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Selfish Response to Epidemic Propagation

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***Abstract*— An epidemic spreading in a network calls for a decision on the part of the network members: They should decide whether to protect themselves or not. Their decision depends on the trade off between their perceived risk of being infected and the cost of being protected. The network members can make decisions repeatedly, based on information that they receive about the changing infection level in the network.**

We study the equilibrium states reached by a network whose members increase (resp. decrease) their security deployment when learning that the network infection is higher (resp. lower). Our main result is that as the learning rate of the members increases, the equilibrium level of infection increases. We demonstrate this result both when members are strictly rational and when they are not. We characterize the domains of attraction of the equilibrium points. We validate our conclusions with simulations on human mobility traces.

I. INTRODUCTION

Epidemiology research has made extensive use of disease spreading models to study how a virus propagates in a human population [7]. Shortly after the appearance of self-replicating malicious programs in computers, aptly named *computer viruses*, security researchers turned to epidemic models to study the propagation of these programs [10]. More recently, the proliferation of capable mobile devices, like smartphones, made mobile networks a fertile ground for spreading malware [8]. The propagation characteristics of malware in such networks have been studied and counter-measures have been proposed [14], [3].

Countermeasures to an infection can be centrally enforced, or the decision for their adoption can be left to individual agents such as individual home computer users, companies, or people in a society. Centralized enforcing is more likely to work in tightly controlled environments, such as within a company network where the users are obliged to abide by the company security policy. However, when it is up to individual agents to invest in protection against infection [9], there appear contradicting incentives. Although agents want to be safe against real or virtual viruses, they would prefer to avoid investing in security: Security not only costs money, but it usually also reduces the utility of the network by, for example, isolating the agent from the rest of the network, or it reduces the utility of the device by, for example, slowing it down [16]. Another counterincentive is that the security of a network agent exhibits *positive externalities* with respect to the decisions of others: If others patch their computers, everyone becomes more secure, even those who do not patch their own computer. If others are vaccinated, everyone becomes safer, even those who are not vaccinated. Therefore, agents have an incentive to free-ride on the security investments of others, reaping the benefits without paying the costs. More background on computer network security and individual incentives can be found in two recent books [1], [4].

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In this paper, we model individuals' *changing* responses that depend myopically on the fluctuating infection level in an ongoing epidemic. We combine the epidemic propagation with a game theoretic description of the user behavior into an Ordinary Differential Equation (ODE) model. We find that the network reaches an endemic equilibrium, that is, an equilibrium where the infection persists. We reach the counterintuitive conclusion that the higher the learning rate (the rate at which users learn what the infection level is), the higher the infection level at the equilibrium.

The rest of the paper is organized as follows: We first describe our model for the evolution of the network state, comprising an epidemic propagation component and a user behavior component. We study the case of users with a strictly rational behavior, then users with non-strictly rational behavior. We end the paper with the empirical validation of our conclusions through simulations on human mobility traces. Due to space limitations we omit several of the proofs, and the case of users with heterogeneous behavior.

II. MODEL FOR EPIDEMIC PROPAGATION AND USER BEHAVIOR

A. Epidemic Propagation

There are N users in the network. Each user can be in one of three states:

- Susceptible, denoted by S: The user is not currently deploying security and is not infected.
- Infected, denoted by I: The user has been infected by the virus, and will spread it to any susceptible user he makes contact with.
- Protected, denoted by P: The user is deploying security and is therefore immune to the virus.

The number and fraction of users in each state are denoted, respectively, by N_S, N_I, N_P and S, I, P. It follows that $N_S + N_I + N_P = N$ and $S + I + P = 1$. The state of the network is $x = (S, I, P)$, and the set of possible states is $X = \left(\frac{N_S}{N}, \frac{N_I}{N}, \frac{N_P}{N}\right) \subseteq \frac{1}{N}\mathbb{N}^3$.

