2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

An Improved Algorithm for Partial Fraction Expansion Based Frequency Weighted Balanced Truncation

Wan Mariam Wan Muda, Victor Sreeram, Herbert Ho-Ching Iu

Abstract— **In this paper, we present an improvement to frequency weighted balanced truncation technique based on well-known partial fraction expansion. The method yields stable reduced-order model for double-sided weightings. A numerical example and comparison with other well-known techniques shows that a significant approximation error reduction can be achieved using this improvement.**

I. Introduction

Frequency weighted balanced truncation was first introduced by Enns [1] based on a modification of balanced truncation [2]. The method may use input weighting, output weighting or both. The stability of the reduced order model is guaranteed only when one weighting is present. The original Lin and Chiu's technique [3] and its generalization [4] provide a simple modification to Enns' technique to guarantee the stability in the case of double-sided weightings provided that there are no pole-zero cancellations between the original system and the weights [5]. Another modification to Enns' technique was proposed by Wang et al. [6] which not only guarantees stability in the case of double-sided weightings but also yields simple and elegant error bounds.

Inspired by the method [7], Al-Saggaf and Franklin [8], [9] proposed a frequency weighted balanced truncation technique based on partial fraction expansion. However, their method [8], [9] has significant limitations: (i) it can be used with single-sided weight only, (ii) the output matrix of the input weight or the input matrix of the output weight must be square and nonsingular and (iii) the original system and the weighting function have to be both strictly proper. Their method was then generalized by Sreeram and Anderson [10] to include double-sided and proper weighting functions. Ghafoor and Sreeram [11] modified their method in [10] by combining it with unweighted balancing [2] to obtain simple and elegant error bounds. Although the method gives a low approximation error but is adhoc with no theoretical justification. Improved technique was proposed by Sahlan and Sreeram [12] which is conceptually simple and elegant. Errors obtained although slightly lower than the Enns' method but may be considered still very large.

In this paper, we present further improvements to partial fraction expansion technique which yields substantial approximation error reduction compared to Enns' technique. The method is also elegant with simple and easily computable error bounds. The method is illustrated by an example.

Wan Mariam Wan Muda, Victor Sreeram, Herbert Ho-Ching Iu are with School of Electrical, Electronic and Computer Engineering, University of Western Australia, 35 Stirling Highway, Crawley, Western Australia 6009, Australia. {wmariam,sreeram,herbert}@ee.uwa.edu.au

II. Preliminaries

This section reviews some of the well-known frequency weighted balanced truncation techniques. Let $G(s)$, $V(s)$ and $W(s)$ be the stable original system and the stable input and output weights respectively. Let $\{A,B,C,D\}$, $\{A_v,B_v,C_v,D_v\}$ and $\{A_w,B_w,C_w,D_w\}$ be their corresponding minimal realizations respectively. Consider the augmented system $W(s)G(s)V(s)$ represented by the following realization:

$$\begin{aligned} W(s)G(s)V(s) &= \left[\begin{array}{ccc|c} A_w & B_wC & B_wDC_v & B_wDD_v \\ 0 & A & BC_v & BD_v \\ 0 & 0 & A_v & B_v \\ \hline C_w & D_wC & D_wDC_v & D_wDD_v \end{array}\right] \\ &= \left[\begin{array}{cc} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{array}\right] \end{aligned} \quad (1)$$

The controllability and observability Gramians of the augmented realization $\{\tilde{A},\tilde{B},\tilde{C},\tilde{D}\}$ are given by:

$$\tilde{P} = \begin{bmatrix} P_w & P_{12} & P_{13} \\ P_{12}^T & P_E & P_{23} \\ P_{13}^T & P_{23}^T & P_v \end{bmatrix} \quad \tilde{Q} = \begin{bmatrix} Q_w & Q_{12} & Q_{13} \\ Q_{12}^T & Q_E & Q_{23} \\ Q_{13}^T & Q_{23}^T & Q_v \end{bmatrix} \quad (2)$$

where $\tilde{P}$ and $\tilde{Q}$ satisfy the following Lyapunov equations:

$$\tilde{A}\tilde{P} + \tilde{P}\tilde{A}^T + \tilde{B}\tilde{B}^T = 0 \quad (3a)$$
$$\tilde{A}^T\tilde{Q} + \tilde{Q}\tilde{A} + \tilde{C}^T\tilde{C} = 0 \quad (3b)$$

Assuming that there are no pole-zero cancellations in $W(s)G(s)V(s)$, the Gramians, $\tilde{P}$ and $\tilde{Q}$ are positive definite.

