconservation of entropy are two of the most useful and general laws in all sciences. In particular, the second law of thermodynamics is intimately connected to the irreversibility of dynamical processes, that is, the status quo cannot be restored everywhere. This gives rise to an interesting quantity known as entropy. Entropy permeates the whole of nature, and unlike energy, which describes the state of a dynamical system, entropy is a measure of change in the status quo of a dynamical system. Motivated by this observation, in this paper we use the entropy function for deterministic systems as a benchmark to design a semistable controller that minimizes the time-averaging of the "heat" of the dynamical system. We present both state feedback control and output feedback control based on the dissipative systems. Furthermore, we convert the control design into an optimization problem with two linear matrix inequalities.**

I. Introduction

Thermodynamics [1], a classic physics topic, stirs a recent trend to re-design and review the old results in systems and control theory [2]–[16]. It turns out that many available results have a strong connection with thermodynamics and can be interpreted from a physics point of view. Among these results, the seminal work by Haddad *et al* [2] on system thermodynamics makes a great contribution to control/systems theory and provides a possible way to design a thermodynamic-like controller for control systems. Inspired by this breakthrough, the first attempt to realize thermodynamic stabilization using hybrid controllers has been reported in [17] where some conceptual design ideas were formulated and verified by a mechanical RTAC system. However, these ideas lack the rigorous foundation to justify their effectiveness. For example, the convergence analysis in [17] is only valid for Lyapunov stability, not sufficient for proving asymptotic stability. Motivated by [17], the authors in [5] developed a rigorous general framework for hybrid controllers proposed in [17]. Furthermore, they proposed a novel hybrid dynamic compensator which is designed in such a way that the total energy of the closed-loop system is consistent with the basic thermodynamic principles. In this case, the whole system behaves like a thermodynamic system so that thermodynamic stabilization is achieved for control systems by decreasing the total energy monotonically.

One of the most distinct features for thermodynamic systems is *energy equipartition* [2], [15] where the total energy is uniformly distributed among interconnected subsystems so that the temperatures of all subsystems achieve the same, which is the zeroth law of thermodynamics. From the dynamical systems point of view, this property is related to the notion of *semistability* for dynamical systems having a continuum of equilibria [18]. Specifically, for a semistable system, every trajectory that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. If we think all the energy equipartition states are the equilibria of thermodynamic systems, then the energy equipartition problem becomes semistability analysis of thermodynamic systems.

There are many important applications for which semistability is the most appropriate stability property of interest. A classical example is the synchronization of multiple weakly coupled oscillators to a common frequency. Recently, significant results have been obtained on semistability in consensus problems for networked agents [18]–[24]. An example of such a problem is for a group of networked autonomous vehicles to converge to a common heading, and for the network to respond to a small perturbation with only a corresponding small change to the common heading. Other recent results in semistability theory can be found in [25]–[27].

The contribution of this paper is to develop a first, novel thermodynamic framework for semistabilization of linear dynamical systems. The thermodynamic principles and concepts are introduced into control systems design so that control theory and system thermodynamics are combined to create some new perspective for the controller design based on our knowledge of physics. This attempt is comparable to the effort of putting energy back in control [28] which tries to use fundamental concepts in science and engineering practice to design controllers by viewing dynamical systems as energy-transformation devices. To do this, we restrict our attention to dissipative systems [9], [10] since not only are these systems widespread in engineering, but also dissipative systems have a clear physical connection with thermodynamics as pointed out by [10] and [2]. Based on the theory developed in [2], we define the notion of *entropy* for linear dynamical systems. Then we present state feedback and output feedback design frameworks for time-averaging minimum entropy control or time-averaging minimum "heat" control. A nice example is used to illustrate the basic idea of our controller design. The main result characterizes our optimal control problem into an optimization problem with two linear matrix inequalities.

The organization of this paper is as follows. Section II introduces the notation and the linear system used in the

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paper. Section III elaborates the design motivation and idea for state feedback and output feedback thermodynamic control. To this end, first we define the notion of entropy for deterministic systems. Then the state feedback thermodynamic control design is presented under state dissipatible systems. Next, we generalize the state feedback case to the output feedback thermodynamic control case by using the dynamic compensator and output dissipatibility. Finally, we draw some conclusions in Section IV.

