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On Higher Order Derivatives of Lyapunov Functions

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Abstract— **This note is concerned with a class of differential inequalities in the literature that involve higher order derivatives of Lyapunov functions and have been proposed to infer asymptotic stability of a dynamical system without requiring the first derivative of the Lyapunov function to be negative definite. We show that whenever a Lyapunov function satisfies these conditions, we can explicitly construct another (standard) Lyapunov function that is positive definite and has a negative definite first derivative. Our observation shows that a search for a standard Lyapunov function parameterized by higher order derivatives of the vector field is less conservative than the previously proposed conditions. Moreover, unlike the previous inequalities, the new inequality can be checked with a convex program. This is illustrated with an example where sum of squares optimization is used.**

I. HIGHER ORDER DERIVATIVES OF LYAPUNOV FUNCTIONS

Consider the dynamical system

$$\dot{x} = f(x), \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ has an equilibrium point at the origin (i.e., $f(0) = 0$), and satisfies the standard assumptions for existence and uniqueness of solutions; see e.g. [1, Chap. 3]. By higher order derivatives of a Lyapunov function $V(x) : \mathbb{R}^n \to \mathbb{R}$ we mean the time derivatives of V along the trajectories of (1) given by $\dot{V}(x) = \langle \frac{\partial V(x)}{\partial x}, f(x)\rangle, \ddot{V}(x) = \langle \frac{\partial \dot{V}(x)}{\partial x}, f(x)\rangle, \ldots, V^{(m)}(x) = \langle \frac{\partial V^{(m-1)}(x)}{\partial x}, f(x)\rangle$. In [2], Butz showed that existence of a three times continuously differentiable Lyapunov function $V(x)$ satisfying

$$\tau_2 \dddot{V}(x) + \tau_1 \ddot{V}(x) + \dot{V}(x) < 0 \tag{2}$$

for all $x \neq 0$ and for some nonnegative scalars τ_1, τ_2 implies global asymptotic stability of the origin of (1).[1] Note that unlike the standard condition $\dot{V}(x) < 0$, condition (2) is not jointly convex in the scalars τ_i and the parameters of the Lyapunov function $V(x)$. Therefore, computational techniques based on convex optimization cannot be used to search for a Lyapunov function satisfying (2). In [3], Heinen and Vidyasagar adapted the condition of Butz to establish a result on boundedness of the trajectories. More recently, Meigoli and Nikravesh [4], [5] have generalized the result of Butz to derivatives of higher order and to the case of time-varying systems. A simplified version of their result that is most relevant for our purposes deals with a differential inequality of the type

$$V^{(m)}(x) + \tau_{m-1} V^{(m-1)}(x) + \cdots + \tau_1 \dot{V}(x) < 0. \tag{3}$$

It is shown in [4] that if the corresponding characteristic polynomial

$$p(s) = s^m + \tau_{m-1} s^{m-1} + \cdots + \tau_1 s$$

is Hurwitz (and some additional standard assumptions hold), then the inequality in (3) proves global asymptotic stability. It is later shown in [5, Cor. 1] that this condition can be weakened to $p(s)$ having nonnegative coefficients. We will show that no matter what types of conditions on $V(x)$ and the scalars $\tau_{m-1}, \ldots, \tau_1$ are placed, if the system is globally asymptotically stable and the inequality (3) holds (which is in particular the case if inequality (3) is used to establish global asymptotic stability), then we can explicitly extract a standard Lyapunov function from it. This will follow as a corollary of the following simple and general fact.

Theorem 1.1: Consider a system $\dot{x} = f(x)$ that is known to be globally asymptotically stable. Suppose there exists a continuously differentiable function $W(x)$ whose derivative $\dot{W}(x)$ along the trajectories is negative definite and satisfies $W(0) = 0$. Then, $W(x)$ must be positive definite.

Proof: Assume by contradiction that there exists a nonzero point $\bar{x} \in \mathbb{R}^n$ such that $W(\bar{x}) \leq 0$. We evaluate the Lyapunov function $W(x)$ along the trajectory of the system starting from the initial condition $\bar{x}$. The value of the Lyapunov function is nonpositive to begin with and will strictly decrease because $\dot{W}(x) < 0$. Therefore, the value of the Lyapunov function can never become zero. On the other hand, since we know that the vector field is globally asymptotically stable, trajectories of the system must all go to the origin, where we have $W(0) = 0$. This gives us a contradiction. ■

Corollary 1.1: Consider a globally asymptotically stable dynamical system $\dot{x} = f(x)$. Suppose that the higher order differential inequality in (3) holds for some scalars $\tau_1, \ldots, \tau_{m-1}$ and for some m times continuously differentiable Lyapunov function $V(x)$ with $V(0) = 0$. Then,

$$W(x) = V^{(m-1)}(x) + \tau_{m-1} V^{(m-2)} + \cdots + \tau_2 \dot{V}(x) + \tau_1 V(x) \tag{4}$$

is continuously differentiable and positive definite and its derivative $\dot{W}(x)$ is negative definite.

Proof: Continuous differentiability of $W(x)$ and negative definiteness of $\dot{W}(x)$ follow from condition (3). Since $f(0) = 0$, we have that $\dot{V}^{(m-1)}(0) = \cdots = \dot{V}(0) = 0$. This together with the assumption that $V(0) = 0$ implies that $W(0) = 0$. Positive definiteness of $W(x)$ follows from Theorem 1.1. ■

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[1]Throughout, when we state a condition involving $V^{(m)}(x)$, there is always the implicit assumption that the vector field $f(x)$ is $m-1$ times differentiable.