A. Enns' Technique

Expanding $(2,2)$ blocks of (3) yield the following equations:

$$AP_E + P_EA^T + X_E = 0 \quad (4a)$$
$$A^TQ_E + Q_EA + Y_E = 0 \quad (4b)$$

where

$$\begin{aligned} X_E &= BC_vP_{23}^T + P_{23}C_v^TB^T + BD_vD_v^TB^T \\ Y_E &= C^TB_w^TQ_{12} + Q_{12}^TB_wC + C^TD_w^TD_wC \end{aligned}$$

Diagonalizing the weighted Gramians $\{P_E,Q_E\}$ yields:

$$T_E^{-1}P_ET_E^{-T} = T_E^TQ_ET_E = diag(\sigma_1,\sigma_2,\ldots,\sigma_r,\sigma_{r+1},\ldots,\sigma_n)$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > \sigma_{r+1} \geq \ldots \geq \sigma_n > 0$.

978-1-4577-0079-8/11/$26.00 ©2011 AACC

Transforming and partitioning the original system realization, we have

$$\left[\begin{array}{c|c} T_E^{-1}AT_E & T_E^{-1}B \\ \hline CT_E & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$$

Enns' reduced-order model is then given by $G_E(s) = \{A_{11}, B_1, C_1, D\}$.

Essentially, Enns' technique is based on diagonalizing simultaneously the solutions of Lyapunov equations as given in (4). However, Enns' technique cannot guarantee the stability of reduced order models as X_E and Y_E may not be positive semidefinite. Several modifications to Enns' technique are proposed in the literature to overcome the stability problem.

B. Sreeram and Anderson's Partial Fraction Expansion Based Technique

Sreeram and Anderson generalized the partial fraction expansion based technique proposed in [8], [9] to include proper weighting functions [10]. The technique first transforms the augmented system realization (1) into a block diagonal form by the following transformation matrix:

$$\tilde{T} = \begin{bmatrix} I & -Y & R \\ 0 & I & X \\ 0 & 0 & I \end{bmatrix} \tag{5}$$

Note that even though the technique [10] considers only strictly proper original systems, the derivation presented in this paper is generalized to include proper original systems as these equations will be required in the main section of the paper. Transforming the augmented system realization (1), we have:

$$\begin{aligned} W(s)G(s)V(s) &= \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \\ &= \begin{bmatrix} \tilde{T}^{-1}\tilde{A}\tilde{T} & \tilde{T}^{-1}\tilde{B} \\ \tilde{C}\tilde{T} & \tilde{D} \end{bmatrix} \\ &= \left[\begin{array}{ccc|c} A_w & X_{12} & X_{13} & X_1 \\ 0 & A & X_{23} & X_2 \\ 0 & 0 & A_v & B_v \\ \hline C_w & Y_1 & Y_2 & D_wDD_v \end{array}\right] \\ &= \hat{W}(s) + \hat{G}(s) + \hat{V}(s) \\ &= \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \end{aligned} \tag{6}$$

where

$$\begin{aligned} X_{12} &= YA - A_wY + B_wC = 0 && \text{(7a)} \\ X_{23} &= AX - XA_v + BC_v = 0 && \text{(7b)} \\ X_{13} &= A_wR - RA_v + B_wCX + YAX + B_wDC_v \\ &\quad + YBC_v - YXA_v = 0 && \text{(7c)} \\ X_1 &= B_wDD_v + YBD_v - YXB_v - RB_v && \text{(7d)} \\ X_2 &= BD_v - XB_v && \text{(7e)} \\ Y_1 &= D_wC - C_wY && \text{(7f)} \\ Y_2 &= D_wCX + D_wDC_v + C_wR && \text{(7g)} \\ \hat{D} &= D_wDD_v && \text{(7h)} \end{aligned}$$

Remark 1: In (7c), matrix R exists if and only if $A_v \neq A_w$.

Instead of balancing and truncating the original system $\{A, B, C\}$, the method balances and truncates the new system $\{A, X_2, Y_1\}$ to obtain the reduced-order models.

Note that the frequency weighted error can be large with this method. However, the error can be reduced for strictly proper original system and the weights ($D = 0, D_v = 0$ and $D_w = 0$) if the reduction error is made to have zeros at the poles of input weight or output weight as shown in [10].