II. Mathematical Preliminaries

The notion we use in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\overline{\mathbb{R}}_+$ denotes the set of nonnegative numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n\times m$ real matrices, $(\cdot)^{\mathrm{T}}$ denotes transpose, $(\cdot)^{\#}$ denotes the group generalized inverse, and I_n or I denotes the $n \times n$ identity matrix. Furthermore, we write $\|\cdot\|$ for the Euclidean vector norm, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ for the range space and the null space of a matrix A (or an operator A), $\mathrm{tr}(\cdot)$ for the trace operator, and $A \geq 0$ (resp., $A > 0$) to denote the fact that the Hermitian matrix A is positive definite (respectively, semidefinite).

In this paper, we consider a linear time-invariant system $\mathcal{G}$ given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \tag{1}$$

$$y(t) = Cx(t) + Du(t), \tag{2}$$

where for each $t \geq 0$, $x(t) \in \mathbb{R}^n$ denotes the state vector, $u(t) \in \mathbb{R}^m$ denotes the control input, $y(t) \in \mathbb{R}^l$ denotes the system output, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$, and $D \in \mathbb{R}^{l\times m}$. Throughout the paper, we make the following standing assumption:

Assumption 2.1: $\mathcal{G}$ is completely reachable.

III. A Framework for Thermodynamic Semistabilization

In this section, we present two frameworks for semistabilization of linear systems. In particular, a state feedback framework and an output feedback framework are proposed based on the notion of entropy. Hence, before we state our design methods, we need to define entropy for linear systems.

A. Entropy for Deterministic Systems

The following definition of state dissipatibility is needed in the paper. This notion defines a class of dynamical systems which can be dissipative by means of feedback control. The detailed discussion of passifiability on affine nonlinear systems can be found in [29]. Here we do not restrict our systems to passifiable systems.

Definition 3.1: $\mathcal{G}$ is called *state dissipatible* if there exists K such that $u = Kx + v$ and the system $\mathcal{G}_{\mathrm{s}}$

$$\dot{x} = (A + BK)x + Bv, \tag{3}$$

$$y = (C + DK)x + Dv, \tag{4}$$

is passive with respect to the supply rate $r(v, y)$, where $r(\cdot,\cdot)$ is continuous. If $r(v, y) = v^{\mathrm{T}}y$, then $\mathcal{G}$ is called *state passifiable*.

Remark 3.1: For the case where $\mathcal{G}$ can be rendered passive, one can use the conditions developed in [29] to check state passifiability of $\mathcal{G}$ and to find out K. An alternative method to find out K is to use the KYP lemma.

The following result is a direct consequence of the KYP lemma and dissipative systems.

Lemma 3.1: For a state dissipatible system $\mathcal{G}$, there exist a continuously differentiable, nonnegative function $V_{\mathrm{s}}(x)$ called the *storage function* and two continuous functions $\ell(x)$ and $\mathcal{W}(x)$ such that $\dot{V}_{\mathrm{s}}(x) = r(v, y) - [\ell(x) + \mathcal{W}(x)v]^{\mathrm{T}}[\ell(x) + \mathcal{W}(x)v]$.

Define $d(x, v) \triangleq [\ell(x) + \mathcal{W}(x)v]^{\mathrm{T}}[\ell(x) + \mathcal{W}(x)v]$ and $\mathrm{d}Q(t) \triangleq [r(v(t), y(t)) - d(x(t), v(t))]\mathrm{d}t$. This $Q(t)$ is similar to the notion of "heat" in thermodynamics, which corresponds to the net energy stored in the system (useful work supplied to the system minus the energy dissipation from the system). Motivated by [2], we have a *Clausius equality* for $\mathcal{G}$ being state dissipatible.

Proposition 3.1: Consider the dynamical system $\mathcal{G}$. Assume $\mathcal{G}$ is state dissipatible. Then for $t_{\mathrm{f}} \geq t_0 \geq 0$ and $v \in \mathcal{V}$ such that $V_{\mathrm{s}}(x(t_{\mathrm{f}})) = V_{\mathrm{s}}(x(t_0))$,

$$\int_{t_0}^{t_{\mathrm{f}}} \frac{r(v(t), y(t)) - d(x(t), v(t))}{c + V_{\mathrm{s}}(x(t))}\mathrm{d}t = \oint \frac{\mathrm{d}Q(t)}{c + V_{\mathrm{s}}(x(t))} = 0, \tag{5}$$

where $c > 0$.

Based on Proposition 3.1, we have the definition of entropy for deterministic systems.

Definition 3.2: For a state dissipatible system $\mathcal{G}$, a function $S : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$S(x(t_2)) \geq S(x(t_1)) + \int_{t_1}^{t_2} \frac{\mathrm{d}Q(t)}{c + V_{\mathrm{s}}(x(t))} \tag{6}$$

for any $t_2 \geq t_1 \geq 0$ and $v \in \mathcal{V}$ is called the *entropy* function of $\mathcal{G}$.