The evolution of the network state x is described as a Continuous Time Markov Process, as follows. With each user

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Event	Effect Δx
Meeting between S and I	$\frac{1}{N}(-1,+1,0)$
Update of S	$\frac{1}{N}(-p_{SP}(x),0,+p_{SP}(x))$
Update of P	$\frac{1}{N}(+p_{PS}(x),0,-p_{PS}(x))$
Disinfection of I	$\frac{1}{N}(0,-1,+1)$

TABLE I: Possible events and their effect on the network state

a Poisson alarm clock of rate $\beta+\gamma+\delta$ is associated. When the clock of user i rings – say at time t – one of three events happens:

- M With probability $\frac{\beta}{\beta+\gamma+\delta}$, user i has a *meeting* with another user, chosen uniformly at random. If the meeting is between a Susceptible and an Infected user, the Susceptible user becomes Infected. Otherwise nothing happens.
- U With probability $\frac{\gamma}{\beta+\gamma+\delta}$, user i receives an *update* about the network state x, and he has the opportunity to revise his current strategy if his state is S or P. If i's state is S, he switches to P with probability $p_{SP}(x)$. If i's state is P, he switches to S with probability $p_{PS}(x)$. If i is Infected, nothing happens.
- D With probability $\frac{\delta}{\beta+\gamma+\delta}$, user i has a *disinfection* opportunity. That is, if i is Infected, he becomes disinfected, and we assume he becomes Protected. If i is not Infected, nothing happens.

Table I summarizes the possible events and their effect on the network state.

We consider the large population scenario, i.e., the limit $N \longrightarrow \infty$. Kurtz [11] and Ljung [13] have shown that when $N \longrightarrow \infty$, the Continuous Time Markov Process described previously converges to a deterministic function, which is the solution to a system of Ordinary Differential Equations:

$$\frac{d}{dt}S = -\beta SI - \gamma S p_{SP}(x) + \gamma P p_{PS}(x) \tag{1a}$$
$$\frac{d}{dt}I = \beta SI - \delta I \tag{1b}$$
$$\frac{d}{dt}P = \delta I + \gamma S p_{SP}(x) - \gamma P p_{PS}(x) \tag{1c}$$

Eliminating P, as $S+I+P=1$, the system becomes

$$\frac{d}{dt}S = -\beta SI - \gamma S p_{SP}(x) + \gamma(1-S-I) p_{PS}(x) \tag{2a}$$
$$\frac{d}{dt}I = \beta SI - \delta I, \tag{2b}$$

together with $P = 1-S-I$. The state space is $D = (S,I), 0 \le S, I \le 1, S+I \le 1$, and it is bounded. This system is two dimensional and autonomous. Note that for $\gamma = 0$, the model is identical to the standard SIR epidemic model [7] (R stands for Recovered).

We will denote the righthand side of the system (2) by $F(x)$, and we will slightly abuse the notation for x to be $x = (S,I), x \in D$. So, the system (2) will be written

$$\frac{d}{dt}x = F(x). \tag{3}$$

B. User Behavior

As can be seen from the epidemic propagation model, the only point at which the users can make a choice is at an update event. We assume that there is a cost c_I associated with becoming Infected, and a cost c_P associated with becoming Protected. It holds that $c_I > c_P > 0$. There is no cost for being Susceptible. Note that these costs need not be the actual costs; what influences the decisions of users are the costs as perceived by the users.

If we assume that each user behaves strictly rationally, the choice between Susceptible and Protected depends on which state minimizes the user's expected cost. Specifically, given the aforementioned model of random pair meetings, a user's expected cost at a particular network state $x = (S,I)$ is c_P if he chooses to be Protected and Ic_I if he chooses to be Susceptible risking infection. Therefore, the user's decision would be S if $Ic_I < c_P$, and P if $Ic_I > c_P$. In this case, the functions $p_{SP}(x)$ and $p_{PS}(x)$ would be step functions of I:

$$p_{SP}(x) = p_{SP}(I) = 1\{Ic_I > c_P\} \tag{4}$$
$$p_{PS}(x) = p_{PS}(I) = 1\{Ic_I < c_P\}. \tag{5}$$

If $Ic_I = c_P$, then both choices are optimal, and any randomization between them is also optimal. So, when $Ic_I = c_P$, the functions $p_{SP}(I)$ and $p_{PS}(I)$ are multivalued. For convenience, we define

$$I^* \equiv \frac{c_P}{c_I}. \tag{6}$$

Note that if we were to set I^* to a value larger than 1, then p_{SP} would always be equal to 0, and p_{PS} would always be equal to 1. In that case, our model would be identical to the SIRS model [7].

