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Cooperative Control Based on Distributed Estimation of Network Connectivity

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Abstract— **This paper addresses distributed estimation and control problems for directed network, whose connectivity can be captured by the first left eigenvector associated with the dominant eigenvalue of the communication matrix. In particular, distributive estimation scheme is designed such that the first left eigenvector as well as the expected consensus vector are estimated online employing the same communication network among the systems. Based on the estimation results, distributed cooperative controls with adaptive gains are synthesized to improve convergence of the overall system by making the time derivative of the cooperative control Lyapunov function more negative over time, such that the convergence to the expected consensus value can be enhanced. Simulation results demonstrate the effectiveness of the proposed design.**

I. Introduction

Cooperative control theory and its applications has received significant amount of interests in the past decades, leading to breakthroughs in formation control [1], [2], attitude synchronization [3], [4], [5], and flocking [6], [7]. Implementation of cooperative control requires sharing of local information among a group of networked systems, whose interactions can be captured either by graph laplacian [8] or by communication matrix [2], it is thus of both theoretical and practical interests to find a systematic strategy, preferably distributively, to improve the network performance, which will be investigated in this paper.

It is known that the first left eigenvector (i.e., associated with dominant eigenvalue) of a nonnegative, row-stochastic matrix D characterizes the connectivity of the network $I-D$ or its corresponding graph. Specifically, if the network is undirected or symmetric, the first left eigenvector is trivial and the convergence rate can be conservatively captured by the Fiedler value [9], which has been rigorously addressed in the open literatures. Several approaches are available on estimating the eigenvectors and eigenvalues of undirected network. For instance, a decentralized orthogonal iteration approach is proposed in [10] to estimate the leading k eigenvectors, but this proposed method is not scalable and also requires a centralized initialization. In [11], eigenvalues of Laplacian matrix are estimated using fast Fourier transform (FFT) by constructing distributed oscillators whose states oscillate at frequencies corresponding to the eigenvalues of graph Laplacian, however the FFT technique is not appropriate for real-time implementation and for handling switching topologies. Other numerical solutions to finding Fiedler eigenvalue are [12], [13] in which it shows clearly that direct computation of Fiedler value requires global information of the network. The best available result in estimating Fiedler value is arguably [14] in which distributed nonlinear dynamic estimators are designed using a decentralized power iteration approach to real-time estimate components of the Fiedler eigenvector and the Fiedler eigenvalue of an undirected and connected network, though the estimators require estimation of several other consensus values and they may not be able to handle fast changing topologies.

In addition, optimizing Fiedler value directly or indirectly is another noteworthy strategy in improving network convergence. Successful treatments in this venue include applications of centralized semi-definite programming (SDP) approaches [15][16] or decentralized supergradient method [17], or indirectly by introducing additional node(s) whose location is optimized based on the global information of the network [18].

However, all of the aforementioned results are restricted to undirected and connected networks. To the best of our knowledge, little is available on distributively estimating connectivity of directed networks or on improving their performance. In this paper, we focus upon distributed estimation and control problems of directed networks. Specifically, distributed estimators are provided to estimate the first left eigenvector and expected consensus vector, based on which the gains in standard linear cooperative controls are distributively and adaptively adjusted so that the time derivative of cooperative control Lyapunov function becomes more negative and hence performance of the overall system is enhanced. The proposed estimation and adaptation scheme provide a systematic way of synthesizing cooperative control laws for varying and directed networks, its effectiveness is verified by theoretical proofs and numerical examples.

II. Problem Formulation

Consider n linear systems whose dynamics are described by

$$\dot{y}_i = u_i, \tag{1}$$

where $y_i \in \Re^m$ is the output of the ith system, and $u_i \in \Re^m$ is the control to be designed. Information sharing among the group of the systems is through a local communication network whose status is described by piecewise-constant binary communication matrix $S(t)$. Specifically, there is a time sequence $\{t_k : k \in \aleph\}$ such that $S(t) = S(t_k)$ for all

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$t \in [t_k, t_{k+1})$, where $\aleph = \{0, 1, ..., \infty\}$,

$$S(t_k) = \begin{bmatrix} 1 & s_{12}(t_k) & \dots & s_{1n}(t_k) \\ s_{21}(t_k) & 1 & \dots & s_{2n}(t_k) \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1}(t_k) & s_{n2}(t_k) & \dots & 1 \end{bmatrix}, \tag{2}$$

$s_{ij}(t) = 1$ if information of $y_j(t)$ is received by the ith system, and $s_{ij}(t) = 0$ if otherwise. Hence, the neighboring set of the ith system is defined as $\mathcal{N}_i$, such that for any $k \in \mathcal{N}_i$, we have $s_{ik}(t) = 1$.