The standard entropy in thermodynamics satisfies the following inequality

$$\mathrm{d}S \geq \frac{\mathrm{d}Q}{T_{\mathrm{e}}}, \tag{7}$$

where Q is the heat of the system and T_{e} is the temperature of the system that supplies heat (for instance heat bath). The Clausius formulation of the second law of thermodynamics implies that the total entropy of Universe never decreases. From this point, it is clear that heat never flows from a cold system to a hot system spontaneously. The Kelvin-Planck statement easily follows from there. Note that it follows from (7) that in our definition $c + V_{\mathrm{s}}(x)$ plays the role of "temperature" in dynamical systems, which has been pointed out by [2]. On the other hand, the Lyapunov measure in [30] is defined as the inverse of a Lyapunov function. Hence, $1/(c + V_{\mathrm{s}}(x))$ in our definition can be understood as a *temperature measure* for dynamical systems. This point of view is important since it can be used to define *almost everywhere energy equipartition* as a way to weaken the notion of energy equipartition and to develop the system thermodynamic theory in the probability space.

The following result is concerned with the continuity of entropy functions.

Lemma 3.2: Consider a state dissipatible system $\mathcal{G}$. Let $S : \mathbb{R}^n \to \mathbb{R}$ be an entropy function of $\mathcal{G}$. Then $S(\cdot)$ is continuous on $\mathbb{R}^n$.

Next, we give an explicit form of a continuously differentiable entropy function.

Proposition 3.2: For a state dissipatible system $\mathcal{G}$, the function $S : \mathbb{R}^n \to \mathbb{R}$ given by

$$S(x) \triangleq \log\left[c + V_{\rm s}(x)\right] - \log c, \tag{8}$$

where $c > 0$, is a continuously differentiable entropy function of $\mathcal{G}$.

Remark 3.2: In [2], the authors proved the uniqueness of the entropy function for power balance equations. Whether or not there exists a unique continuously differentiable entropy function for $\mathcal{G}$ remains an open problem. Similar remarks hold for Proposition 3.3 below.

Next, we introduce the notion of output dissipatibility.

Definition 3.3: $\mathcal{G}$ is called *output dissipatible* if there exists a dynamic compensator $\mathcal{G}_{\rm c}$ given by

$$\dot{x}_{\rm c} = A_{\rm c}x_{\rm c} + B_{\rm c}u_{\rm c}, \tag{9}$$

$$y_{\rm c} = C_{\rm c}x_{\rm c}, \tag{10}$$

denoted by $(A_{\rm c}, B_{\rm c}, C_{\rm c})$, where $x_{\rm c} \in \mathbb{R}^{n_{\rm c}}$, $u_{\rm c} \in \mathbb{R}^{m_{\rm c}}$, $y_{\rm c} \in \mathbb{R}^{l_{\rm c}}$, $A_{\rm c} \in \mathbb{R}^{n_{\rm c}\times n_{\rm c}}$, $B_{\rm c} \in \mathbb{R}^{n_{\rm c}\times m_{\rm c}}$, and $C_{\rm c} \in \mathbb{R}^{l_{\rm c}\times n_{\rm c}}$, such that $u = -y_{\rm c}$ and the closed-loop system $\tilde{\mathcal{G}}$ given by

$$\begin{bmatrix} \dot{x} \\ \dot{x}_{\rm c} \end{bmatrix} = \begin{bmatrix} A & -BC_{\rm c} \\ 0 & A_{\rm c} \end{bmatrix} \begin{bmatrix} x \\ x_{\rm c} \end{bmatrix} + \begin{bmatrix} 0 \\ B_{\rm c} \end{bmatrix} u_{\rm c}, \tag{11}$$

$$y = \begin{bmatrix} C & -DC_{\rm c} \end{bmatrix} \begin{bmatrix} x \\ x_{\rm c} \end{bmatrix}, \tag{12}$$

is dissipative with respect to the supply rate $r(u_{\rm c}, y)$, where $r(\cdot,\cdot)$ is continuous. If $r(u_{\rm c}, y) = u_{\rm c}^{\rm T}y$, then $\mathcal{G}$ is called *output passifiable*.

Let $\tilde{x} \triangleq [x^{\rm T}, x_{\rm c}^{\rm T}]^{\rm T}$. Similar to the state dissipatibility case, here we can define the entropy function for the closed-loop system consisting of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$.

