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Stability of Switched Stochastic Dynamical Systems driven by Brownian Motion and Markov Modulated Compound Poisson Process

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Abstract— **Stability conditions of continuous-time switched stochastic dynamical systems driven by a Brownian motion and a Markov modulated compound Poisson process are provided. The mode signal, which manages the transition between subsystems, is modeled as a Markov chain. The state variables of the switched stochastic system are subject to jumps of random size occurring at random instances. The intensity of the occurrences, as well as the size of these jumps are modulated by the mode signal. A comparison approach is employed to show the almost sure asymptotic stability of the zero solution. Finally, an illustrative numerical example is presented to demonstrate the efficacy of our results.**

I. Introduction

Many real life processes from finance, physics and engineering fields are subject to noise and random environmental variations. Stochastic hybrid systems extend deterministic hybrid systems by including stochasticity in the dynamics to describe these processes accurately. There has been increasing amount of studies regarding the stability of various classes of stochastic hybrid systems. Particularly, the stability of switched stochastic systems has attracted attention and been explored in several studies [1]–[8]. In addition, researchers have recently shown interest in the stability of switched stochastic systems with state jumps [9]–[12].

In this paper, we explore stability conditions of switched stochastic dynamical systems that incorporate several types of random elements. First, the dynamics include Brownian motion. Second, we model the mode signal as a Markov chain. In addition to Brownian motion and the probabilistic mode signal, the state of the dynamical system is subject to jumps of random size occurring at random instances. We model the occurrences and size of the state jumps by employing a compound Poisson process modulated by the mode signal. Specifically, the intensity of occurrences of state jumps depends on the active mode. Moreover, the state jump size, at a particular state jump instance, is modeled as a random variable distributed by a cumulative distribution function assigned for the active mode. As a result, the distribution of state jump sizes and the intensity of the occurrences of state jumps have different characteristics for each mode. Stochastic models with probabilistic state jumps can describe systems that face randomly occurring sharp and sudden dynamical changes. We employ a comparison approach to derive sufficient conditions of stability. In the literature, comparison approach has been used in several studies to investigate conditions of existence/uniqueness [13], and conditions of stability [4], [13], [14]. In our paper, we derive almost sure asymptotic stability conditions by analyzing a comparison system, which is another switched stochastic dynamical system. The comparison system that we analyze is special in the sense that state jumps occur exactly at mode switching instances. We first investigate the stability of this comparison system, then we use our results to obtain stability conditions of switched stochastic systems driven by the Brownian motion and the Markov modulated compound Poisson process.

The paper is organized as follows. In Section II, the notation used in the paper is explained; a review of Markov chains, Markov modulated compound Poisson processes, and the definition of almost sure asymptotic stability are given. In Section III, we present the mathematical model of switched stochastic dynamical systems driven by the Brownian motion and the Markov modulated compound Poisson process; furthermore, we explain the comparison approach that we use to obtain sufficient conditions of stability. We present a numerical example in Section IV to demonstrate the utility of the results. Finally, we conclude the paper in Section V.

II. Mathematical Preliminaries

In this section we introduce notation, several definitions, and some key results concerning stochastic dynamical systems that are necessary for developing the main results of this paper. Specifically, $\mathbb{R}$ and $\mathbb{R}^+$ respectively denote the set of real numbers and positive real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices, $\mathbb{N}$ and $\mathbb{N}_0$ respectively denote positive and nonnegative integers. $\|\cdot\|$ is the Euclidean vector norm. Furthermore, we write $(\cdot)^{\mathrm{T}}$ for transpose and $\mathrm{tr}(\cdot)$ for trace of a matrix, I_n for the identity matrix of dimension n. A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be positive definite if $V(x) > 0, \quad x \neq 0$, $V(0) = 0$ and proper if $\lim_{\|x\|\to\infty} V(x) = \infty$. Finally, ∇V denotes the vector of the first order spatial derivatives of a twice continuously differentiable scalar V, that is, $\nabla V = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \ldots, \frac{\partial V}{\partial x_n}\right]$ and $\nabla(\nabla V)$ denotes the matrix of the second-order spatial derivatives of V, that is,

$$\nabla(\nabla V) = \begin{bmatrix} \frac{\partial^2 V}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 V}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_n \partial x_n} \end{bmatrix}.$$