To account for users that cannot be assumed to be strictly rational, or their perception of the cost is not crisp (e.g., they are not sure about the exact values of c_I and c_P), or they take the network state report to be not completely accurate, we consider a different scenario for the functions $p_{SP}(\cdot)$ and $p_{PS}(\cdot)$. We assume that they can be arbitrary functions of I, as long as the former is non-decreasing with I and the latter is non-increasing with I.

In what follows, first we will consider the case that $p_{SP}(\cdot)$ and $p_{PS}(\cdot)$ are discontinuous step functions and actually multivalued at the discontinuity, and then that they are continuously differentiable.

III. THE USERS ARE STRICTLY RATIONAL

The best response correspondence dictates the shape of $p_{SP}(I)$ and $p_{PS}(I)$:

$$p_{SP}(I) = \begin{cases} 0, & I < I^* \\ [0,1], & I = I^* \\ 1, & I > I^* \end{cases} \quad p_{PS}(I) = \begin{cases} 1, & I < I^* \\ [0,1], & I = I^* \\ 0, & I > I^* \end{cases}. \tag{7}$$

As $F(x)$ (recall (3)) is set-valued, we have to solve the differential inclusion

$$\frac{d}{dt}x \in F(x), x \in D. \tag{8}$$

The system can also be viewed as a switched non-linear system [12], or as a positive and compartmental system [2], [6] because it is characterized by nonnegative solutions for nonnegative initial conditions. Notice, however, that, unlike such systems, our system only has point equilibria rather than a continuum of equilibria.

We define a partition of the state space D into three domains: $D^- = D \cap \{(S,I), I < I^*\}$, $D^+ = D \cap \{(S,I), I > I^*\}$, and $L = D \cap \{(S,I) : I = I^*\}$. The domain L will also be referred to as the discontinuity line.

A. Existence of solutions

A solution for this differential inclusion [5] is an absolutely continuous vector function $x(t)$ defined on an interval J for which $\frac{d}{dt}x(t) \in F(x(t))$ almost everywhere on J. From the theory of differential inclusions we know that a solution of (8) exists if, for every $x \in D$, the *basic conditions* apply: The set $F(x)$ is nonempty, bounded, closed, convex, and the function F is upper semicontinuous.

The basic conditions apply in our case and therefore a solution exists.

B. Uniqueness of solutions

In general, because the righthand side of (8) is multivalued, even though two solutions at time t_0 are both at the point x_0, they may not coincide on an interval $t_0 \leq t \leq t_1$ for any $t_1 > t_0$. If any two solutions that coincide at t_0 also coincide until some $t_1 > t_0$, then we say that *right uniqueness holds at* (t_0, x_0). Left uniqueness at (t_0, x_0) is defined similarly (with $t_1 < t_0$), and (right or left) uniqueness in a domain holds, if it holds at each point of the domain.

The solution is unique in D^- and in D^+ because F has continuous partial derivatives there.

We next show which of the solutions of (8) lying on the line of discontinuity L can be uniquely continued in the direction of increasing t. We see that all solutions can be uniquely continued, except those that start at the point $(S,I) = \left(\frac{\delta}{\beta}, I^*\right)$. Those latter solutions all start at the same point and then diverge, but none of them can ever approach that point again in the positive direction of time. So, if we ignore the initial point of those solutions, all solutions can be uniquely continued.

C. Stationary points

The stationary points are found by solving for x the inclusion $0 \in F(x)$.

1) Stationary points above the discontinuity line: There can be no stationary points in the domain D^+. The system becomes

$$\frac{d}{dt}S = -\beta SI - \gamma S \tag{9a}$$

$$\frac{d}{dt}I = \beta SI - \delta I. \tag{9b}$$

From the first equation, we see that S has to be zero. But then the second equation implies that I also has to be zero, which is not an admissible value for I as $I = 0$ cannot be above the discontinuity line.