Definition 3.4: For an output dissipatible system $\mathcal{G}$, a function $\mathcal{S} : \mathbb{R}^n \times \mathbb{R}^{n_{\rm c}} \to \mathbb{R}$ satisfying

$$\mathcal{S}(\tilde{x}(t_2)) \geq \mathcal{S}(\tilde{x}(t_1)) + \int_{t_1}^{t_2} \frac{{\rm d}\tilde{Q}(t)}{c + V_{\rm s}(\tilde{x}(t))} \tag{13}$$

for any $t_2 \geq t_1 \geq 0$ and $u_{\rm c} \in \mathcal{U}_{\rm c}$ is called the *entropy* function of $\tilde{\mathcal{G}}$.

The following result is immediate from the above definition.

Proposition 3.3: For an output dissipatible system $\mathcal{G}$, if $V_{\rm s}(\tilde{x})$ is the storage function, then the function $\mathcal{S} : \mathbb{R}^n \times \mathbb{R}^{n_{\rm c}} \to \mathbb{R}$ given by

$$\mathcal{S}(\tilde{x}) \triangleq \log\left[c + V_{\rm s}(\tilde{x})\right] - \log c, \tag{14}$$

where $c > 0$, is a continuously differentiable entropy function of $\tilde{\mathcal{G}}$.

B. State Feedback Thermodynamic Control

To begin our controller design, we consider the system $\mathcal{G}$ given by (1) and (2) with $C = I_n$ and $D = 0$. In this case, the output equation becomes $y = x$. The cost functional is given by

$$J = \lim_{t\to\infty} \left[\frac{1}{t}\int_0^t S(x(\sigma)){\rm d}\sigma\right]. \tag{15}$$

Hence, J is interpreted as the time-average of the entropy function for the deterministic case of $\mathcal{G}$.

The control aim here is to design a state feedback controller $u = Kx + v$ and $v = Ly = Lx$ so that the closed-loop system is *semistable* [19], [31] and the cost functional J is minimized, that is, the time-average of the entropy function is minimized. The physical meaning of this control aim is to minimize the amount of the heat stored in the system so that the system is stable at some energy level and this energy level is determined by the initial energy level and how much of control effort we want to put in the system.

The main feature of this optimal control problem is semistability instead of asymptotic stability in the literature. Semistability is defined in terms of continuum of equilibria for dynamical systems, which distinguishes thermodynamic systems from usual dynamical systems [2]. It has been shown in [2] that for an isolated thermodynamic system, it is semistable rather than asymptotically stable in the sense that every trajectory of the system which starts in a neighborhood of a Lyapunov stable equilibrium converges to a possibly different Lyapunov stable equilibrium. Some relevant results on semistable control of coupled systems are reported in [32]. Here we use the thermodynamic ideas developed in [2], [5] to propose a novel framework for semistabilization of linear systems.

Before we discuss our thermodynamic controller design, a fundamental question regarding the above control problem is the following: Is this control problem a well-defined problem? This question implies two subquestions as follows: 1) Is J finite? and 2) if $J < \infty$, does there exist K such that J is minimized? The first subquestion is answered by the following lemma.

Lemma 3.3: Consider the linear control system $\mathcal{G}$. If the closed-loop system is semistable, then $-\infty < J < \infty$.

To answer the second subquestion, we give an example and consider the continuously differentiable entropy function given by Proposition 3.2. However, before we present this example, we need several lemmas. The first lemma gives an estimate of the entropy function.

Lemma 3.4: For the entropy function $S(\cdot)$ given by (8), we have

$$\frac{V_{\rm s}(x)}{c + V_{\rm s}(x)} \leq S(x) \leq \frac{1}{c}V_{\rm s}(x), \quad x \in \mathbb{R}^n. \tag{16}$$

The second lemma is like a squeezing lemma about optimality of J by looking at optimality of upper-bound and lower-bound functions.

Lemma 3.5: Consider a state dissipatible system $\mathcal{G}$. Suppose we design a state feedback controller $u = Kx$ for

$\mathcal{G}$ such that $\dot{V}_{\rm s}(x(t)) \leq 0$ for all $t \geq 0$ and the closed-loop system is semistable. If there exists K^* such that the following cost functional

$$\mathcal{J} = \lim_{t\to\infty}\left[\frac{1}{t}\int_0^t V_{\rm s}(x(\sigma)){\rm d}\sigma\right] \quad (17)$$

is minimized and $\min_{K^*}\mathcal{J} \equiv 0$, then J is also minimized by K^* and $\min_{K^*} J \equiv 0$. Alternatively, if $0 \not\equiv \min_{K^*}\mathcal{J} < \infty$, then $J = S(x_{\rm e})$ where $x_{\rm e} = \lim_{t\to\infty} x(t)$.