Although the time sequence of $\{t_k : k \in \aleph\}$ and the topological changes of $S(t)$ are not known apriori or predictable or prescribed/modeled in any specific way. For the systems described by (1), linear cooperative controls for the ith system are of the form

$$\begin{aligned} u_i(t) &= \sum_{j=1}^{n} \frac{s_{ij}(t)\alpha_{ij}(t)}{\sum_{l=1}^{n} s_{il}(t)\alpha_{il}(t)}[y_j(t) - y_i(t)] \\ &\triangleq \sum_{j=1}^{n} d_{ij}(t)[y_j(t) - y_i(t)], \end{aligned} \tag{3}$$

where $\alpha_{ij}(t) > 0$ are scalar piecewise-constant control gains, and changes of $s_{ij}(t)$ are instantaneously detected according information reception by the ith system.

Substituting (3) into (1) yields the overall networked system

$$\dot{y} = [-I_{nm} + D(t) \otimes I_m]y, \tag{4}$$

where $y^T = [y_1^T \ y_2^T \ ... \ y_n^T] \in \Re^{nm}$, $I_l \in \Re^{l \times l}$ is the identity matrix, $\otimes$ denotes the Kronecker product, and $D(t) = [d_{ij}(t)] \in \Re^{n \times n}$ is a non-negative, piecewise-constant, row-stochastic and diagonally positive matrix.

It is well established that consensus $(y_i - y_j) \to 0$ for all $i \neq j$ can be achieved if $S(t_k)$ is uniformly sequentially complete or its corresponding graph is cumulatively connected [2], [19]. Specifically, the convergence rate (defined as rate of convergence to a consensus) of system (4) is determined not only by the cumulative information flow $S(t)$ (and equivalently $D(t)$), but also the state of the systems within [18]. Clearly, choices of constant gains for $\alpha_{ij}(t)$ are the simplest and, in order to improve performance, control gains $\alpha_{ij}(t)$ should be adjusted online according to both locally available state and topological information. Let $\gamma(t) \in \Re^n$ denotes the first left eigenvector of matrix $D(t)$, such that

$$D^T(t)\gamma(t) = \gamma(t), \text{ and } \gamma^T(t)\mathbf{1}_n = 1 \tag{5}$$

where $\mathbf{1}_n \in \Re^n$ is a vector of 1s. The leader set of network $D(t)$ is defined as $\mathcal{L}$, such that for any $j \in \mathcal{L}$, we have $\gamma^T(t)e_j > 0$.

Then, the expected consensus vector can be given by

$$\sigma(t) = [\gamma^T(t) \otimes I_m]y(t) = [\sigma_1^T \ ... \ \sigma_n^T]^T \tag{6}$$

where σ_i is the expected consensus vector at the i system.

Furthermore, to quantitatively analyze the convergence rate, we can use the following cooperative control Lyapunov function [19]: for $t \in [t_k, \ t_{k+1})$,

$$V_c(t) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \beta_{ij}[y_i(t) - y_j(t)]^2 \tag{7}$$

where $\beta_{ij} = \gamma^T e_i \gamma^T e_j$ if the $j \in \mathcal{L}$, and $\beta_{ij} = 0$ if otherwise, $e_i \in \Re^n$ is unit vector in which all entries are zeros except for 1 as the ith entry. The objective of this paper is threefold: (i) Develop a distributed algorithm to estimate the left eigenvector γ in (5); (ii) design a distributed algorithm to estimate the expected consensus vector σ_i as of (6); and (iii) Synthesize an online adaptive scheme to adjust the gains in the linear cooperative control law so that convergence to the consensus is improved. The proposed estimation-control algorithm works for any unknown time sequence of $\{t_k : k \in \aleph\}$ and any topological changes of $S(t_k)$ provided that the following simple assumption holds.

Assumption 1: Time sequence $\{t_k : k \in \aleph\}$ has the property that $(t_{k+1} - t_k) \geq \tau$ for some known constant $\tau > 0$.*

III. Distributed Estimation of Network Connectivity

In this section, the left eigenvector as well as the expected consensus vector of system (4) are estimated online by each of the systems. Before proceeding further, the following lemmas are provided, illustrating the relationship among connectivity of network matrix D, convergence of its corresponding system, and the first left eigenvector γ. Their proof are omitted here since they combine the results in [19].

***Lemma** 1*: Consider diagonally-positive row-stochastic matrix $D \in \Re_+^{n \times n}$ with $D^T\gamma = \gamma$.

(i) Suppose that D is irreducible. Then, γ is positive and unique.