C. Sahlan and Sreeram's Partial Fraction Expansion Based Technique

As in the previous method, Sahlan and Sreeram's method [12] involves decomposing the augmented system $W(s)G(s)V(s)$ into $\hat{W}(s) + \hat{G}(s) + \hat{V}(s)$ (see equation (6)) using partial fraction expansion. These terms are then recombined to obtain a new augmented system $\overline{W}(s)\overline{G}(s)\overline{V}(s)$ such that

$$W(s)G(s)V(s) = \hat{W}(s) + \hat{G}(s) + \hat{V}(s) = \overline{W}(s)\overline{G}(s)\overline{V}(s) \tag{8}$$

where $\overline{G}(s) = \{A, \overline{B}, \overline{C}, \overline{D}\}$ is the new original system, and $\overline{V}(s) = \{A_v, B_v, \overline{C}_v, \overline{D}_v\}$ and $\overline{W}(s) = \{A_w, \overline{B}_w, C_w, \overline{D}_w\}$ are the new input and output weights respectively. The new parameters in the above equations are given by

$$\begin{aligned} \overline{B}_w &= \begin{bmatrix} B_w & A_w & I \end{bmatrix} && \text{(9a)} \\ \overline{D}_w &= \begin{bmatrix} D_w & C_w & 0 \end{bmatrix} && \text{(9b)} \\ \overline{B} &= \begin{bmatrix} B & -X & AX \end{bmatrix} && \text{(9c)} \\ \overline{C} &= \begin{bmatrix} C \\ -Y \\ YA \end{bmatrix} && \text{(9d)} \\ \overline{D} &= \begin{bmatrix} D & 0 & CX \\ 0 & 0 & R \\ YB & -R-YX & YAX \end{bmatrix} && \text{(9e)} \\ \overline{C}_v &= \begin{bmatrix} C_v \\ A_v \\ I \end{bmatrix} && \text{(9f)} \\ \overline{D}_v &= \begin{bmatrix} D_v \\ B_v \\ 0 \end{bmatrix} && \text{(9g)} \end{aligned}$$

Using the matrices defined in (9), the equations in (7) can now be expressed as:

$$\begin{aligned} X_{12} &= \overline{B}_w\overline{C} && \text{(10a)} \\ X_{23} &= \overline{B}\,\overline{C}_v && \text{(10b)} \\ X_{13} &= \overline{B}_w\overline{D}\,\overline{C}_v && \text{(10c)} \\ X_1 &= \overline{B}_w\overline{D}\,\overline{D}_v && \text{(10d)} \\ X_2 &= \overline{B}\,\overline{D}_v && \text{(10e)} \\ Y_1 &= \overline{D}_w\overline{C} && \text{(10f)} \\ Y_2 &= \overline{D}_w\overline{D}\,\overline{C}_v && \text{(10g)} \\ \hat{D} &= \overline{D}_w\overline{D}\,\overline{D}_v && \text{(10h)} \end{aligned}$$

Remark 2: From equation (8), we have

$$W(s)G(s)V(s) = \overline{W}(s)\overline{G}(s)\overline{V}(s)$$

This relation is valid even if

$$W(s)G(s)V(s) \neq \hat{W}(s)+\hat{G}(s)+\hat{V}(s)$$

which is the case when R in (7c) does not exist (see *Remark 1*).

Diagonalizing the weighted Gramians $\{\overline{P},\overline{Q}\}$ of the new system $\overline{G}(s)=\{A,\overline{B},\overline{C},\overline{D}\}$ which satisfy

$$A\overline{P}+\overline{P}A^T+\overline{B}\,\overline{B}^T = 0 \quad (11a)$$
$$A^T\overline{Q}+\overline{Q}A+\overline{C}^T\overline{C} = 0 \quad (11b)$$

yields

$$T_{SS}^{-1}\overline{P}T_{SS}^{-T}=T_{SS}^T\overline{Q}T_{SS}=diag(\sigma_1,\sigma_2,\ldots,\sigma_r,\sigma_{r+1},\ldots,\sigma_n) \quad (12)$$

where $\sigma_1\geq\sigma_2\geq\ldots\geq\sigma_r>\sigma_{r+1}\geq\ldots\geq\sigma_n>0$. Instead of reducing $G(s)$, in this technique the new original system $\overline{G}(s)$ is reduced using balanced truncation to obtain $\overline{G}_r(s)$. The final reduced-order model $G_r(s)$ is obtained by simply deleting the extra rows in $\overline{C}_r$, extra columns in $\overline{B}_r$ and both extra rows and columns in $\overline{D}_r$. Since the realization $\{A,\overline{B},\overline{C}\}$ is minimal and the weighted Gramians $\{\overline{P},\overline{Q}\}$ satisfy the Lyapunov equations (11), the technique yields stable models in the case of double-sided weightings. Although the method is simple and elegant, approximation error reduction obtained from this technique is very small and is often negligible.