2) Stationary points below the discontinuity line: We look for stationary points in the domain D^-. The system becomes

$$\frac{d}{dt}S = -\beta SI + \gamma(1 - S - I) \tag{10a}$$

$$\frac{d}{dt}I = \beta SI - \delta I, \tag{10b}$$

which is identical to the SIRS case except that the domain is not the whole state space, it is only D^-.

This system has the solutions:

$$X_0 = (S_0, I_0) = (1, 0) \tag{11}$$

$$X_1 = (S_1, I_1) = \left(\frac{\delta}{\beta}, \frac{1 - \frac{\delta}{\beta}}{1 + \frac{\delta}{\gamma}}\right). \tag{12}$$

The second solution, X_1, is admissible if and only if $X_1 \in D^-$, i.e.,

$$\frac{\delta}{\beta} \leq 1, \tag{13}$$

and also

$$\frac{1 - \frac{\delta}{\beta}}{1 + \frac{\delta}{\gamma}} < I^*. \tag{14}$$

Note that if $\frac{\delta}{\beta} = 1$, then X_0 and X_1 coincide. Also, it is not surprising that X_1 is the equilibrium point of the corresponding SIRS model. That is, I^* does not play an explicit role in this case, as long as (14) holds.

3) Stationary points on the discontinuity line: We look for stationary points on the discontinuity line $I = I^*$, that is, we solve the inclusion $0 \in F(S, I^*)$ for S. The system becomes

$$\frac{d}{dt}S = -\beta SI^* + [-\gamma S, \gamma(1 - S - I^*)] \tag{15a}$$

$$\frac{d}{dt}I = \beta SI^* - \delta I^*. \tag{15b}$$

Since $I^* > 0$, $\frac{d}{dt}I$ is zero only when $S = \frac{\delta}{\beta}$. We then have to check if it is possible to make $\frac{d}{dt}S$ equal to zero, that is, if $0 \in F(\frac{\delta}{\beta}, I^*)$. We find that it is possible when I^* is such that

$$I^* \leq \frac{1 - \frac{\delta}{\beta}}{1 + \frac{\delta}{\gamma}}. \tag{16}$$

In that case, the stationary point is

$$X_2 = (S_2, I_2) = \left(\frac{\delta}{\beta}, I^*\right). \tag{17}$$

In general, there are many combinations of $p_{SP}(I^*)$ and $p_{PS}(I^*)$ that make $\frac{d}{dt}S$ equal to zero, but there is always one with $p_{SP}(I^*) = 0$. In that case, $p_{PS}(I^*) = \frac{\delta I^*}{\gamma(1 - \frac{\delta}{\beta} - I^*)}$.

To summarize, X_0 exists always. If $\delta < \beta$, one more equilibrium point exists: X_1 if $I^* > \frac{1-\frac{\delta}{\beta}}{1+\frac{\delta}{\gamma}}$, or X_2 otherwise.

D. Local Asymptotic Stability

1) Stability of X_0 and X_1: We show that, when $\frac{\delta}{\beta} \geq 1$, X_0 is asymptotically stable. When $\frac{\delta}{\beta} < 1$, X_0 is a saddle point, and if X_1 exists it is asymptotically stable. These results follow by the evaluation of the Jacobian at the points X_0 and X_1, and by checking the sign of its eigenvalues.

2) Stability of X_2: To show that the stationary point on the discontinuity line is asymptotically stable we will use Theorem 1 below [5, §19, Theorem 3]. To use this theorem we transform the system so that the line of discontinuity is the horizontal axis, the stationary point is $(0,0)$, and the trajectories have a clockwise direction for increasing t.

We set $x = \frac{\delta}{\beta} - S$ and $y = I - I^*$. The domains D, D^-, D^+ become $G = \{(x,y)|x \le \frac{\delta}{\beta}, y \ge -I^*, y - x \le 1 - I^* - \frac{\delta}{\beta}\}$, $G^- = G \cap \{(x,y)|y < 0\}$, and $G^+ = G \cap \{(x,y)|y > 0\}$. Then, the system can be written as

$$\frac{dx}{dt} = P^-(x,y) \tag{18a}$$

$$\frac{dy}{dt} = Q^-(x,y) = -\beta x(y + I^*) \tag{18b}$$

for $(x,y) \in G^-$, and

$$\frac{dx}{dt} = P^+(x,y) \tag{19a}$$

$$\frac{dy}{dt} = Q^+(x,y) = -\beta x(y + I^*) \tag{19b}$$

for $(x,y) \in G^+$.