Lemma 3.5 implies that if we want to consider semistable optimal control regarding the cost functional J, then we can consider semistable optimal control using a new cost functional $\mathcal{J}$. Dealing with $\mathcal{J}$ may be easier in many cases since, by dissipativity theory [10], [33], it is always a quadratic cost functional instead of a logarithmic cost functional.

Since state feedback thermodynamic control based on J is not always well defined, to make it a valid optimal control problem, we restrict our design to a class of admissible solutions.

Lemma 3.6: Consider a state dissipatible system $\mathcal{G}$. Suppose we design a state feedback controller $u = Kx$ for $\mathcal{G}$ such that $\dot{V}_{\rm s}(x(t)) \leq 0$ for all $t \geq 0$ and the closed-loop system is semistable. Let $\mathcal{K}$ be the admissible set for which $\min_{K\in\mathcal{K}}\mathcal{J}$ is well defined. Then for the entropy function $S(\cdot)$ given by (8), $\arg\min_{K\in\mathcal{K}} J = \arg\min_{K\in\mathcal{K}}\mathcal{J}$.

Now we have the main result for designing a state feedback thermodynamic controller for a state dissipatible system $\mathcal{G}$.

Theorem 3.1: Consider the system $\mathcal{G}$ given by (1) and (2) with $C = I_n$ and $D = 0$. Assume $\mathcal{G}$ is state dissipatible. Then solving the following optimal control problem (labeled as *Problem I*)

$$\min_{K\in\mathcal{K}} J = J(K, x(0))$$
$$\text{subject to} \quad u = Kx, \quad A + BK \text{ is semistable},$$

where $\mathcal{K}$ denotes the admissible set, J is given by (15), and $S(\cdot)$ is given by (8), is equivalent to solving another optimal control problem (labeled as *Problem II*)

$$\min_{K\in\mathcal{K}} \mathcal{J} = \mathcal{J}(K, x(0))$$
$$\text{subject to} \quad u = Kx, \quad A + BK \text{ is semistable}.$$

Theorem 3.1 states that Problems I and II are equivalent. Since the quadratic cost functional for Problem II is much easier to deal with than the nonlinear cost functional for Problem I, the state feedback thermodynamic control design becomes solving the equivalent Problem II. Hence, from now on, we only focus on Problem II.

As we mentioned before, the supply rate $r(u, y)$ for many control systems is a quadratic function in terms of u and y ($r(u, y) = y^{\rm T}Qy + 2y^{\rm T}Su + u^{\rm T}Ru$) [10]. The most notable example is the passive system. In this case, it follows from Theorem 5.9 of [34] that the linear system $\mathcal{G}$ possesses a quadratic storage function $V_{\rm s}(x) = x^{\rm T}Px$, where $P = P^{\rm T} \geq 0$. Hence, $\mathcal{J}$ has the form $\mathcal{J} = \lim_{t\to\infty}(1/t)\int_0^t x^{\rm T}(s)Px(s){\rm d}s$.

Lemma 3.7: For a state dissipatible system $\mathcal{G}$, we assume $r(u, y)$ is a quadratic function of $[u^{\rm T}, y^{\rm T}]^{\rm T}$. Suppose we design the controller $u = Kx$ such that $\dot{V}_{\rm s}(x(t)) \leq 0$ for all $t \geq 0$ and $\mathcal{A} \triangleq A + BK$ is semistable. Then for $\mathcal{J}$ given by (17), there exists a symmetric $P \geq 0$ such that

$$\begin{aligned}\mathcal{J} &= x^{\rm T}(0)[I_n - \mathcal{A}^{\rm T}(\mathcal{A}^{\rm T})^{\#}]P[I_n - \mathcal{A}\mathcal{A}^{\#}]x(0) \quad (18)\\ &= {\rm tr}[I_n - \mathcal{A}^{\rm T}(\mathcal{A}^{\rm T})^{\#}]P[I_n - \mathcal{A}\mathcal{A}^{\#}]V, \quad (19)\end{aligned}$$

where $V \triangleq x(0)x^{\rm T}(0)$.