In the next section, we present an improvement to this technique **to obtain a significant weighted error reduction not reported so far with any technique**.

III. Main Results

The proposed method can be explained using the following steps.

Step 1 The augmented system $W(s)G(s)V(s)$ is decomposed using partial fraction expansion to obtain $\hat{W}(s)+\hat{G}(s)+\hat{V}(s)$. This step is the same in all three partial fraction expansion techniques [10]–[12] and can be written as follows

$$W(s)G(s)V(s)=\hat{W}(s)+\hat{G}(s)+\hat{V}(s)$$

Step 2 The block diagonalized augmented system $\hat{W}(s)+\hat{G}(s)+\hat{V}(s)$ is reconstructed to find a new augmented system $\overline{W}(s)\overline{G}(s)\overline{V}(s)$. This step is the same as in [12] and is written as

$$\hat{W}(s)+\hat{G}(s)+\hat{V}(s)=\overline{W}(s)\overline{G}(s)\overline{V}(s)$$

Step 3 Intermediate reduced order model $\overline{G}_r(s)=\overline{C}_r(sI-A_r)^{-1}\overline{B}_r+\overline{D}_r$ is obtained from $\overline{G}(s)$ by using balanced truncation. This step is same as in [12].

Step 4 which is the final step is different to the technique of [12]. In [12], the final reduced order model is obtained by directly deleting the extra rows in $\overline{C}_r$, extra columns in $\overline{B}_r$ and extra rows and columns in $\overline{D}_r$. In the proposed method, if $G_r(s)=C_r(sI-A_r)^{-1}B_r+D_r$ is the final reduced-order model then the matrices C_r,B_r and D_r are chosen such that

$$W(s)G_r(s)V(s)=\overline{W}(s)\overline{G}_r(s)\overline{V}(s)$$

To find the final reduced-order model $G_r(s)$ in the proposed technique, let $G_r(s)=C_r(sI-A_r)^{-1}B_r+D_r$ with $D_r=D$, then the augmented system $W(s)G_r(s)V(s)$ is given by:

$$W(s)G_r(s)V(s) = \left[\begin{array}{ccc|c} A_w & B_wC_r & B_wDC_v & B_wDD_v \\ 0 & A_r & B_rC_v & B_rD_v \\ 0 & 0 & A_v & B_v \\ \hline C_w & D_wC_r & D_wDC_v & D_wDD_v \end{array}\right] = \begin{bmatrix} \tilde{A}_r & \tilde{B}_r \\ \tilde{C}_r & \tilde{D}_r \end{bmatrix}$$

Let $\tilde{T}_r=\begin{bmatrix} I & -Y_r & R_r \\ 0 & I & X_r \\ 0 & 0 & I \end{bmatrix}$ be the transformation matrix required to take the augmented system to a block diagonal form, then

$$W(s)G_r(s)V(s) = \begin{bmatrix} \tilde{T}_r^{-1}\tilde{A}_r\tilde{T}_r & \tilde{T}_r^{-1}\tilde{B}_r \\ \tilde{C}_r\tilde{T}_r & \tilde{D}_r \end{bmatrix} = \left[\begin{array}{ccc|c} A_w & X_{12,r} & X_{13,r} & X_{1,r} \\ 0 & A_r & X_{23,r} & X_{2,r} \\ 0 & 0 & A_v & B_v \\ \hline C_w & Y_{1,r} & Y_{2,r} & D_wDD_v \end{array}\right] \quad (13)$$

where

$$\begin{aligned} X_{12,r} &= Y_rA_r-A_wY_r+B_wC_r=0 && (14a)\\ Y_{1,r} &= D_wC_r-C_wY_r && (14b)\\ X_{23,r} &= A_rX_r-X_rA_v+B_rC_v=0 && (14c)\\ X_{2,r} &= B_rD_v-X_rB_v && (14d)\\ X_{13,r} &= A_wR_r-R_rA_v+B_wC_rX_r+Y_rA_rX_r+B_wDC_v \\ &\quad +Y_rB_rC_v-Y_rX_rA_v=0 && (14e)\\ X_{1,r} &= B_wDD_v+Y_rB_rD_v-Y_rX_rB_v-R_rB_v \\ Y_{2,r} &= D_wC_rX_r+D_wDC_v+C_wR_r && (14f)\\ \hat{D}_r &= D_wDD_v && (14g) \end{aligned}$$