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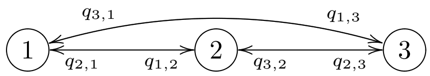

Fig. 1. Transition diagram of a 3-state Markov chain

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. A filtration $\{\mathcal{F}_t\}_{t\geq 0}$ on this probability space is a family of σ-algebras such that

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}, \quad 0 \leq s < t.$$

A stochastic process $\{x(t)\}_{t\geq 0}$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if the random variable $x_t : \Omega \to \mathbb{R}^n$ is $\mathcal{F}_t$-measurable, that is,

$$\{\omega \in \Omega : x_t(\omega) \in B\} \in \mathcal{F}_t$$

for all Borel sets $B \subset \mathbb{R}^n$.

A. Markov Chains and Markov Modulated Compound Poisson Processes

A finite-state Markov chain is a piecewise-constant and right-continuous stochastic process that takes values from a finite set $I \triangleq \{1, 2, \ldots, M\}$. Mathematically, it is the collection of I-valued, $\mathcal{F}_t$-adapted random variables $\{r(t)\}_{t\geq 0}$, with $r(0) = r_0 \in I$. It is characterized by a generator matrix $Q \in \mathbb{R}^{M\times M}$, which determines the transition rates between each pair of states $i, j \in I$ such that

$$\mathrm{P}[r(t+\Delta t) = j | r(t) = i] = \begin{cases} q_{i,j}\Delta t + o(\Delta t), & i \neq j, \\ 1 + q_{i,j}\Delta t + o(\Delta t), & i = j, \end{cases}$$

where $q_{i,j}$ denotes the element on the ith row, jth column of the matrix Q. Note that $q_{i,j} \geq 0$, $i \neq j$ and $q_{i,i} = -\sum_{j\neq i} q_{i,j}$, $i \in I$. A Markov chain can be represented by a state transition diagram. For example, a 3-state Markov chain is represented by a graph of 3 nodes as shown in Fig. 1. The nodes in the figure represent the states of the Markov chain, the arrowed edges represent a possible transition between the states in the direction of the arrows, and the labels on the edges indicate the transition rates between the paired states. A Markov chain is called "irreducible" if it is possible to reach from any state to another state with one or more transitions. Thus, a Markov chain is irreducible if there exist a directed path from each node to another node in the state transition diagram. For example, the Markov chain presented in Fig. 1 is irreducible provided $q_{i,j}$, $i, j \in \{1, 2, 3\}$, are nonzero. For all finite-state, irreducible Markov chains there exists a unique stationary probability distribution $\pi \in \mathbb{R}^M$ such that $\pi^{\mathrm{T}} Q = 0$, $\pi_i > 0$, $i \in I$, and $\sum_{i\in I} \pi_i = 1$ [15].

A Markov modulated compound Poisson process is a pure jump stochastic process, mathematically defined as a collection of $\mathbb{R}$-valued, $\mathcal{F}_t$-adapted random variables $\{Z(t)\}_{t\geq 0}$, given by