The partial derivatives of $P^\pm$, that is, of P^+ and of P^-, are denoted by $P^\pm_x, P^\pm_{xx}, P^\pm_y$ etc., and similarly for $Q^\pm$. We define two quantities $A^\pm$ in terms of the functions $P^\pm$, $Q^\pm$ and their derivatives at the point $(0,0)$:

$$A^\pm = \frac{2}{3}\left(\frac{P^\pm_x + Q^\pm_y}{P^\pm} - \frac{Q^\pm_{xx}}{2Q^\pm_x}\right). \tag{20}$$

Theorem 1: Let the conditions

$$Q^- = Q^+ = 0, P^- < 0, P^+ > 0 \tag{21}$$

$$Q^-_x < 0, Q^+_x < 0 \tag{22}$$

be fulfilled at the point $(0,0)$. Then, $A^+ - A^- < 0$ implies that the zero solution is asymptotically stable, whereas $A^+ - A^- > 0$ implies that the zero solution is unstable.

All the conditions of Theorem 1 are satisfied in our case, together with $A^+ - A^- < 0$. The condition $P^- < 0$ is equivalent to (16), i.e., the condition on I^* that causes the stationary point to be on the line of discontinuity. All the other conditions are straightforward to verify.

Therefore, the stationary point $(S,I) = (\frac{\delta}{\beta}, I^*)$ is asymptotically stable.

E. Domains of attraction

From Theorem 6, §13 [5] we know that for autonomous systems on the plane, it holds that if a half trajectory T^+ is bounded, then its ω-limit set $\Omega(T^+)$ contains either a stationary point or a closed trajectory. Recall that the ω-limit set of a half trajectory $T^+(x = \phi(t), t_0 \le t < \infty)$ is the set of all points q for which there exists a sequence $t_1, t_2, \dots$ tending to ∞ such that $\phi(t_i) \longrightarrow q$ as $i \longrightarrow \infty$.

In this section, we show that there are no solutions that are closed trajectories. So we can conclude that all system trajectories converge to equilibrium points. When there is more than one equilibrium point, we show which trajectories converge to which point.

The main result is that for any half trajectory T^+, its ω-limit set $\Omega(T)$ can only contain equilibrium points, that is, $X_0 = (1,0)$, $X_1 = (S_1, I_1) = \left(\frac{\delta}{\beta}, \frac{1-\frac{\delta}{\beta}}{1+\frac{\delta}{\gamma}}\right)$, or $X_2 = (\frac{\delta}{\beta}, I^*)$.

We will find the following two functions useful:

$$E(S,I) = S - S_1\ln(S) + I + \frac{\gamma}{\beta}\ln(I), \tag{23}$$

$$M(S,I) = S - (S_1 + \frac{\gamma}{\beta})\ln(S + \frac{\gamma}{\beta}) + I - I_1\ln(I), \tag{24}$$

It holds that $E(S,I)$ is constant on trajectories in the area D^+, and $M(S,I)$ is decreasing along trajectories in the area D^-.

Assume that there exists a half trajectory T^+ whose limit set $\Omega(T)$ contains a closed trajectory Γ. By successively eliminating properties of such a trajectory, we will prove that it cannot exist. Note that Lemma 1 below is trivial if $(\frac{\delta}{\beta}, I^*)$ is an equilibrium point.

Lemma 1: The point $(\frac{\delta}{\beta}, I^*)$ cannot be on Γ.

If $(S,I) = (\frac{\delta}{\beta}, I^*)$ is not on Γ, then on Γ there holds right uniqueness. Also, $\Omega(\Gamma) = \Gamma$. We will continue by proving that Γ cannot have more or fewer than two intersection points with L.