Since we know $\overline{G}_r(s)$ from **Step 3**, $\overline{W}(s)$ and $\overline{V}(s)$ from **Step 2**, we can write the augmented system as

$$\begin{aligned}&\overline{W}(s)\overline{G}_r(s)\overline{V}(s)\\ &= \begin{bmatrix} A_w & \overline{B}_w \\ C_w & \overline{D}_w \end{bmatrix}\begin{bmatrix} A_r & \overline{B}_r \\ \overline{C}_r & \overline{D}_r \end{bmatrix}\begin{bmatrix} A_v & B_v \\ \overline{C}_v & \overline{D}_v \end{bmatrix}\\ &= \left[\begin{array}{ccc|c} A_w & \overline{B}_w\overline{C}_r & \overline{B}_w\overline{D}_r\overline{C}_v & \overline{B}_w\overline{D}_r\overline{D}_v \\ 0 & A_r & \overline{B}_r\overline{C}_v & \overline{B}_r\overline{D}_v \\ 0 & 0 & A_v & B_v \\ \hline C_w & \overline{D}_w\overline{C}_r & \overline{D}_w\overline{D}_r\overline{C}_v & \overline{D}_w\overline{D}_r\overline{D}_v \end{array}\right] \end{aligned} \quad (15)$$

To find $G_r(s)$ such that

$$W(s)G_r(s)V(s)=\overline{W}(s)\overline{G}_r(s)\overline{V}(s)$$

we need to equate equations (13) and (15). This gives

$$X_{12,r} = \overline{B}_w\overline{C}_r \quad (16a)$$
$$Y_{1,r} = \overline{D}_w\overline{C}_r \quad (16b)$$
$$X_{23,r} = \overline{B}_r\overline{C}_v \quad (16c)$$
$$X_{2,r} = \overline{B}_r\overline{D}_v \quad (16d)$$
$$X_{13,r} = \overline{B}_w\overline{D}_r\overline{C}_v \quad (16e)$$
$$X_{1,r} = \overline{B}_w\overline{D}_r\overline{D}_v \quad (16f)$$
$$Y_{2,r} = \overline{D}_w\overline{D}_r\overline{C}_v \quad (16g)$$
$$\hat{D}_r = \overline{D}_w\overline{D}_r\overline{D}_v \quad (16h)$$

and

$$\overline{D}_r = \begin{bmatrix} D & 0 & C_rX_r \\ 0 & 0 & R_r \\ Y_rB_r & -R_r-Y_rX_r & Y_rA_rX_r \end{bmatrix} \quad (17)$$

Rewriting the above equations we get

$$X_{12,r} = Y_rA_r - A_wY_r + B_wC_r = \overline{B}_w\overline{C}_r \quad (18a)$$
$$Y_{1,r} = D_wC_r - C_wY_r = \overline{D}_w\overline{C}_r \quad (18b)$$
$$X_{23,r} = A_rX_r - X_rA_v + B_rC_v = \overline{B}_r\overline{C}_v \quad (18c)$$
$$X_{2,r} = B_rD_v - X_rB_v = \overline{B}_r\overline{D}_v \quad (18d)$$

The equations (18a) and (18b) can be written as

$$\begin{bmatrix} -I\otimes A_w + A_r^T\otimes I & I\otimes B_w \\ -I\otimes C_w & I\otimes D_w \end{bmatrix}\begin{bmatrix} Vec(Y_r) \\ Vec(C_r) \end{bmatrix} = \begin{bmatrix} Vec(\overline{B}_w\overline{C}_r) \\ Vec(\overline{D}_w\overline{C}_r) \end{bmatrix} \quad (19)$$

where $Vec(X)$ denotes the vector formed by stacking the columns of X into one long vector. The coefficient matrix on the left of the above equation has full rank, guaranteeing solvability of the equation when

$$\begin{bmatrix} -A_w+\lambda I & B_w \\ -C_w & D_w \end{bmatrix}$$

has full rank for all $\lambda = \lambda_i(A_r), i = 1,\dots,r$ [13], [14], where $\lambda(X)$ denotes the eigenvalues of X. However, there is a unique solution if and only if the matrix on the left of (19) is square. Similarly X_r and B_r, provided they exist, are uniquely determined if and only if $V(s)$ is square.

Remark 3: The condition that

$$\begin{bmatrix} -A_w+\lambda I & B_w \\ -C_w & D_w \end{bmatrix}$$

have full rank at some λ_i is effectively a condition that $W(\lambda_i)$ have full rank there. This observation follows immediately from the identity:

$$\begin{bmatrix} -A_w+\lambda I & B_w \\ -C_w & D_w \end{bmatrix} = \begin{bmatrix} I & 0 \\ C_w(A_w-\lambda_iI)^{-1} & I \end{bmatrix}\begin{bmatrix} -A_w+\lambda I & B_w \\ 0 & W(\lambda_i) \end{bmatrix}$$

We say effectively, since there remains open the possibility that $W(s)$ could have a pole at λ_i. A similar remark applies to the input weight $V(\lambda_i)$.