$$Z(t) = \sum_{k=1}^{N(t)} \xi_{r(t_k)}(k),$$

for $t \geq 0$ with initial condition $Z(0) = 0$, where $\{N(t)\}_{t\geq 0}$ is a Markov modulated Poisson process that counts the number of jumps that occur in the interval $(0, t]$, $\{r(t) \in I \triangleq \{1, 2, \ldots, M\}\}_{t\geq 0}$ is a finite-state Markov chain, and $\{t_k > 0,\, k \in \mathbb{N}\}$ is the sequence of jump time instances. The intensity parameter of the Markov modulated Poisson process $\{N(t)\}_{t\geq 0}$ is given by the piecewise constant process $\{\lambda_{r(t)}\}_{t\geq 0}$, where $\lambda_i > 0$, $i \in I$. The size of the jump that occurs at time t_k is given by the random variable $\xi_{r(t_k)}(k) \in \mathbb{R}$ which is distributed by the cumulative distribution function $F_{r(t_k)}(\nu) \triangleq \mathrm{P}[\xi_{r(t_k)}(k) \leq \nu]$. Note that both jump sizes and the intensity of occurrences of jumps depend on the Markov chain. The distribution functions F_i, $i \in I$, can be purely continuous or purely discrete functions; they can also be combinations of both. Expectation of a function $\mu : \mathbb{R} \to \mathbb{R}$ of $\xi_{r(t_k)}(k)$ is given by $\mathbb{E}[\mu(\xi_{r(t_k)}(k))] \triangleq \int_{-\infty}^{\infty} \mu(\nu)\mathrm{d}F_{r(t_k)}(\nu)$. The discrete random process $\{\xi_{r(t_k)}(k),\, k \in \mathbb{N}\}$ is independent of $\{N(t)\}_{t\geq 0}$. Furthermore, occurrences of jumps are independent of the Markov chain $\{r(t)\}_{t\geq 0}$. A Markov modulated compound Poisson process has finite number of jumps in a finite time interval, almost surely.

In this study, the mode signal, which manages the transition between subsystems (modes) of the switched system, is modeled as a finite-state Markov chain. Additionally, we assume that the state variable of the switched system is subject to jumps which are modulated by the mode signal of the switched system.

B. Almost Sure Asymptotic Stability

In our analysis we adopt almost sure asymptotic stability notion. The zero solution $x(t) \equiv 0$ of a stochastic system is asymptotically stable almost surely if

$$\mathrm{P}\{\omega \in \Omega : \lim_{t\to\infty} \|x_t(\omega)\| = 0\} = 1, \tag{1}$$

for $t \geq 0$ and $x_0(\omega) = x_0 \in \mathbb{R}^n$. This notion is also called "asymptotic stability with probability one" [16].

III. Stability Analysis for Switched Stochastic Dynamical Systems with State Jumps

In this section we provide sufficient conditions for almost sure asymptotic stability of switched stochastic dynamical systems with state jumps. First, we give the mathematical model of a continuous-time multi-dimensional switched stochastic system with state jumps where state jump intensities and jump sizes are modulated by the mode signal of the system.

A. Mathematical Model

Consider the switched stochastic dynamical system driven by the Brownian motion and the Markov modulated compound Poisson process given by

$$\mathrm{d}x(t) = f_{r(t)}(x(t))\mathrm{d}t + G_{r(t)}(x(t))\mathrm{d}W(t), \quad t \neq t_k, \tag{2}$$

$$x(t) = \xi_{r(t^-)}(k) J_{r(t^-)}(x(t^-)), \quad t = t_k,\, k \in \mathbb{N}, \tag{3}$$