(ii) Suppose that D is reducible and is the following 2-by-2 lower triangular canonical form†

$$PDP^T = \left[\begin{array}{c:c} E_{11} & 0 \\ \hdashline E_{21} & E_{22} \end{array}\right] \triangleq E_\angle, \tag{8}$$

where E_{11} is irreducible and P is a permutation matrix. Then, if E_{22} is irreducible, and $E_\angle$ is lower triangularly complete (i.e., $E_{21} \neq 0$), $\gamma^T = [\gamma_{E_{11}}^T \ 0]P$ is unique, where $\gamma_{E_{11}}$ is positive such that $\gamma_{E_{11}}^T E_{11} = \gamma_{E_{11}}^T$. If E_{22} is irreducible but $E_{21} = 0$, then γ is not unique, and its linearly-independent choices are $\gamma^T = [\gamma_{E_{11}}^T \ 0]P$ and $\gamma^T = [0 \ \gamma_{E_{22}}^T]P$, where $\gamma_{E_{22}}$ is positive such that $\gamma_{E_{22}}^T E_{22} = \gamma_{E_{22}}^T$. If E_{22} is reducible, the properties of γ can similarly be argued after permutation and lower triangulation of E_{22}.

***Lemma** 2*: Consider row-stochastic matrix $D \in \Re_+^{n \times n}$ with $\gamma^T D = \gamma^T$. Then, for any $\mu > 0$ and $t > 0$,

$$\gamma^T e^{\mu(-I+D)t} = \gamma^T.$$

*In the event that rapid changes of $S(t)$ are present and that inequality of $(t_{k+1} - t_k) \geq \tau$ is violated occasionally, these changes would be accommodated by the transient of the proposed distributed estimation schedule. Clearly, if $(t_{k+1} - t_k) \geq \tau$ does not hold most of the time, there is little chance for any online estimation scheme of network connectivity to work.

†It is straightforward to extend the result to matrix D that has an l-by-l lower triangular canonical form.

Furthermore, if D is irreducible or reducible but lower triangularly complete,

$$\lim_{t\to\infty} e^{\mu(-I+D)t} = \frac{1}{\gamma^T \mathbf{1}_n}\mathbf{1}_n\gamma^T, \tag{9}$$

If D is reducible and lower triangularly incomplete, there is a permutation matrix P such that the limit of $Pe^{\mu(-I+D)t}P^T$ is block diagonal and each of those diagonal blocks has the aforementioned properties.

A. Distributed Estimation of Left Eigenvector

Topological changes in $S(t)$ are not known either apriori or real-time by all the systems, nor is the corresponding time sequence $\{t_k : k \in \aleph\}$. What is known instantaneously to the ith system is whether $s_{ij}(t)$ or $\alpha_{ij}(t)$ or their weighted average $d_{ij}(t)$ experiences a discontinuous jump at certain specific time instance t_k. If there is, the ith system knows that certain communication link(s) to itself has changed the status at time t_k ($\hat{t}_k$ is the estimation of t_k), but such a change is generally directed and not known to any other system. In fact, this direct measurement individually done by the ith system is the only information available about topological condition of the communication network as well as the gains of the overall networked system. This information is then used to construct the following distributed estimator: if $d_{ij}(t)$ does not experience any change at time t for all j (that is, $d_{ij}(t^+) = d_{ij}(t^-)$) and if $d_{ij}(t^+) > 0$ and $\hat{\gamma}_j(t)$ is not reset at time t (that is, $\hat{\gamma}_j(t^+) \neq e_j$), then

$$\dot{\hat{\gamma}}_i(t) = \mu \sum_{j=1}^{n} d_{ij}(t)[\hat{\gamma}_j(t) - \hat{\gamma}_i(t)]; \tag{10}$$

if otherwise,

$$\hat{\gamma}_i(t^+) = e_i; \tag{11}$$

where $\hat{\gamma}_i(t_0) = e_i$, $\mu > 1$ is the estimation gain to be specified, $d_{ij}(t)$ is the same as that in (3), and $\hat{\gamma}_j(t)$ is transmitted through the same communication network as $y_j(t)$. In other words, if the ith row of communication matrix $S(t)$ has any binary change at time t or if $\alpha_{ij}(t)$ is adjusted to a different value for some j, the local estimate $\hat{\gamma}_i(t)$ is reset to be e_i. Furthermore, if $s_{ki}(t) = 1$, then any reset of $\hat{\gamma}_i(t)$ to e_i is detected by the kth system so that $\hat{\gamma}_k(t)$ is also reset to e_k. If the network matrix $S(t)$ is irreducible, a reset of local eigenvector estimate by one system will propagate into resets of local eigenvector estimates by all the systems. If there is no topological change or gain change at time t, the local estimate of the network left eigenvector evolves according to differential equation (10). Performance of the proposed left-eigenvector estimation algorithm is summarized into the following theorem.