Remark 4: Note that if the weights $W(s)$ and $V(s)$ have full row and column rank respectively, the requirement for them to have this property for the particular values of $\lambda = \lambda_i(A_r)$ will be generally satisfied.

Theorem 3.1: If $G(s) = \{A,B,C,D\}$ is stable and minimal then $G_r(s) = \{A_r,B_r,C_r,D\}$ obtained from the proposed method is also stable and minimal.

Proof: It has been proved in [12] that for a stable and minimal original system $G(s) = \{A,B,C,D\}$, the new realization $\overline{G}(s) = \{A,\overline{B},\overline{C},\overline{D}\}$ is also stable and minimal. Since $\overline{G}_r(s) = \overline{C}_r(sI-A_r)^{-1}\overline{B}_r + \overline{D}_r$ is obtained by balanced truncation of $\overline{G}(s)$, stability of $\overline{G}_r(s)$ follows immediately. Since $G_r(s) = C_r(sI-A_r)^{-1}B_r + D_r$ is the reduced order model obtained using the proposed technique has the same A_r as $\overline{G}_r(s)$, the stability is guaranteed for stable original systems.

Theorem 3.2: If $G_r(s) = \{A_r,B_r,C_r,D\}$ is a r^{th} order model of the given original system $G(s)$ and $\overline{G}_r(s) = \{A_r,\overline{B}_r,\overline{C}_r,\overline{D}_r\}$ is a r^{th} order model of the new system $\overline{G}(s)$, then

$$\|W(s)(G(s)-G_r(s))V(s)\|_\infty = \|\overline{W}(s)(\overline{G}(s)-\overline{G}_r(s))\overline{V}(s)\|_\infty$$

Proof: From **Step 1** and **Step 2** of the proposed method

$$W(s)G(s)V(s) = \overline{W}(s)\overline{G}(s)\overline{V}(s) \quad (20)$$

From **Step 4** of the proposed method

$$W(s)G_r(s)V(s) = \overline{W}(s)\overline{G}_r(s)\overline{V}(s) \quad (21)$$

Substracting (21) from (20) we have

$$W(s)(G(s)-G_r(s))V(s) = \overline{W}(s)(\overline{G}(s)-\overline{G}_r(s))\overline{V}(s)$$

Corollary 1:

$$\begin{aligned} & \|W(s)(G(s)-G_r(s))V(s)\|_\infty \\ = & \|\overline{W}(s)(\overline{G}(s)-\overline{G}_r(s))\overline{V}(s)\|_\infty \\ \le & 2\|\overline{V}(s)\|_\infty\|\overline{W}(s)\|_\infty \sum_{i=r+1}^{n}\sigma_i \end{aligned}$$

where σ_i are the singular values of $\overline{G}(s)$.

Remark 5: If the reduced order model $G_r(s)$ is obtained without frequency weighting, then $V(s) = W(s) = I$. The following result of [1], [15] can be obtained easily:

$$\|(G(s)-G_r(s))\|_\infty \le 2\sum_{i=r+1}^{n}\sigma_i$$

Algorithm

A step-by-step algorithm for the proposed method can be obtained as follows:

1) Given a stable and minimal $G(s)$, $V(s)$ and $W(s)$, compute Y and X from (7a) and (7b) respectively.
2) Compute the fictitious input and output matrices $\overline{B}$ and $\overline{C}$ from (9c) and (9d) respectively.
3) Calculate the transformation matrix, T which balance $\{A,\overline{B},\overline{C}\}$ to diagonalize the Gramians:
$$T^{-1}\overline{P}T^{-T} = T^T\overline{Q}T = diag\{\sigma_1,\sigma_2,\dots,\sigma_n\}$$
4) Compute the frequency weighted balanced realization

$$\left[\begin{array}{c|c} T^{-1}AT & T^{-1}\overline{B} \\ \hline \overline{C}T & \overline{D} \end{array}\right] = \left[\begin{array}{cc|c} A_r & A_{12} & \overline{B}_r \\ A_{21} & A_{22} & B_2 \\ \hline \overline{C}_r & C_2 & \overline{D} \end{array}\right].$$

5) Solve (18a) to (18d) for X_r, Y_r, B_r, C_r.
6) A r^{th} order model can be obtained as $G_r(s) = \{A_r, B_r, C_r, D\}$.
7) Calculate the weighted error =

$$\|W(s)(G(s)-G_r(s))V(s)\|_\infty \tag{22}$$

Remark 6: In the above algorithm, the values of R and R_r do not have any affect on the approximation errors. The matrices only determine the values of X_{13} and $X_{13,r}$ respectively. In other words, the equation $\|W(s)(G(s)-G_r(s))V(s)\|_\infty = \left\|\overline{W}(s)(\overline{G}(s)-\overline{G}_r(s))\overline{V}(s)\right\|_\infty$ is true due to *Remark 2*.