for $t \geq 0$ with initial conditions $x(0) = x_0$, $r(0) = r_0$, where $\{x(t)\}_{t\geq 0}$ is the $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted state vector and $\{W(t)\}_{t\geq 0}$ is an $\mathbb{R}^l$-valued $\mathcal{F}_t$-adapted Wiener process. The dynamical system described by (2), (3) is assumed to have $M \geq 1$ number of subsystems (modes). The transition between the modes is characterized by the piecewise constant mode signal $\{r(t) \in I \triangleq \{1,2,\ldots,M\}\}_{t\geq 0}$, which is assumed to be an irreducible Markov chain with generator matrix $Q \in \mathbb{R}^{M\times M}$ with a stationary probability distribution $\pi \in \mathbb{R}^M$. The continuous dynamics of the modes are described by the vector- and the matrix-valued functions $f_i : \mathbb{R}^n \to \mathbb{R}^n$ and $G_i : \mathbb{R}^n \to \mathbb{R}^{n\times l}$ with $f_i(0) = 0$, $G_i(0) = 0$, $i \in I$, respectively. The state of the system is discontinuous at state jump instances, $\{t_k > 0, k \in \mathbb{N}\}$. At these time instances, the state moves to a random point in the state space according to the equation (3), where $\{\xi_{r(t_k^-)}(k) > 0, k \in \mathbb{N}\}$ is a discrete stochastic process composed of $\mathbb{R}^+$-valued random variables distributed by the cumulative distribution functions $F_{r(t_k)}(\nu) \triangleq \mathrm{P}[\xi_{r(t_k)}(k) \leq \nu]$. The cumulative distribution functions for all modes, F_i, $i \in I$ are assumed to have positive support, that is, there exist $\theta_i > 0$, $i \in I$ such that $\mathrm{P}[\xi_i(k) < \theta_i] = 0$, $i \in I$, $k \in \mathbb{N}$. Furthermore, for the vector-valued functions $J_i : \mathbb{R}^n \to \mathbb{R}^n$, $i \in I$, we assume $J_i(x(t)) = 0$, only when $x(t) = 0$. Hence, at state jump instances, non-zero states are not reset to the origin, although the state size may decrease. The occurrences of state jumps are governed by the Markov modulated Poisson process $\{N_{\mathrm{J}}(t) \in \mathbb{N}_0\}_{t\geq 0}$ with the mode dependent intensity $\{\lambda_{r(t)}\}_{t\geq 0}$, where $\lambda_i > 0$ denotes the intensity of occurrences of state jumps when ith mode is active. Note that the distribution of state jump sizes and the intensity of the occurrences of state jumps differ for each mode.

We assume that the stochastic processes $\{N_{\mathrm{J}}(t)\}_{t\geq 0}$ and $\{r(t)\}_{t\geq 0}$ are independent of $\{W(t)\}_{t\geq 0}$. Furthermore, the stochastic processes $\{f_i(x(t))\}_{t\geq 0}$ and $\{G_i(x(t))\}_{t\geq 0}$, $i \in I$ are assumed to be adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ to ensure that the Ito integrals with these terms are well defined. For existence of a unique solution, it is assumed that all modes of the switched system satisfy linear growth and local Lipschitz continuity conditions.

B. Stability Analysis via a Comparison Approach

We will now show almost sure asymptotic stability of the system (2), (3) by employing a comparison approach. Specifically, we will obtain sufficient stability conditions for the dynamical system (2), (3) by analyzing the stability of a "comparison system", which is another switched stochastic dynamical system with state jumps. The comparison system that we analyze is special in the sense that state jumps occur exactly on mode switching instances.

Consider the comparison system,

$$\mathrm{d}v(t) = \bar{f}_{\bar{r}(t)}(v(t))\mathrm{d}t + \bar{G}_{\bar{r}(t)}(v(t))\mathrm{d}W(t),\ t \neq \tau_s, \tag{4}$$

$$v(t) = \eta_{\bar{r}(t^-),\,\bar{r}(t)}(s)\bar{J}_{\bar{r}(t^-),\bar{r}(t)}(v(t^-)),\ t = \tau_s,\ s \in \mathbb{N}, \tag{5}$$

for $t \geq 0$, $v(0) = v_0$, $\bar{r}(0) = \bar{r}_0$, where $\{v(t) \in \mathbb{R}^n\}_{t\geq 0}$ is the state vector, $\{W(t) \in \mathbb{R}^l\}_{t\geq 0}$ is the Wiener process, $\{\bar{r}(t) \in \bar{I} \triangleq \{1,2,\ldots,\bar{M}\}\}_{t\geq 0}$ is the mode signal which is assumed to be an irreducible Markov chain with the generator matrix $\bar{Q}$ with a stationary probability distribution $\bar{\pi} \in \mathbb{R}^{\bar{M}}$. The mode signal $\bar{r}(t)$ randomly chooses a value from the index set $\bar{I}$ at mode switching time instances, $\{\tau_s > 0, s \in \mathbb{N}\}$. Additionally, at mode switching instances, the state is subject to state jumps according to the equation (5), where $\{\eta_{\bar{r}(\tau_s^-),\bar{r}(\tau_s)}(s) > 0, s \in \mathbb{N}\}$ is a discrete stochastic process composed of $\mathbb{R}^+$-valued random variables distributed by the cumulative distribution functions $\bar{F}_{\bar{r}(\tau_s^-),\bar{r}(\tau_s)}(y) \triangleq \mathrm{P}[\eta_{\bar{r}(\tau_s^-),\bar{r}(\tau_s)}(s) \leq y]$. The cumulative distribution functions $\bar{F}_{i,j}$, $i,j \in \bar{I}$ are assumed to have positive support, that is, there exist $\bar{\theta}_{i,j} > 0$, $i,j \in \bar{I}$ such that $\mathrm{P}[\eta_{i,j}(s) < \bar{\theta}_{i,j}] = 0$, $i,j \in \bar{I}$, $s \in \mathbb{N}$. Furthermore, for the vector-valued functions $\bar{J}_{i,j} : \mathbb{R}^n \to \mathbb{R}^n$, $i,j \in \bar{I}$, we assume $\bar{J}_{i,j}(v(t)) = 0$, only when $v(t) = 0$. Note that the state jumps occur exactly on mode switching instances, and jump sizes depend on two modes: the mode right before the switch (denoted by $\bar{r}(\tau_s^-)$), and the mode that becomes active right after the switch (denoted by $\bar{r}(\tau_s)$).