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This corollary shows that instead of imposing the inequality in (3) along with conditions on $V(x)$ and the scalars $\tau_1,\ldots,\tau_{m-1}$ (such as positive definiteness of $V(x)$ and nonnegativity of τ_is) as proposed by the works in [2], [4], [5], we are better off imposing no conditions on $V(x)$ and the scalars τ_i individually but instead require

$$V^{(m-1)}(0)+\tau_{m-1}V^{(m-2)}(0)+\cdots+\tau_2\dot{V}(0)+\tau_1V(0)=0, \quad (5)$$

and

$$V^{(m-1)}(x)+\tau_{m-1}V^{(m-2)}(x)+\cdots+\tau_2\dot{V}(x)+\tau_1V(x)>0 \quad (6)$$

$$V^{(m)}(x)+\tau_{m-1}V^{(m-1)}(x)+\cdots+\tau_2\ddot{V}(x)+\tau_1\dot{V}(x)<0 \quad (7)$$

for all $x\neq 0$. In other words, we simply impose the standard Lyapunov conditions on a Lyapunov function of the specific structure in (4).[2] By Corollary 1.1, the latter approach is always less conservative than the former.

Now to get around the issue of non-convexity of inequalities (5)-(7) in the decision variables τ_i and the parameters of $V(x)$, we can simply search for different functions $V_1(x),\ldots,V_m(x)$ (with no sign conditions on them individually), such that

$$V_m^{(m-1)}(0)+V_{m-1}^{(m-2)}(0)+\cdots+\dot{V}_2(0)+V_1(0)=0 \quad (8)$$

and

$$V_m^{(m-1)}(x)+V_{m-1}^{(m-2)}(x)+\cdots+\dot{V}_2(x)+V_1(x)>0 \quad (9)$$

$$V_m^{(m)}(x)+V_{m-1}^{(m-1)}(x)+\cdots+\ddot{V}_2(x)+\dot{V}_1(x)<0 \quad (10)$$

for all $x\neq 0$. These three conditions are convex and it should be clear that if conditions (5)-(7) are satisfied for some function $V(x)$ and some scalars τ_i, then conditions (8)-(10) are satisfied with $V_i=\tau_iV$ for $i=1,\ldots,m-1$ and $V_m=V$.

We refer the reader to [6], [7] for more discussion and also for a discrete time analogue of these results.

II. An example

The following example shows the potential advantages of using higher order derivatives of Lyapunov functions. It also demonstrates the use of convex optimization for imposing constraints of the type (8)-(10). We assume the reader is familiar with the sum of squares (sos) relaxation of polynomial nonnegativity and its formulation as a linear matrix inequality. See [8], [9].

Example 2.1: Consider the following polynomial dynamics

$$\begin{aligned}\dot{x_1} &= -0.8x_1^3-1.5x_1x_2^2-0.4x_1x_2-0.4x_1x_3^2-1.1x_1\\ \dot{x_2} &= x_1^4+x_3^6+x_1^2x_3^4\\ \dot{x_3} &= -0.2x_1^2x_3-0.7x_2^2x_3-0.3x_2x_3-0.5x_3^3-0.5x_3.\end{aligned}$$

If we use SOSTOOLS [10] to search for a standard quadratic Lyapunov function $V(x)$ that is a sum of squares and for which $-\dot{V}(x)$ is a sum of squares, the search will be infeasible. If needed, this can be turned into a proof (using duality of semidefinite programming) that no such Lyapunov function exists. Instead, we search for $V_1(x)$ and $V_2(x)$ such that $\dot{V}_2(0)+V_1(0)=0$, and $\dot{V}_2(x)+V_1(x)$ and $-(\ddot{V}_2(x)+\dot{V}_1(x))$ are both sums of squares.[3] In principle, we can start with a linear parametrization for $V_1(x)$ and $V_2(x)$ since there is no positivity constraint on them directly. For this example, a linear parametrization will be infeasible. However, if we search for a linear function $V_2(x)$ and a quadratic function $V_1(x)$, SOSTOOLS and the semidefinite programming solver SeDuMi [11] find

$$\begin{aligned}V_1(x) &= 0.47x_1^2+0.89x_2^2+0.91x_3^2\\ V_2(x) &= 0.36x_2.\end{aligned}$$

Therefore, the origin is asymptotically stable. The standard Lyapunov function constructed from $V_1(x)$ and $V_2(x)$ will be the following sextic polynomial:

$$\begin{aligned}&W(x)=\dot{V}_2(x)+V_1(x)=\\ &0.36x_1^4+0.36x_1^2x_3^4+0.47x_1^2+0.89x_2^2+0.36x_3^6+0.91x_3^2.\end{aligned}$$

It is easy to see that $W(x)-\frac{1}{4}(x_1^2+x_2^2+x_3^2)$ is a sum of squares. This confirms positivity of $W(x)$ and also shows that it is radially unbounded. Therefore, the origin is in fact globally asymptotically stable.

One could of course forget about higher order derivatives all together and directly search for a standard degree six polynomial Lyapunov function using SOSTOOLS. A simple calculation shows however that this search would have had 68 more decision variables than our search for the quadratic and linear functions $V_1(x)$ and $V_2(x)$.

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[2]We can see from (4) that $W(x)$ is a Lyapunov function that has the vector field $f(x)$ and its derivatives in its parametrization. This is in some sense reminiscent of Krasovskii's method, where the vector field $f(x)$ is used in the parametrization of a Lyapunov function. See e.g. [1, p. 183].

[3]It is easy to check whether a sum of squares polynomial is strictly positive as opposed to merely nonnegative. This is the case throughout this example.