Lemma 2: A closed trajectory Γ that does not pass through the point $(\frac{\delta}{\beta}, I^*)$ can have neither more than two nor fewer than two intersection points with the discontinuity line L. If it has two intersection points, they cannot be on the same side of $(\frac{\delta}{\beta}, I^*)$.

Lemma 3: A closed trajectory Γ cannot intersect the discontinuity line L on exactly two points that are on opposite sides of the point $(\frac{\delta}{\beta}, I^*)$.

From the previous lemmata, we conclude that there can be no closed trajectory Γ. Therefore, all trajectories have to converge to equilibrium points.

F. Conclusion

The total fraction $I = \frac{1-\frac{\delta}{\beta}}{1+\frac{\delta}{\gamma}}$ of Infected at the system equilibrium increases with the update rate γ, until I becomes equal to the threshold I^*. The reason for this increase is that, when the equilibrium value of I is below I^*, the trajectories will eventually be completely contained in the domain D^- (below I^*). In this domain, every time a Protected is being informed about the value of I will choose to become Susceptible, thus fueling the infection. In parallel, no Susceptible will choose to become Protected. The larger the value of γ, the shorter time a user will spend being Protected, thus the smaller the fraction of Protected. However, a smaller fraction of Protected implies a larger fraction of Infected, as the fraction of Susceptible at equilibrium is necessarily $\frac{\delta}{\beta}$, i.e., it is independent of γ.

When the quantity $\frac{1-\frac{\delta}{\beta}}{1+\frac{\delta}{\gamma}}$ exceeds I^*, the equilibrium value of I is limited to I^*; further increases of γ have no effect. The explanation is that, as soon as the instantaneous value of I exceeds I^*, Susceptible users switch to Protected, and Protected users stay Protected, thus bringing the infection level below I^*. However, there is no equilibrium point for the system in the domain D^-, so the only possible equilibrium value of I is I^*. For $I = I^*$ there are in general many combinations of $p_{SP}(I^*)$ and $p_{PS}(I^*)$ that lead to an equilibrium, including one with $p_{SP}(I^*) = 0$ and $p_{PS}(I^*) > 0$. That combination means that no Susceptible users become Protected, but some Protected become Susceptible.

IV. The users are not strictly rational

For the case of non-strictly-rational users, the behavior functions $p_{SP}(I)$ and $p_{PS}(I)$ are continuously differentiable, and we require that $\frac{d}{dI}p_{SP}(I) > 0$ and $\frac{d}{dI}p_{SP}(I) < 0$. Other than that, the two functions are arbitrary.

A. Stationary points

The equilibrium points of the system are found by solving for x the equation $F(x) = 0$:

$$\frac{d}{dt}S = 0 = -\beta SI - \gamma S p_{SP}(I) + \gamma(1 - S - I)p_{PS}(I) \tag{25a}$$

$$\frac{d}{dt}I = 0 = \beta SI - \delta I \tag{25b}$$

From (25b) we see that either $I = 0$ or $S = \frac{\delta}{\beta}$.

- Equilibrium point X_0
  Substituting $I = 0$ into (25a), we have that $X_0 = (S_0, I_0) = \left(\frac{p_{PS}(0)}{p_{SP}(0)+p_{PS}(0)}, 0\right)$. These values of (S_0, I_0) are always admissible since they are always non-negative and at most equal to 1.
  Recalling the meaning of $p_{PS}(0)$ and $p_{SP}(0)$, we can reasonably expect that $p_{PS}(0) = 1$ and $p_{SP}(0) = 0$: Protected have no reason to remain Protected, and Susceptible have no reason to become Protected, when there is no infection in the network. In this case, X_0 is the point $(1, 0)$.
- Equilibrium point X_1
  Substituting $S = \frac{\delta}{\beta}$ into (25a), we see that I has to satisfy

  $$g(I) \equiv -\delta I - \frac{\gamma\delta}{\beta}p_{SP}(I) + \gamma\left(1 - \frac{\delta}{\beta} - I\right)p_{PS}(I) = 0. \tag{26}$$