The stability of the switched stochastic system with state jumps that occur on mode switching instances given by (4), (5) can be analyzed using a Lyapunov-like function $\bar{V} : \mathbb{R}^n \to \mathbb{R}$. In the statement of the following results, let $\mathrm{L}_i\bar{V}(v) \triangleq \nabla\bar{V}(v)\bar{f}_i(v) + \frac{1}{2}\mathrm{tr}\big(\bar{G}_i(v)\bar{G}_i^{\mathrm{T}}(v)\nabla\big(\nabla\bar{V}(v)\big)\big)$, $i \in \bar{I}$.

Theorem 3.1: Consider the nonlinear stochastic switched system (4), (5). If there exist a twice continuously differentiable, positive definite, and proper function $\bar{V} : \mathbb{R}^n \to \mathbb{R}$, scalars $\bar{\zeta}_i \in \mathbb{R}$, $\bar{\alpha}_i \geq 0$, $\bar{\beta}_i \geq 0$, $i \in \bar{I}$ and positive definite functions $\bar{\mu}_{i,j} : \mathbb{R} \to \mathbb{R}$, $i,j \in \bar{I}$ such that

$$\mathrm{L}_i\bar{V}(v) \leq \bar{\zeta}_i\bar{V}(v), \quad i \in \bar{I}, \tag{6}$$

$$\bar{\alpha}_i\bar{V}(v) \leq \|\nabla\bar{V}(v)\bar{G}_i(v)\| \leq \bar{\beta}_i\bar{V}(v), \quad i \in \bar{I}, \tag{7}$$

$$\bar{V}(y\bar{J}_{i,j}(v)) \leq \bar{\mu}_{i,j}(y)\bar{V}(v), \quad i,j \in \bar{I}, \tag{8}$$

$$\sum_{i\in\bar{I}}\bar{\pi}_i\Big(\sum_{j\neq i}\bar{q}_{i,j}\bar{\gamma}_{i,j} + \bar{\zeta}_i - \frac{\bar{\alpha}_i^2}{2}\Big) < 0, \tag{9}$$

where $\bar{\gamma}_{i,j} \triangleq \int_{-\infty}^{\infty}\ln(y)\mathrm{d}\bar{F}_{i,j}(y)$, $i,j \in \bar{I}$, then the zero solution $v(t) \equiv 0$ of the system (4), (5) is asymptotically stable almost surely.