***Theorem** 1*: Consider cooperative system (4) and let $\gamma(t)$ be the unity left eigenvector of network matrix $D(t)$ and associated with eigenvalue $\lambda(D(t)) = 1$. Then, under assumption 1, $\gamma(t)$ can be estimated distributively by estimation algorithm (10) and (11). Specifically, for any time sequence $\{t_k : k \in \aleph\}$ satisfying the assumption and for any topological changes in $S(t_k)$, output $\hat{\gamma}_i(t)$ of distributed estimator (10) and (11) over time interval $[t_k,\ t_{k+1})$ converges to either the network's unique left eigenvector $\gamma(t_k)$ or one of its linearly independent components (as specified by lemma 1). In addition, the convergence rate of distributed estimators can be made arbitrarily fast by choosing a large value of μ (say, $\mu \geq 40/\tau$).

Proof: It follows that, whenever switching laws (11) are not active, the combined dynamics of all the distributed estimators can be expressed in a matrix form as

$$\dot{\hat{\bar{\gamma}}}(t) = \mu[-I_{n^2} + D(t)\otimes I_n]\hat{\bar{\gamma}}(t) \tag{12}$$

where $\hat{\bar{\gamma}}(t) = [\hat{\gamma}_1^T\ \hat{\gamma}_2^T\ \ldots\ \hat{\gamma}_n^T]^T \in \Re^{n^2}$. The proof is done by an induction with respect to time sequence $\{t_k : k \in \aleph\}$.

First, consider the interval $t \in [t_0,\ t_1)$. It follows from initial conditions of $\hat{\gamma}_i(t_0) = e_i$, from (12), and from lemma 2 that, letting $\gamma(t_0)$ denote (one of) left eigenvector(s) defined by (5),

$$\bar{\gamma}^T(t_0)[D(t_0)\otimes I_n] = \bar{\gamma}^T(t_0).$$

Therefore, from (12)

$$\hat{\bar{\gamma}}(t) \quad = \quad e^{\mu\{[-I_n + D(t_0)]\otimes I_n\}(t-t_0)}\hat{\bar{\gamma}}(t_0) \tag{13}$$

Invoking lemma 2, and because of the fact

$$e^{A\bigoplus B} = e^{I_m\otimes A + B\otimes I_n} = e^B \otimes e^A \tag{14}$$

where $\bigoplus$ denotes the Kronecker sum, A and B are square matrices of any order.
Hence,

$$\lim_{\mu\to\infty} e^{\mu\{[-I_n + D(t_0)]\otimes I_n\}(t-t_0)} = \left[\mathbf{1}_n\gamma^T(t_0)\right]\otimes I_n \tag{15}$$

Consequently,

$$\hat{\bar{\gamma}}(t) \to \left[\mathbf{1}_n\gamma^T(t_0)\right]\otimes I_n\hat{\bar{\gamma}}(t_0) = \mathbf{1}_n \otimes \gamma(t_0), \tag{16}$$

It is clear that $\hat{\gamma}_i(t) \to \gamma(t_0)$ can be ensured. In addition, it follows from lemma 1 that, depending upon the topological property of $D(t_0)$, $\gamma(t_0)$ may not be unique and hence convergence of $\hat{\gamma}_i(t) \to \gamma(t_0)$ will render their corresponding (linearly independent) left eigenvector for each estimator. The convergence rate is specified by $e^{-\mu(1-\lambda_D)(t-t_0)}$ where λ_D is the Fiedler eigenvalue of $D(t_0)$, and the convergence time can be made smaller than $\tau/4$ by simply increasing μ according to $\mu \geq 40/\tau$.