Remark 7: To reduce the approximation error, the matrices $\overline{B}$ and $\overline{C}$ used in the proposed algorithm can be made to be functions of free parameters α and β as follows:

$$\begin{aligned} \overline{B} &= \left[\begin{array}{ccc} B & -\alpha X & AX \end{array}\right] \\ \overline{C} &= \left[\begin{array}{c} C \\ -\beta Y \\ YA \end{array}\right] \end{aligned}$$

To ensure that equations in (10) are valid, we need to have

$$\begin{aligned} \overline{C}_v &= \left[\begin{array}{c} C_v \\ \frac{A_v}{\alpha} \\ I \end{array}\right] \\ \overline{B}_w &= \left[\begin{array}{ccc} B_w & \frac{A_w}{\beta} & I \end{array}\right] \end{aligned}$$

Note that, α and β can be any scalar values other than zeros. By varying the scalars α and β, one can easily reduce the weighted approximation errors.

IV. Example

Consider the one-link flexible robot arm controller reduction problem [16] as in *Example 2* of [11]. The transfer function of the flexible robot arm from the motor voltage signal to angular position of a load mass is given by

$$G(s) = \frac{4445.7}{s^4 + 28.3s^3 + 364.1s^2 + 2386.9s}$$

A convex optimization based fifth-order controller transfer function is given by

$$K(s) = \frac{s^5 + 3.1s^4 + 4.4s^3 + 3.2s^2 + 1.3s + 0.2}{s^5 + 3s^4 + 4.3s^3 + 3s^2 + 1.2s + 0.2}$$

The input weight $V(s) = (I + G(s)K(s))^{-1}$ and output weight $W(s) = (I + G(s)K(s))^{-1}G(s)$ are used as the frequency weighted models.

Simulation result is shown in Table I. The figures in the last column gives approximation error improvement (in percentage) of the proposed technique over Enns' technique. From the table, even though the reduction error for the proposed method is slightly higher for order 1, but it gives a significant improvement for order 2 and order 3.

Fig. 1 and Fig. 2 show the frequency weighted model reduction error versus parameters (α and β) for order 1 and order 3 respectively. For order 1, the reduction errors are generally lower for lower values of α and β. For order 3, the reduction errors are smaller for higher values of β.

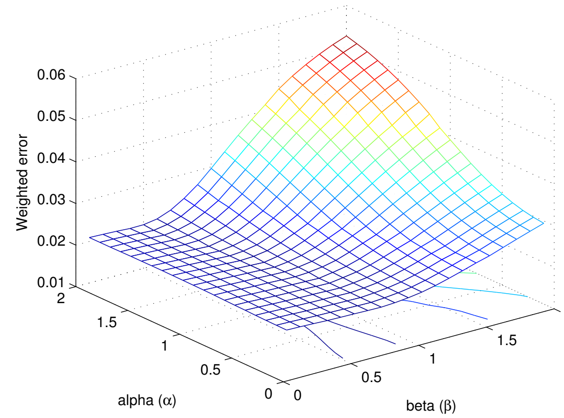

Fig. 1. Frequency weighted error versus parameters (α and β) for order 1

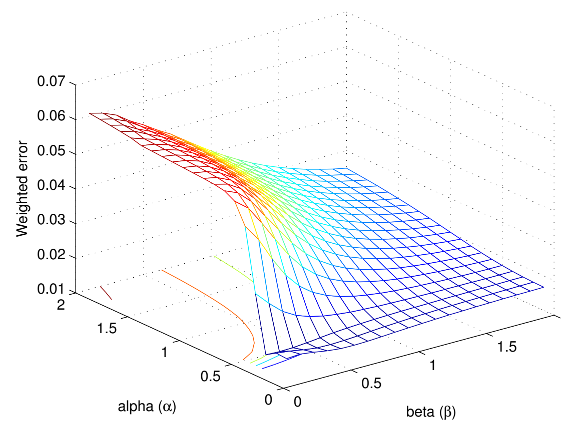

Fig. 2. Frequency weighted error versus parameters (α and β) for order 3

V. Conclusions

An improved frequency weighted balanced truncation based on partial fraction expansion is presented. The method guarantees the stability of reduced order models for double-sided weights. The approximation error can be reduced by varying user chosen free parameters α and β. The results of the example indicate a significant improvement over the existing techniques [1], [11], [12].