Proof: All subsystems of the switched system (4), (5) are described by multi-dimensional Ito stochastic differential equations. We can employ Ito formula to obtain

$$\mathrm{d}\bar{V}(v(t)) = \mathrm{L}_i\bar{V}(v(t))\mathrm{d}t + \nabla\bar{V}(v(t))\bar{G}_i(v(t))\mathrm{d}W(t), \tag{10}$$

which determines the evolution of $\bar{V}(v(t))$, between consequent switching instances, when the ith mode is active. Now consider the function $\ln\bar{V}(v(t))$, which is well-defined for non-zero values of the state, since $\bar{V}$ is a positive definite function. We use Ito formula once again to compute

$$\begin{aligned}\mathrm{d}\ln\bar{V}(v(t)) = {} & \frac{1}{\bar{V}(v(t))}\mathrm{L}_i\bar{V}(v(t))\mathrm{d}t \\ & - \frac{1}{2\bar{V}^2(v(t))}\|\nabla\bar{V}(v(t))\bar{G}_i(v(t))\|^2\mathrm{d}t \\ & + \frac{1}{\bar{V}(v(t))}\nabla\bar{V}(v(t))\bar{G}_i(v(t))\mathrm{d}W(t).\end{aligned} \tag{11}$$

During the time interval $[\tau_0, \tau_1)$ where $\tau_0 = 0$ and τ_1 denotes the first mode switching instance, the system evolves according to the dynamics of the mode $\bar{r}_0$. We integrate (11) over this time interval to obtain

$$\begin{aligned}
&\ln \bar{V}(v(\tau_1^-)) \\
&\quad = \ln \bar{V}(v(0)) + \int_0^{\tau_1} \frac{1}{\bar{V}(v(\tau))} \mathrm{L}_{\bar{r}(\tau)} \bar{V}(v(\tau)) \mathrm{d}\tau \\
&\quad\quad - \int_0^{\tau_1} \frac{1}{2\bar{V}^2(v(\tau))} \|\nabla \bar{V}(v(\tau)) \bar{G}_{\bar{r}(\tau)}(v(\tau))\|^2 \mathrm{d}\tau \\
&\quad\quad + \int_0^{\tau_1} \frac{1}{\bar{V}(v(\tau))} \nabla \bar{V}(v(\tau)) \bar{G}_{\bar{r}(\tau)}(v(\tau)) \mathrm{d}W(\tau), \quad (12)
\end{aligned}$$

By (6) and (7), it follows that

$$\begin{aligned}
&\ln \bar{V}(v(\tau_1^-)) \\
&\quad \leq \ln \bar{V}(v(0)) + \int_0^{\tau_1} (\bar{\zeta}_{\bar{r}(\tau)} - \frac{\bar{\alpha}^2_{\bar{r}(\tau)}}{2}) \mathrm{d}\tau \\
&\quad\quad + \int_0^{\tau_1} \frac{1}{\bar{V}(v(\tau))} \nabla \bar{V}(v(\tau)) \bar{G}_{\bar{r}(\tau)}(v(\tau)) \mathrm{d}W(\tau). \quad (13)
\end{aligned}$$

At mode switching instance τ_1, the state is subject to a jump according to equation (5). By using (8), we obtain

$$\begin{aligned}
&\ln \bar{V}(v(\tau_1)) \\
&\quad = \ln \bar{V}\big(\eta_{\bar{r}(\tau_0),\bar{r}(\tau_1)}(1) \bar{J}_{\bar{r}(\tau_0),\bar{r}(\tau_1)}(v(\tau_1^-))\big) \\
&\quad \leq \ln \bar{\mu}(\eta_{\bar{r}(\tau_0),\bar{r}(\tau_1)}(1)) \bar{V}(v(\tau_1^-)) \\
&\quad = \ln \bar{\mu}(\eta_{\bar{r}(\tau_0),\bar{r}(\tau_1)}(1)) + \ln \bar{V}(v(\tau_1^-)) \\
&\quad \leq \ln \bar{\mu}(\eta_{\bar{r}(\tau_0),\bar{r}(\tau_1)}(1)) + \ln \bar{V}(v(0)) \\
&\quad\quad + \int_0^{\tau_1} (\bar{\zeta}_{\bar{r}(\tau)} - \frac{\bar{\alpha}^2_{\bar{r}(\tau)}}{2}) \mathrm{d}\tau \\
&\quad\quad + \int_0^{\tau_1} \frac{1}{\bar{V}(v(\tau))} \nabla \bar{V}(v(\tau)) \bar{G}_{\bar{r}(\tau)}(v(\tau)) \mathrm{d}W(\tau). \quad (14)
\end{aligned}$$