Now, assume that the estimators work up to the time instant $t = t_k^-$. Although t_k is not known apriori, time instant t_k becomes known to the ith system if $s_{il}(t)$ has a binary change at $t = t_k$ and for some l. Then, at $t = t_k^+$, the ith system invokes switching law (11) and $\hat{\gamma}_i(t_k)$ is reset, and so does the jth system for those j with $s_{ji}(t_k^+) = 1$. If $D(t_k^+)$ is irreducible, this chain of resetting will instantaneously propagate to all the systems. If $D(t_k^+)$ is lower triangularly complete and the ith system corresponds to the first block of its lower triangular canonical form, the chain of resetting will also instantaneously propagate to all the systems. If $D(t_k^+)$ is lower triangularly complete but the ith system does not correspond to the first block in its lower triangular canonical form, the chain of resetting will only instantaneously propagate to those systems corresponding to the same and lower block rows, but those reset "initial" conditions have no effect on the left-eigenvector estimation (since the corresponding entries in the left eigenvector are always zero). If $D(t_k^+)$ is lower triangularly incomplete and the ith system corresponds to one of the isolated diagonal block in the lower triangular canonical form, then the chain of resetting will instantaneously propagate only to those systems corresponding to the same block row of $D(t_k^+)$. In all the cases, this chain of resetting ensures that all the distributed estimators have appropriate "initial" conditions to estimate the left eigenvector(s).

After the resetting is done and for $t \in (t_k^+,\ t_{k+1})$, the combined dynamics of all the distributed estimators are given again by (12). Using the same argument as those of (12) up to (16), we have

$$\dot{\hat{\gamma}}(t) \to \mathbf{1}_n \otimes \gamma(t_k^+),$$

and the limit can be achieved within $t \in (t_k^+,\ t_k + \tau/4)$.

The proof is completed by noting the above inductive argument. $\square$.

With first left eigenvector, the topological properties of the network is known explicitly at each system. Specifically, straightforward decision can be drawn on connectivity of the overall network and the social standing of any particular system. This enables each of the systems to take any corrective measure in a high-level control of the communication network.

B. Estimation of the Consensus Value

In this section, the expected consensus value $\sigma(t)$ for system (4) over any time interval $[t_k,\ t_{k+1})$ is estimated by a parallelled distributed scheme. Moreover, the consensus estimator $\hat{\sigma}_i(t)$ also needs reset properly in order to accommodate the possible topological changes in the network matrix, likewise to the resetting of $\hat{\gamma}_i(t)$. In particular, all the estimators reset their states in the same manner as in (11). That is, if there is no change detected in $d_{ij}(t), \forall j$, and if $d_{ij}(t^+) > 0$ and $\hat{\sigma}_j$ is not reset at time t, then

$$\dot{\hat{\sigma}}_i(t_k) = \mu \sum_{j=1}^{n} d_{ij}(t_k)[\hat{\sigma}_j(t_k) - \hat{\sigma}_i(t_k)]; \tag{17}$$

Otherwise,

$$\hat{\sigma}_i(t) = y_i(t); \tag{18}$$

where $\hat{\sigma}_i(t_0) = y_i(t_0)$.

Hence, once there is any binary change in $S(t)$ or α_{ij} experiences any adjustment or its connected neighbor resets its estimator, the ith system will synchronize $\hat{\sigma}_i$ with the current output y_i. The following theorem concludes the performance of the proposed consensus estimator:

***Theorem** 2*: Consider networked system (4) whose consensus vector is given by (6). Then, under assumption 1 and $\mu > 40/\tau$, $\sigma(t)$ can be estimated distributively by (17) and (18). In particular, for any time sequence $\{t_k : k \in \aleph\}$ and for any topological changes in $S(t_k)$, distributed estimator $\hat{\sigma}_i$ converges to the system's unique consensus value if $D(t)$ is irreducible or reducible but lower triangularly complete. If $D(t)$ is lower triangularly incomplete, $\hat{\sigma}_i$ is not unique since $\gamma(t)$ is not unique, and $\hat{\sigma}_i$ converges to the expected consensus vector associated with each block, as indicated in lemma 1.

Proof: If there is no resetting of $\hat{\sigma}_i$, the closed-loop system for consensus estimator is

$$\dot{\hat{\sigma}}(t) = \mu[-I_{nm} + D(t) \otimes I_m]\hat{\sigma}(t) \tag{19}$$

where $\hat{\sigma}(t_k) = [\hat{\sigma}_1(t_k)^T\ \hat{\sigma}_2(t_k)^T\ \ldots\ \hat{\sigma}_n(t_k)^T]^T \in \Re^{nm}$ is the overall consensus estimator.

The proof can be done by introducing the time sequence $\{t_k : k \in \aleph\}$. In particular, for the interval $t \in [t_0,\ t_1)$,

$$\hat{\sigma}(t) = e^{\mu[-I_{mn}+D(t_0)\otimes I_m](t-t_0)}\hat{\sigma}(t_0)$$

Therefore, after complying with lemma 2, we have

$$\lim_{\mu\to\infty} \hat{\sigma}(t) \to \{[\mathbf{1}_n\gamma^T(t_0)] \otimes I_m\}\hat{\sigma}(t_0) = \sigma(t)$$

where $\sigma(t)$ is defined in (6).