Remark 3.3: If we define an operator $\mathcal{L}_A$ given by

$$\mathcal{L}_A(P) \triangleq \mathcal{A}^{\rm T}P + P\mathcal{A}, \quad (20)$$

where $P = P^{\rm T}$. Then it follows from Proposition 4.1 of [16] that $\mathcal{N}(\mathcal{L}_A) = \mathcal{R}(\mathcal{A}_A)$ and $\mathcal{N}(\mathcal{A}_A) = \mathcal{R}(\mathcal{L}_A)$. In other words, these properties imply that 1) a quadratic function is an integral of motion of

$$\dot{x} = \mathcal{A}x \quad (21)$$

if and only if it is the average of some quadratic function along the solutions of $\dot{x} = \mathcal{A}x$ and 2) a quadratic function has zero average along trajectories of $\dot{x} = \mathcal{A}x$ if and only if it is the Lie derivative of some quadratic function along the trajectories of $\dot{x} = \mathcal{A}x$. See [16] for the detailed discussion.

Since we are considering a state dissipatible system $\mathcal{G}$, without loss of generality, we assume that $\mathcal{G}$ is dissipative with respect to the supply rate $r(u, y)$. Furthermore, we assume that $r(u, y) = y^{\rm T}Qy + 2y^{\rm T}Su + u^{\rm T}Ru$, where Q and R are symmetric. Then it follows from Theorem 5.9 of [34] that there exist $P = P^{\rm T} \geq 0$ and $L \in \mathbb{R}^{p\times n}$ such that

$$A^{\rm T}P + PA - C^{\rm T}QC + L^{\rm T}L = 0. \quad (22)$$

The following definition is needed in this paper. This definition is due to [35].

Definition 3.5: Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times n}$. The pair (A, C) is *semiobservable* if

$$\bigcap_{k=1}^{n}\mathcal{N}(CA^{k-1}) = \mathcal{N}(A). \quad (23)$$

Lemma 3.8 ([35]): Consider the linear dynamical system given by (21). Then (21) is semistable if and only if for every semiobservable pair $(\mathcal{A}, R)$ with positive semidefinite R, there exists a symmetric $n \times n$ matrix $\hat{P} > 0$ such that

$$\mathcal{A}^{\rm T}\hat{P} + \hat{P}\mathcal{A} + R = 0. \quad (24)$$

Such a $\hat{P}$ is not unique.

The next result characterizes state feedback thermodynamic control as an optimization problem involving two linear matrix inequalities.

Theorem 3.2: Consider the linear control system $\mathcal{G}$. Assume that $\mathcal{G}$ is dissipative with respect to the supply rate $r(u, y)$. Furthermore, assume that $r(u, y) = y^{\rm T}Qy + 2y^{\rm T}Su + u^{\rm T}Ru$, where Q and R are symmetric matrices. Then solving Problem I is equivalent to solving the following optimal control problem

$$\min_{K\in\mathcal{K}} \mathcal{J} = \mathrm{tr}[I_n - \mathcal{A}^{\mathrm{T}}(\mathcal{A}^{\mathrm{T}})^{\#}]P[I_n - \mathcal{A}\mathcal{A}^{\#}]V$$
$$\text{subject to} \quad u = Kx, \quad (\mathcal{A}, W) \text{ is semiobservable,}$$
$$\mathcal{A}^{\mathrm{T}}P + P\mathcal{A} - C^{\mathrm{T}}QC + L^{\mathrm{T}}L = 0, \quad P \geq 0,$$
$$\mathcal{A}^{\mathrm{T}}\hat{P} + \hat{P}\mathcal{A} + W = 0, \quad \hat{P} > 0.$$

Remark 3.4: To guarantee $(\mathcal{A}, W)$ is semiobservable, one can take $W = \mathcal{A}^{\mathrm{T}}M\mathcal{A}$ where $M > 0$ is an arbitrary symmetric matrix. To see this, note that in this case $\mathcal{N}(W\mathcal{A}^{k-1}) = \mathcal{N}(\mathcal{A}^{\mathrm{T}}M\mathcal{A}^k) = \mathcal{N}(\mathcal{A}^k)$ for every $k = 1, \ldots, n$. Furthermore, note that $\mathcal{N}(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A}^k)$ for every $k = 1, \ldots, n$, then it follows that $\bigcap_{k=1}^{n} \mathcal{N}(W\mathcal{A}^{k-1}) = \bigcap_{k=1}^{n} \mathcal{N}(\mathcal{A}^k) = \mathcal{N}(\mathcal{A})$, which, by Definition 3.5, implies semiobservability.

The optimal control problem given by Theorem 3.2 is complicated in a way that the cost functional involves $\mathcal{A}^{\#}$ and $\mathcal{A}$ is a function of K. Next, we simplify this cost functional by noting the following fact.

