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Unbiased Minimum-variance Filtering for Delayed Input Reconstruction

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***Abstract*— The unknown inputs in a dynamical system may represent unknown external drivers, input uncertainty, state uncertainty, or instrument faults and thus unknown-input reconstruction has several important applications. In this paper, we consider delayed state estimation and input reconstruction. That is, we develop filters that recursively use current measurements to estimate past states and reconstruct past inputs. By introducing this delay, recursive input reconstruction is viable for a potentially broader class of systems.**

1. Introduction

State estimation for system with unknown inputs have been widely considered (see [4], [9], [10] and references therin). The unknown inputs in a dynamical system may represent unknown external drivers, input uncertainty, state uncertainty, or instrument faults. Thus unknown-input reconstruction has several important applications in uncertainty estimation and fault detection. Input reconstruction also has applications in filtering and coding theory. In some early work, input reconstruction is achieved through system inversion [12], [8]. More recently, methods for input reconstruction using optimal filters are developed in [13], [6], [5], [3]. Unbiased minimum-variance filters for discrete-time stochastic systems with arbitrary unknown inputs are considered in [7], [4], [9], [10]. However, all of these approaches use current measurements to estimate the states or reconstruct the input at the same time step, and apply to a restricted class of systems. Recent results on input and state observability suggest that by allowing a delay in the estimation process, input reconstruction is possible for a broader class of systems [2], [11]. Therefore, in this note we consider recursive state estimation and input reconstruction at time step k using measurements at time step $k+1$.

2. Filter

Consider the state space system:

$$x_{k+1} = Ax_k + Bu_k + He_k + w_k \tag{2.1}$$

$$y_k = Cx_k + Du_k + v_k \tag{2.2}$$

$x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^l$, $e_k \in \mathbb{R}^p$, are the state, known input, measurement, unknown input vectors, respectively, $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^l$ are zero-mean white process and measurement noise, respectively, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $H \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{l\times n}$, $D \in \mathbb{R}^{l\times m}$. Without loss of generality, we assume $l \le n$ and $p \le n$.

First we consider the simplifications $B = 0$, and $D = 0$. Note that the filter derivation is independent of B and D matrices, and thus the assumption of non-zero B and D matrices is for convenience alone. Next, without loss of generality, we assume $\mathrm{rank}(H) = p$.

For the state-space system (2.1), (2.2), we consider a filter of the form

$$\hat{x}_{k|k+1} = \hat{x}_{k|k} + L_k(y_{k+1} - C\hat{x}_{k+1|k}), \tag{2.3}$$

where

$$\hat{x}_{k|k} = A\hat{x}_{k-1|k}, \quad \hat{x}_{k+1|k} = A^2\hat{x}_{k-1|k}. \tag{2.4}$$

The unique feature of the above filter equations is that estimates are computed with a delay as newer data is used to estimate older states. That is, $\hat{x}_{k|k+1}$ is the state estimate at time step k given data up to time step $k+1$.

Finally, we define the state estimation error as

$$\varepsilon_k \triangleq x_k - \hat{x}_{k|k+1}, \tag{2.5}$$

and the error covariance matrix as

$$P_{k|k+1} \triangleq \mathbb{E}[\varepsilon_k \varepsilon_k^T]. \tag{2.6}$$

3. Unbiasedness

Definition 3.1. The filter (2.3), (2.4) is unbiased if $\hat{x}_{k|k+1}$ is an unbiased estimate of the state x_k.

Definition 3.1 implies that the filter (2.3), (2.4) is unbiased if and only if $\mathbb{E}[x_k - \hat{x}_{k|k+1}] = 0$. Next, we note that

$$\begin{aligned}\varepsilon_k =& x_k - \hat{x}_{k|k+1}\\ =&(A - L_kCA^2)\varepsilon_{k-1} + (H - L_kCAH)e_{k-1}\\ &- L_kCHe_k + w_{k-1} - L_k(CAw_{k-1}\\ &+ Cw_k + v_{k+1})\end{aligned} \tag{3.1}$$

Theorem 3.1. Let L_k be such that the filter (2.3), (2.4) is unbiased. Then

$$H - L_kCAH = 0, \tag{3.2}$$

and

$$L_kCH = 0. \tag{3.3}$$

Proof: By definition, filter (2.3), (2.4) is unbiased if and only if $\mathbb{E}[x_k - \hat{x}_{k|k+1}] = \mathbb{E}[\varepsilon_k] = 0$. Then it follows from (3.1) that

$$\begin{aligned}\mathbb{E}[\varepsilon_k] =&\mathbb{E}[(A - L_kCA^2)\varepsilon_{k-1} + (H - L_kCAH)e_{k-1}\\ &- L_kCHe_k + w_{k-1} - L_k(CAw_{k-1}\\ &+ Cw_k + v_{k+1})] = 0.\end{aligned} \tag{3.4}$$

Since (3.4) must hold for arbitrary e_k and e_{k-1}, it follows that (3.2) and (3.3) must hold. □

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Corollary 3.1. Let L_k be such that the filter (2.3), (2.4) is unbiased. Then, the following conditions hold

$i)$ $p \leq l$,
$ii)$ $\mathrm{rank}(CAH) = p$,
$iii)$ $\mathrm{rank}(CH) \leq l - p$,
$iv)$ $\mathrm{rank}(L_k) \geq p$, for all k.