Therefore, $\hat{\sigma}_i(t) \to \sigma_i(t)$ and it is valid for any time interval $[t_k,\ t_{k+1})$ if $\hat{\sigma}(t_k) = y(t_k)$. In addition, the convergence rate of (19) is again specified by $e^{-\mu(1-\lambda_D)(t-t_0)}$, and if $\mu \geq 40/\tau$, the convergence time can be made smaller than $\tau/4$. Consequently, the proof can be done by applying the same induction procedure for any time interval $[t_k,\ t_{k+1})$. $\square$

The performance of the proposed estimators are examined in the following simple example:

Example 1: Consider time subsequence $\{t_k :\ k = 1,2,3\}$ and suppose that $D(t) = D_k$ for $t \in [5(k-1),\ 5k)$, where

$$D_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix} D_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$
$$D_3 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

The distributed observers proposed in this section are implemented with $\tau = 1$, $m = 1$, $\mu = 40$ and $y_0 = [15\ 0\ -30]^T$. Specifically, the expected left eigenvector and consensus value for system 3 at each switch is

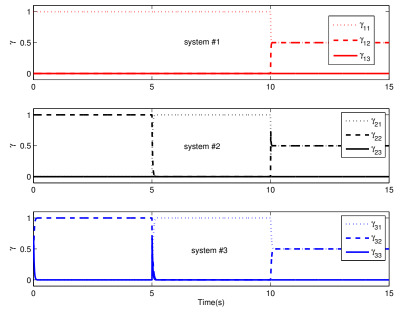
$$\gamma_3(D_1) = [0\ 1\ 0]^T, \gamma_3(D_2) = [1\ 0\ 0]^T, \gamma_3(D_3) = [0.5\ 0.5\ 0]^T,$$
$$\sigma_3(D_1) = 0,\ \sigma_3(D_2) = 15,\ \sigma_3(D_3) = 15$$

Performance of the observers are illustrated in figure 1, where γ_{ij}^* is the jth component of $\hat{\gamma}_i$. It can be easily verified that the expected γ_i and σ_i can be achieved at each system, which are consistent with the results in theorems 1 and 2. And, the convergence is prompt.

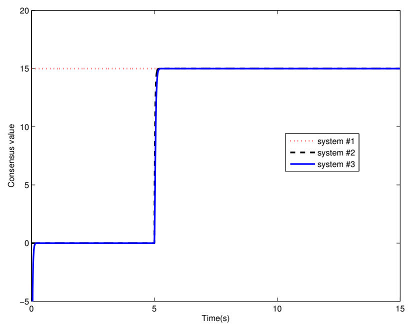
$\triangle$

IV. Adaptive Gains to Improve Convergence Rate

In this section, a distributive strategy is provided to explore the full potential of improving the network convergence by adjusting the cooperative control gain $\alpha_{ij}(t)$ adaptively based on the successful estimation of left eigenvector and consensus value. Without loss of generality, in what follows,



(a) left eigenvector



(b) Consensus vector

Fig. 1: Performance of real-time estimation

we assume the output of each system is a scalar, that is $m=1$, the case of $m>1$ can be derived analogously.

Before proceeding further, the following lemma is provided to illustrate the implication of non-zero entries $d_{ij}(t)$ on the network convergence. That is, how to adjust $d_{ij}(t)$ (by adjusting associated $\alpha_{ij}(t)$) at time t_*, such that $\dot{V}_c(t_*^+) < \dot{V}_c(t_*^-)$.

Lemma 3: Consider networked system (4) with $m=1$ and $S(t)$ is constant over $t \in [t_k,\ t_{k+1})$ for $k \in \aleph$. Suppose for the ith system, gains $d_{ij}(t)$ remain as constant until, at $t=t_*$, $d_{ii}(t)$ and another non-zero entry $d_{i\ell_i^*}(t)$ are adjusted to $d_{ii}(t_*^+) - \varepsilon_i$ and $d_{i\ell_i^*}(t_*^+) + \varepsilon_i$, respectively at some $t_* \in [t_k,\ t_{k+1})$. Then, the convergence of the overall system is improved provided ℓ_i^* and ε_i are chosen as follows: If system i belongs to the leader group, that is $i \in \mathscr{L}$,

$$\ell_i^* \in \mathscr{N}_i \Longrightarrow |y_i(t_*) - y_{\ell_i^*}(t_*)| = \max_{j\in\mathscr{N}_i} |y_i(t_*) - y_j(t_*)| \tag{20}$$