Lemma 3.9: If $\mathcal{A}$ is semistable, then $Y \triangleq I_n - \mathcal{A}\mathcal{A}^{\#}$ is the unique matrix satisfying $\mathcal{N}(Y) = \mathcal{R}(\mathcal{A})$, $\mathcal{R}(Y) = \mathcal{N}(\mathcal{A})$, and $\mathcal{N}(\mathcal{A}) \subseteq \mathcal{N}(Y - I_n)$.

Corollary 3.1: Consider the linear control system $\mathcal{G}$. Assume that $\mathcal{G}$ is dissipative with respect to the supply rate $r(u, y)$. Furthermore, assume that $r(u, y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru$, where Q and R are symmetric matrices. Then solving Problem I is equivalent to solving the following optimal control problem

$$\min_{K\in\mathcal{K}} \mathcal{J} = \mathrm{tr}\, Y^{\mathrm{T}}PYV$$
$$\text{subject to} \quad u = Kx, \quad (\mathcal{A}, W) \text{ is semiobservable,}$$
$$\mathcal{N}(Y) = \mathcal{R}(\mathcal{A}),\ \mathcal{R}(Y) = \mathcal{N}(\mathcal{A}),$$
$$\mathcal{N}(\mathcal{A}) \subseteq \mathcal{N}(Y - I_n),$$
$$\mathcal{A}^{\mathrm{T}}P + P\mathcal{A} - C^{\mathrm{T}}QC + L^{\mathrm{T}}L = 0, \quad P \geq 0,$$
$$\mathcal{A}^{\mathrm{T}}\hat{P} + \hat{P}\mathcal{A} + W = 0, \quad \hat{P} > 0.$$

C. Output Feedback Thermodynamic Control

Now we go back to the original form of $\mathcal{G}$. Here $C \neq I_n$ and $D \neq 0$. The control aim here is to design a dynamic compensator $\mathcal{G}_{\mathrm{c}}$ denoted by $(A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}})$ and $u_{\mathrm{c}} = Ky$ so that the closed-loop system is *semistable* [19], [31] and the cost functional J is minimized. In this case, J is given by

$$J = \lim_{t\to\infty} \left[\frac{1}{t}\int_0^t \mathcal{S}(\tilde{x}(\sigma))\mathrm{d}\sigma\right], \tag{25}$$

where $\mathcal{S}(\tilde{x})$ is given by (14).

To reformulate our optimal control problem, suppose the closed-loop system $\tilde{\mathcal{G}}$ is given by

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u_{\mathrm{c}}, \tag{26}$$
$$y = \tilde{C}\tilde{x}. \tag{27}$$

Clearly, $\mathcal{G}$ is output dissipatible if and only if $\tilde{\mathcal{G}}$ is dissipative. By choosing the dynamic compensator $\mathcal{G}_{\mathrm{c}}$ such that $\tilde{\mathcal{G}}$ is output dissipatible, one can follow the similar arguments as the state feedback thermodynamic control case to obtain some similar results as follows.

Proposition 3.4: Consider the closed-loop system $\tilde{\mathcal{G}}$. Assume $\tilde{\mathcal{G}}$ is dissipative. Then solving the following optimal control problem

$$\min_{(K,A_{\mathrm{c}},B_{\mathrm{c}},C_{\mathrm{c}})\in\tilde{\mathcal{K}}} J = J(K, A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}}, \tilde{x}(0))$$
$$\text{subject to} \quad u_{\mathrm{c}} = Ky, \quad \tilde{A} + \tilde{B}K \text{ is semistable,}$$

where $\tilde{\mathcal{K}}$ denotes the admissible set, J is given by (25), and $\mathcal{S}(\cdot)$ is given by (14), is equivalent to solving another optimal control problem (labeled as *Problem III*)

$$\min_{(K,A_{\mathrm{c}},B_{\mathrm{c}},C_{\mathrm{c}})\in\tilde{\mathcal{K}}} \mathcal{J} = \mathcal{J}(K, A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}}, \tilde{x}(0))$$
$$\text{subject to} \quad u_{\mathrm{c}} = Ky, \quad \tilde{A} + \tilde{B}K \text{ is semistable,}$$

where $\mathcal{J}$ is given by

$$\mathcal{J} = \lim_{t\to\infty} \left[\frac{1}{t}\int_0^t V_{\mathrm{s}}(\tilde{x}(\sigma))\mathrm{d}\sigma\right], \tag{28}$$