References

[1] D. F. Enns, "Model reduction with balanced realizations: an error bound and a frequency weighted generalization," in *Proc. of the 23rd IEEE Conf. on Decision and Control*, Las Vegas, USA, Dec. 1984, pp. 127–132.

[2] B. C. Moore, "Principal component analysis in linear systems: controllability, observability, and model reduction," *IEEE Trans. Autom. Control*, vol. 26, pp. 17–32, 1981.

[3] A. Lin and T. Chiu, "Model reduction via frequency weighted balanced realization," *Control Theory and Adv. Tech.*, vol. 8, pp. 341–351, 1992.

TABLE I

WEIGHTED MODEL REDUCTION ERRORS

	Enns' [1]	GS's [11]			SS's [12]			Proposed Method			
Order	Error	α_{GS}	β_{GS}	Error	α_{SS}	β_{SS}	Error	α	β	Error	%
1	1.9213×10^{-2}	0.3	0.1	1.9177×10^{-2}	100	100	1.9126×10^{-2}	0.1	0.8	1.9800×10^{-2}	-3.06
2	1.7891×10^{-2}	2	2	1.4971×10^{-2}	8	10	1.5475×10^{-2}	0.7	0.9	1.4127×10^{-2}	21.04
3	2.1923×10^{-2}	2.5	0.25	1.5115×10^{-2}	10	10	1.5817×10^{-2}	0.3	0.2	1.4302×10^{-2}	34.76
4	3.0830×10^{-4}	0.25	1.5	2.8624×10^{-4}	100	100	3.2893×10^{-4}	2.0	0.8	3.0582×10^{-4}	0.80

[4] V. Sreeram, B. D.O. Anderson, and A. G. Madievski, "New results on frequency weighted balanced reduction technique," in *Proc. American Control Conf.*, Seattle, USA, 1995, pp. 4004–4009.

[5] A. Varga and B. D.O. Anderson, "Accuracy-enhancing methods for balancing-related frequency-weighted model and controller reduction," *Automatica*, vol. 39, pp. 919–927, 2003.

[6] G. Wang, V. Sreeram, and W. Liu, "A new frequency-weighted balanced truncation method and an error bound," *IEEE Trans. Autom. Control*, vol. 44, pp. 1734–1737, 1999.

[7] A. Latham and B. D.O. Anderson, "Frequency weighted optimal hankel norm approximation of stable transfer functions," *Syst. Control Lett.*, vol. 5, pp. 229–236, 1986.

[8] U. M. Al-Saggaf and G. F. Franklin, "On model reduction," *Proc. of 23th IEEE Conf. on Decision and Control*, pp. 1064–1069, 1986.

[9] ——, "Model reduction via balanced realization: an extension and frequency weighted technique," *IEEE Trans. Autom. Control*, vol. 33, pp. 687–692, 1988.

[10] V. Sreeram and B. D.O. Anderson, "Frequency weighted balanced reduction technique: A generalization and an error bound," *Proc. of the 34th IEEE Conf. on Decision and Control*, vol. 4, pp. 3576–3581, 1995.

[11] A. Ghafoor and V. Sreeram, "Partial-fraction expansion based frequency weighted model reduction technique with error bounds," *IEEE Trans. Autom. Control*, vol. 52, no. 10, pp. 1942 –1948, oct. 2007.

[12] S. Sahlan and V. Sreeram, "New results on partial fraction expansion based frequency weighted balanced truncation," in *Proc. American Control Conf.*, jun. 2009, pp. 5695 –5700.

[13] K. Zhou, "Frequency weighted model $\mathcal{L}_\infty$ norm and optimal hankel norm model reduction," *IEEE Trans. Autom. Control*, vol. 40, no. 10, pp. 1687–1699, Oct. 1995.

[14] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. New Jersey, USA: Prentice Hall, 1996.

[15] K. Glover, "All optimal hankel norm approximations of linear multivariable systems and their $\mathcal{L}_\infty$ error bounds," *International Journal of Control*, vol. 39, pp. 1115–1193, 1984.

[16] W. Khaisongkram and D. Banjerdpongchai, "MATLAB based GUIs for linear controller design via convex optimization," *Comp. Appl. Eng. Educ.*, vol. 11, pp. 13–24, 2003.