$$\varepsilon_i = \begin{cases} 0 & \text{if} \quad \delta_i = 0 \\ K_a d_{ii}(t_*^-) & \text{if} \quad \delta_i > 0 \\ -K_a d_{i\ell_i^*}(t_*^-) & \text{if} \quad \delta_i < 0 \end{cases} \tag{21}$$

with $\delta_i = [y_i(t_*) - y_{\ell_i^*}(t_*)][y_i(t_*) - \sigma_i(t_*)]$. If system i belongs to the follower group, that is $i \notin \mathscr{L}$,

$$\ell_i^* \in \mathscr{N}_i \Longrightarrow |y_{\ell_i^*}(t_*) - \sigma_i(t_*)| = \min_{j\in\mathscr{N}_i} |y_i(t_*) - \sigma_i(t_*)| \tag{22}$$

$$\varepsilon_i = \begin{cases} 0 & \text{if} \quad \delta_i' \geq 0 \\ K_a d_{ii}(t_*^-) & \text{if} \quad \delta_i' < 0 \end{cases} \tag{23}$$

where $\delta_i' = |y_{\ell_i^*}(t_*) - \sigma_i(t_*)| - |y_i(t_*) - \sigma_i(t_*)|$.

Proof: After the adjustments of $d_{ii}(t_*)$ and $d_{i\ell_i^*}(t_*)$ are accomplished at every system, network matrix $D(t_*^+)$ becomes

$$D(t_*^+) = D(t_*^-) - \sum_{i=1}^{n} \varepsilon_i e_i e_i^T + \sum_{i=1}^{n} \varepsilon_i e_i e_{\ell_i^*}^T \tag{24}$$

Consider cooperative control Lyapunov function (7) over intervals $[t_k,\ t_*)$ and $[t_*,\ t_{k+1}]$. After direct algebraic calculations of its time derivative along system (4), the implications of gain adjustment can be expressed as

$$\delta\dot{V}_c(t_*) \triangleq \sum_{i=1}^{n} \delta\dot{V}_{c_i}(t_*) = \dot{V}_c(t_*^+) - \dot{V}_c(t_*^-),$$

where, if the network is irreducible (i.e., $\gamma > 0$),

$$\delta\dot{V}_{c_i}(t_*) = -2\varepsilon_i \gamma^T(t_*) e_i [y_i(t_*) - y_{\ell_i^*}(t_*)][y_i - \sigma_i(t_*)]. \tag{25}$$

Hence, we have $\delta\dot{V}_{c_i} < 0$ if and only if $\varepsilon_i[y_i(t_*) - y_{\ell_i^*}(t_*)][y_i - \sigma_i(t_*)] > 0$. It is straightforward to verify that the best choice of ℓ_i^* and ε_i are those according to (21) and (20). In addition, in the case D is reducible but lower triangularly complete, γ is unique but contains zero entries, for $i \in \mathscr{L}$, the same logics in (21) and (20) are still applied. If $i \notin \mathscr{L}$, its state is expected to track the expected consensus value in order for a faster convergence, and

$$\begin{aligned} \frac{d[y_i(t) - \sigma(t_k)]}{dt} &= \sum_{j\in\mathscr{N}_i}^{n} d_{ij}(t)[y_j(t) - \sigma(t_k)] \\ &\quad - [1 - d_{ii}(t)][y_i(t) - \sigma(t_k)]. \end{aligned} \tag{26}$$

Therefore, $d_{ii}(t)$ can be adjusted adaptively to make the term $[1 - d_{ii}(t)][y_i(t) - \sigma(t_k)]$ more prominent, such that $y_i(t) \to \sigma_i(t)$ can be converged faster, adaptation logic (23) and (22) is thus selected. This concludes the proof of lemma 3. □

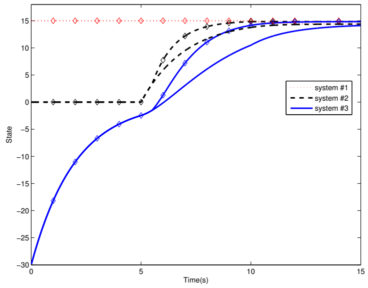
Based on lemma 3, the adaptation of $\alpha_{ij}(t)$ can be designed and implemented adaptively and distributively, its performance is summarized into the following theorem:

Theorem 3: Consider networked system (1) with input (3) and assuming $m=1$. Under assumption 1 and $\gamma_i(t)$ and $\sigma_i(t)$ can be estimated locally. Then, convergence of the network can be improved provided $\alpha_{ij}(t)$ are adjusted as follows: $\alpha_{ij}(t) = \alpha_{ij}(t^-)$ whenever $t \neq \hat{t}_k^i + 0.5l\tau$. Otherwise