  To solve $g(I) = 0$ for I we need to know the two response functions $p_{SP}(I)$ and $p_{PS}(I)$. But even without knowing them, we can still prove that $g(I) = 0$ has a unique solution for $I \in [0, 1]$ under the condition that

  $$\frac{\delta}{\beta} \leq \frac{p_{PS}(0)}{p_{SP}(0) + p_{PS}(0)}. \tag{27}$$

  We first show that $g(I)$ is monotonically decreasing in the interval $[0, 1]$, and then we show that, under the condition (27), $g(0)g(1) \leq 0$. We can then conclude that there is exactly one solution of $g(I) = 0$ in the interval $[0, 1]$.
  Denoting by I_1 the solution of $g(I) = 0$, we can now conclude that $X_1 = (S_1, I_1) = (\frac{\delta}{\beta}, I_1)$ is uniquely determined under (27). The values S_1, I_1 are admissible since they are both between 0 and 1. Note that if (27) does not hold then both $g(0) < 0$ and $g(1) < 0$, so the monotonicity of g in $[0, 1]$ implies that X_1 does not exist. So, (27) is really a necessary and sufficient condition for the existence of X_1.

B. Local Asymptotic Stability

To examine the (local) stability of the equilibrium points X_0 and X_1 we compute the Jacobian of the system (25) and evaluate it at these two points. Local stability at a point is equivalent to the negativity of the eigenvalues of the Jacobian matrix evaluated at that point. So, X_0 is stable when X_1 does not exist, and unstable otherwise. X_1 is always stable.

C. Domains of Attraction

Since the system is two dimensional and F is continuously differentiable, we can use Dulac's criterion to show that the system can have no periodic trajectory.

Theorem 2 (Dulac's criterion): If there exists a continuously differentiable function $h : R^2 \longrightarrow R$ such that $\nabla \cdot (hF)$ is continuous and non-zero on some simply connected domain A, then no periodic trajectory can lie entirely in A.

In our case, the domain A is the state space excluding the line $I = 0$. Note that there can be no periodic trajectory that passes from a point with $I = 0$. We select as function h the function $h(S, I) = \frac{1}{I}$. We compute $\nabla \cdot (hF)$ to be

$$\nabla \cdot (hF) = -\beta - \gamma\frac{p_{SP}(I)}{I} - \gamma\frac{p_{PS}(I)}{I} < 0, \forall(S, I) \in A, \tag{28}$$

which is continuous and non-zero in A. Then, from Dulac's criterion, no periodic trajectory lies entirely in A, and, consequently, the system has no periodic trajectory at all. From the Poincaré-Bendixson theorem, the system can only converge to a periodic trajectory or an equilibrium point; so, we can conclude that every trajectory must converge to an equilibrium point, that is, either to X_0 or X_1.

D. Conclusion

The equilibrium point X_0 is independent of γ. We show now that, at $X_1 = \left(\frac{\delta}{\beta}, I_1\right)$, the equilibrium level of the Infected increases with γ. To this end, we take the derivative $\frac{dI_1}{d\gamma}$ and we see that is always positive.

V. Simulations on Mobility Traces

We validate our conclusions using simulations on human mobility traces. The traces that we use are Bluetooth contacts among 41 devices given to participants in a conference [15]. The traces were collected over a period of approximately 72 hours.

The contact rate β is determined by the traces. Actually, β is a function of time $\beta(t)$, since the number of contacts per time unit fluctuates depending on the time of day. We want to establish whether the fraction of Infected indeed increases for larger values of the update rate γ. For the simulations that follow, we set $\delta = (6hr)^{-1}$, and we plot the system trajectories on the $S-I$ plane (average of 30 simulations) for three different values of γ, $(1hr)^{-1}$, $(6hr)^{-1}$, and $(24hr)^{-1}$. The initial conditions for all simulations were 1 Infected and