Proof. Since the filter (2.3), (2.4) is unbiased, it follows from Theorem 3.1 that (3.2) holds and hence

$$L_k CAH = H. \tag{3.5}$$

Since $\mathrm{rank}(H) = p$, it then follows from (3.5) that $iv)$ holds and

$$\mathrm{rank}(CAH) \geq p. \tag{3.6}$$

Since $CAH \in \mathbb{R}^{l\times p}$, it follows from (3.6) that statement $i)$ holds. Furthermore, it follows from (3.6) and $i)$ that statement $ii)$ holds.

Finally to prove $iii)$, since (3.3) holds, it follows from [1, Proposition 2.5.9, p. 106] that

$$\begin{aligned}\mathrm{rank}(L_k) + \mathrm{rank}(CH) &\leq \mathrm{rank}(L_k CH) + l \\ &= l.\end{aligned} \tag{3.7}$$

Furthermore, using $iv)$, (3.7) becomes

$$p + \mathrm{rank}(CH) \leq l,$$

that is, $\mathrm{rank}(CH) \leq l - p$. □

Corollary 3.2. Let L_k be such that the filter (2.3), (2.4) is unbiased, and let $l = p$. Then, $CH = 0$ and $\mathrm{rank}(L_k) = p$ for all k.

4. Minimum-variance Gain

Next, we determine the filter gain L_k that yields unbiased minimum-variance estimates $\hat{x}_{k|k+1}$ of the states x_k.

Fact 4.1. Let L_k be such that the filter (2.3), (2.4) is unbiased. Then

$$\begin{aligned}P_{k|k+1} = {}&(A - L_k CA^2)P_{k-1|k}(A - L_k CA^2)^{\mathrm{T}} \\ &+ (I - L_k CA)Q_{k-1}(I - L_k CA)^{\mathrm{T}} \\ &+ L_k CQ_k C^{\mathrm{T}} L_k^{\mathrm{T}} + L_k R_{k+1} L_k^{\mathrm{T}}\end{aligned} \tag{4.1}$$

Next, define the cost function J as the trace of the error covariance matrix

$$J(L_k) = \mathrm{tr}\mathbb{E}[\varepsilon_k \varepsilon_k^{\mathrm{T}}] = \mathrm{tr}P_{k|k+1}. \tag{4.2}$$

To derive the unbiased minimum-variance filter gain, we minimize the objective function (4.2) subject to the constraints (3.2) and (3.3). Since from Corollary 3.1, $\mathrm{rank}(CH) \leq l - p$, we first consider the simpler case in which $CH = 0$. In this case, (3.3) is trivially satisfied, thus the only constraint on the filter gain is (3.2).

Theorem 4.1. Suppose $CH = 0$ then the unbiased minimum-variance gain L_k is

$$L_k = \left[T_k A^{\mathrm{T}} C^{\mathrm{T}} + \Phi_k (CAH)^{\mathrm{T}}\right] S_k^{-1}, \tag{4.3}$$

where

$$T_k \triangleq Q_{k-1} + AP_{k-1|k}A^{\mathrm{T}}, \tag{4.4}$$

$$S_k \triangleq CAT_k A^{\mathrm{T}} C^{\mathrm{T}} + CQ_k C^{\mathrm{T}} + R_{k+1}, \tag{4.5}$$

$$\Phi_k \triangleq \left[H - T_k A^{\mathrm{T}} C^{\mathrm{T}} S_k^{-1} CAH\right] \left((CAH)^{\mathrm{T}} S_k^{-1} CAH\right)^{-1} \tag{4.6}$$

5. Input Reconstruction

The filter derived in the previous section provides unbiased minimum-variance estimates of states. Next, we consider using these estimates to reconstruct the unknown inputs.

Proposition 5.1. Let $CH = 0$, and let $\hat{x}_{k|k+1}$ be an unbiased estimate of x_k. Then

$$\hat{e}_{k-1} \triangleq (CAH)^{\dagger} CAL_k (y_{k+1} - C\hat{x}_{k+1|k}) \tag{5.1}$$

is an unbiased estimate of e_{k-1}.

Proof. Refer to [10].

6. Conclusions

In this paper, we developed an unbiased minimum-variance filter that recursively use current measurements to estimate past states and reconstruct past inputs. Future work will focus on both a more general form of this filter when $CH \neq 0$, and for arbitrary delays, that is using measurements at time step $k + l$ to estimate states and unknown-inputs at time step k.

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