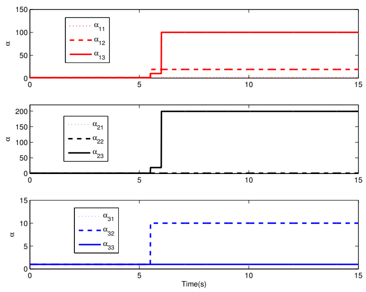
$$\alpha_{ij}(t^+) = \begin{cases} \frac{1}{\kappa_i}\alpha_{ii}(t^-)\left[1 - \frac{\varepsilon_i}{d_{ii}(t^-)}\right] & j = i \\ \frac{1}{\kappa_i}\alpha_{i\ell_i^*}(t^-)\left[1 + \frac{\varepsilon_i}{d_{i\ell_i^*}(t^-)}\right] & j = \ell_i^* \\ \frac{1}{\kappa_i}\alpha_{ij}(t^-) & \text{otherwise} \end{cases} \tag{27}$$

where $\kappa_i = \min\left\{1, \alpha_{ii}(t^-) - \frac{\alpha_{ii}(t^-)\varepsilon_i}{d_{ii}(t^-)}, \alpha_{i\ell_i^*}(t^-) + \frac{\alpha_{ii}(t^-)\varepsilon_i}{d_{i\ell_i^*}(t^-)}\right\}$, $\alpha_{ij}(t_0) = 1$ for all j, t_k is the most recent time instant when the topology changes, $\hat{t}_k^i$ is the estimation of t_k at system i, $l \in \aleph$, ε_i and ℓ_i^* are specified in lemma 3.

Proof: The motivation of adjusting $\alpha_{ij}(t)$ is to achieve an expected ratio between each row of $\alpha_{ij}(t)$ for all $j \in \mathscr{N}_i$, such that the desired $d_{ij}(t)$ can be rendered at each system. Specifically, consider the interval $t \in (\hat{t}_k^i + 0.5l\tau,\ \hat{t}_k^i + 0.5l\tau + 0.5\tau]$, it follows from (27) that there are at most two $\alpha_{ij}(t)$



(a) Consensus speeds of state variables



(b) Time history of gains $\alpha_{ii}(t)$

Fig. 2: Performance with gain adaptations

are adjusted, the sum $[d_{ii}(t) + d_{i\ell_i^*}(t)]$ is continuous, and according to (3)

$$\frac{\alpha_{ii}(t)}{\alpha_{i\ell_i^*}(t)} = \frac{d_{ii}(t)}{d_{i\ell_i^*}(t)}$$

Hence, in order to adjust $d_{ii}(t)$ and $d_{i\ell_i^*}(t)$ to $d_{ii}(t) - \varepsilon_i$ and $d_{i\ell_i^*}(t) + \varepsilon_i$, respectively, the corresponding changes of $\alpha_{ii}(t)$ and $\alpha_{i\ell_i^*}(t)$ can be captured by the adaptation logic (27). Moreover, since the estimation of γ_i and σ_i has already converged (or very close to) to its desired value at t, the convergence rate of the network can be ensured as indicated in lemma 3. This concludes the proof of theorem 3. □

The following example is used to illustrated the proposed adaptation algorithm.

Example 1 (Continued): Consider the communication matrix S_k associated with $D(t) = D_k$ over $t \in [5(k-1),\ 5k)$ for $k = 1, 2, 3$ with $\tau = 1$, $K_a = 0.9$.

Figure 2a shows the comparison between performance under constant gains and that under gain adaptation scheme, where the trajectories under gain adaptation are decorated with diamond markers (versus those without). Specifically, at $t = 15$, the maximal discrepancy between all the states is 0.5 without gain adaptation, and 0 with gain adaptation. Time histories of varying gains α_{ii} is provided in figure 2b, which indicates clearly that estimation convergence and subsequent gain adaptations occur consecutively after each of the topology changes. And, convergence is improved under the proposed gain adaptation scheme.

V. Conclusion

This paper studies distributed estimation and control problems of directed network. A gain adaptation cooperative control scheme is proposed based on distributed estimation of the first left eigenvector and expected consensus vector. Moreover, the proposed estimation scheme parallels to the physical cooperative control system, providing straightforward and real time solution to self-awareness of network structure. Simulation results demonstrate the effectiveness of the proposed scheme.

The proposed distributed gain adaptation scheme is developed using the Lyapunov direct method. Specifically, control gains are adjusted in a distributive and asynchronous manner such that the time derivative of cooperative control Lyapunov function becomes more negative for the purpose of enhancing convergence.

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