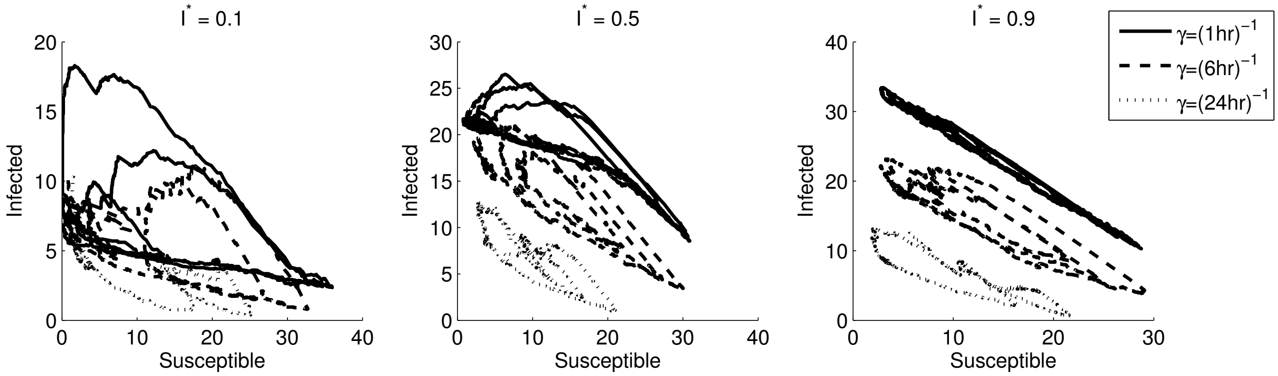

Fig. 1: The trajectory of the system (average of 30 simulations) on the SI plane, when $\delta = (6hr)^{-1}$ and γ takes the values $(1hr)^{-1}$, $(6hr)^{-1}$, and $(24hr)^{-1}$. The thresholds are $I^* = 0.1, 0.5, 0.9$. We see that the network experiences higher numbers of Infected devices for higher values of γ, and for $I^* = 0.1, 0.5$ we also observe the limiting effect of I^*.

40 Susceptible. Each simulation runs until either there are no Infected, or the end of the traces is reached.

We use a piecewise continuous response function

$$p_{SP}(I) = \begin{cases} 0 & I < I^* - \frac{\epsilon}{2} \\ \frac{1}{\epsilon}(I - I^* + \frac{\epsilon}{2}) & I^* - \frac{\epsilon}{2} < I < I^* + \frac{\epsilon}{2} \\ 1 & I > I^* + \frac{\epsilon}{2} \end{cases} \quad (29)$$

and $p_{PS}(I) = 1 - p_{SP}(I)$.

In Figure 1 we plot simulation results for $I^* = 0.1, 0.5, 0.9$, and $\epsilon = 0.001$, omitting an initial transient phase. Since $\beta(t)$ is not constant, the system state oscillates among two equilibrium points, X_0 (when $\beta(t)$ is low enough that $\delta > \beta(t)$) and either X_1 or X_2, depending on whether (14) is satisfied or not $\left(\frac{1-\frac{\delta}{\beta}}{1+\frac{\delta}{\gamma}} < I^*\right)$. Despite these periodicities, we see that for increasing values of γ the system trajectories go through higher values of I, thus confirming our main conclusion that the infection level increases with the update rate. The effect of lowering I^* is that it limits the maximum infection at the equilibrium, so the trajectories are capped at values of I not far above I^*. For lower values of I^*, we see that the effect of γ on the Infected is smaller.

VI. Conclusions

We study the effect of network users being cost-sensitive when deploying security measures. In particular, if users increasingly deploy security when learning that the level of network infection is higher, and retract the deployment when the level of infection drops, then a higher learning rate leads to a higher equilibrium level of infected users.

We reach this same conclusion in two scenarios. Our main scenario is when users are strictly rational cost minimizers, having a discontinuous multi-valued best response behavior. The conclusion does not change when the response function is an arbitrary continuous single-valued function, as long as the function implies that users increasingly choose protection as the level of infection rises. We validate the conclusions both theoretically, using a system of differential inclusions or differential equations, and also with simulations on human mobility traces.

We use the theory of differential inclusions to prove properties (existence, uniqueness, stability) of the system trajectories in the case of multivalued response functions. In the case of uniform user behavior, either continuous or discontinuous, the system is two-dimensional, and we are able to exclude the existence of periodic trajectories and to characterize the domains of attraction for each equilibrium point.

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