Proposition 3.5: Consider the closed-loop system $\tilde{\mathcal{G}}$. Assume that $\tilde{\mathcal{G}}$ is dissipative with respect to the supply rate $r(u_{\mathrm{c}}, y)$. Furthermore, assume that $r(u_{\mathrm{c}}, y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su_{\mathrm{c}} + u_{\mathrm{c}}^{\mathrm{T}}Ru_{\mathrm{c}}$, where Q and R are symmetric matrices. Then solving Problem III is equivalent to solving the following optimal control problem

$$\min_{(K,A_{\mathrm{c}},B_{\mathrm{c}},C_{\mathrm{c}})\in\tilde{\mathcal{K}}} \mathcal{J} = \mathrm{tr}[I_{n+n_{\mathrm{c}}} - \tilde{\mathcal{A}}^{\mathrm{T}}(\tilde{\mathcal{A}}^{\mathrm{T}})^{\#}]P [I_{n+n_{\mathrm{c}}} - \tilde{\mathcal{A}}\tilde{\mathcal{A}}^{\#}]\tilde{V}$$
$$\text{subject to} \quad u_{\mathrm{c}} = K\tilde{x}, \quad (\tilde{\mathcal{A}}, W) \text{ is semiobservable,}$$
$$\tilde{\mathcal{A}}^{\mathrm{T}}P + P\tilde{\mathcal{A}} - \tilde{C}^{\mathrm{T}}Q\tilde{C} + L^{\mathrm{T}}L = 0, \quad P \geq 0,$$
$$\tilde{\mathcal{A}}^{\mathrm{T}}\hat{P} + \hat{P}\tilde{\mathcal{A}} + W = 0, \quad \hat{P} > 0,$$

where $\tilde{\mathcal{A}} \triangleq \tilde{A} + \tilde{B}K$.

Corollary 3.2: Consider the closed-loop system $\tilde{\mathcal{G}}$. Assume that $\tilde{\mathcal{G}}$ is dissipative with respect to the supply rate $r(u_{\mathrm{c}}, y)$. Furthermore, assume that $r(u_{\mathrm{c}}, y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su_{\mathrm{c}} + u_{\mathrm{c}}^{\mathrm{T}}Ru_{\mathrm{c}}$, where Q and R are symmetric matrices. Then solving Problem III is equivalent to solving the following optimal control problem

$$\min_{(K,A_{\mathrm{c}},B_{\mathrm{c}},C_{\mathrm{c}})\in\tilde{\mathcal{K}}} \mathcal{J} = \mathrm{tr}\, \tilde{Y}^{\mathrm{T}}P\tilde{Y}\tilde{V}$$
$$\text{subject to} \quad u_{\mathrm{c}} = K\tilde{x}, \quad (\tilde{\mathcal{A}}, W) \text{ is semiobservable,}$$
$$\mathcal{N}(\tilde{Y}) = \mathcal{R}(\tilde{\mathcal{A}}),\ \mathcal{R}(\tilde{Y}) = \mathcal{N}(\tilde{\mathcal{A}}),$$
$$\mathcal{N}(\tilde{\mathcal{A}}) \subseteq \mathcal{N}(\tilde{Y} - I_{n+n_{\mathrm{c}}}),$$
$$\tilde{\mathcal{A}}^{\mathrm{T}}P + P\tilde{\mathcal{A}} - \tilde{C}^{\mathrm{T}}Q\tilde{C} + L^{\mathrm{T}}L = 0, \quad P \geq 0,$$
$$\tilde{\mathcal{A}}^{\mathrm{T}}\hat{P} + \hat{P}\tilde{\mathcal{A}} + W = 0, \quad \hat{P} > 0,$$

where $\tilde{\mathcal{A}} \triangleq \tilde{A} + \tilde{B}K$.

Hence, the output feedback thermodynamic control design becomes an optimization problem involving two linear matrix inequalities and a dynamic compensator. The design procedure is identical to that of the state feedback thermodynamic control design.

IV. Conclusions

Motivated by the system thermodynamic theory, in this paper we design a semistable controller for the linear dynamical systems based on the notion of entropy for deterministic systems. Specifically, we design a state feedback controller and a dynamic compensator so that the closed-loop system is semistable and the time-averaging of the entropy function is minimized. In other words, the time-averaging of the "heat" in the system is minimized and the system energy reaches certain stable energy level. This framework is the first attempt to discuss the thermal role in control systems design. Apparently, we hope this work will initiate the endeavor towards putting thermodynamics back in control since this thermodynamic idea is aligned with Lyapunov's work on stability which studies the stability of dynamical systems using generalized energy functions.

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