We repeat the calculations until arbitrary time t and use (6), (7), (8) to obtain

$$\begin{aligned}
&\ln \bar{V}(v(t)) \\
&\quad \leq \ln \bar{V}(v(0)) + \sum_{s=1}^{\bar{N}_{\mathrm{S}}(t)} \ln \bar{\mu}(\eta_{\bar{r}(\tau_{s-1}),\bar{r}(\tau_s)}(s)) \\
&\quad\quad + \int_0^t \bar{\zeta}_{\bar{r}(\tau)} - \frac{\bar{\alpha}^2_{\bar{r}(\tau)}}{2} \mathrm{d}\tau \\
&\quad\quad + \int_0^t \frac{1}{\bar{V}(v(\tau))} \nabla \bar{V}(v(\tau)) \bar{G}_{\bar{r}(\tau)}(v(\tau)) \mathrm{d}W(\tau) \quad (15)
\end{aligned}$$

where $\bar{N}_{\mathrm{S}}(t)$ denotes the total number of mode switches in the time interval $(0, t]$. By the strong law of large numbers for irreducible Markov chains [15], [17] we have

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t} \sum_{s=1}^{\bar{N}_{\mathrm{S}}(t)} \ln \bar{\mu}(\eta_{\bar{r}(\tau_{s-1}),\bar{r}(\tau_s)}(s)) \\
&\quad = \sum_{i\in\bar{I}} \bar{\pi}_i \sum_{j\neq i} \bar{q}_{i,j} \bar{\gamma}_{i,j}, \quad (16)
\end{aligned}$$

where $\bar{\gamma}_{i,j} = \int_{-\infty}^{\infty} \ln \mu_{i,j}(y) \mathrm{d}\bar{F}_{i,j}(y)$, $i, j \in \bar{I}$, and

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \bar{\zeta}_{\bar{r}(\tau)} - \frac{\bar{\alpha}^2_{\bar{r}(\tau)}}{2} \mathrm{d}\tau = \sum_{i\in\bar{I}} \bar{\pi}_i (\bar{\zeta}_i - \frac{\bar{\alpha}_i^2}{2}), \quad (17)$$

almost surely. Furthermore, the Ito integral in inequality (15),

$$L(t) = \int_0^t \frac{1}{\bar{V}(v(\tau))} \nabla \bar{V}(v(\tau)) \bar{G}_{\bar{r}(\tau)}(v(\tau)) \mathrm{d}W(\tau) \quad (18)$$

is a local martingale with quadratic variation

$$\begin{aligned}
[L]_t &= \int_0^t \|\frac{1}{\bar{V}(v(\tau))} \nabla \bar{V}(v(\tau)) \bar{G}_{\bar{r}(\tau)}(v(\tau))\|^2 \mathrm{d}\tau \\
&\leq \int_0^t \max_{i\in\bar{I}} \bar{\beta}_i \mathrm{d}\tau = \max_{i\in\bar{I}} \bar{\beta}_i t. \quad (19)
\end{aligned}$$

Consequently, $\lim_{t\to\infty} \frac{1}{t}[L]_t < \infty$. Thus, by using the same approach presented in [2], [6], we can employ the strong law of large numbers for local martingales [16] to show

$$\lim_{t\to\infty} \frac{1}{t} L(t) = 0, \quad (20)$$

almost surely. Moreover, it follows from (15)–(17), and (20) that

$$\lim_{t\to\infty} \frac{1}{t} \ln \bar{V}(v(t)) \leq \sum_{i\in\bar{I}} \bar{\pi}_i \big(\sum_{j\neq i} \bar{q}_{i,j} \bar{\gamma}_{i,j} + \bar{\zeta}_i - \frac{\bar{\alpha}_i^2}{2} \big). \quad (21)$$

Finally, by (9),

$$\mathrm{P}[\lim_{t\to\infty} \bar{V}(v(t)) = 0] = 1, \quad (22)$$

which implies almost sure asymptotic stability of the zero solution. □

We will now show that the switched system (2), (3) can be expressed as the comparison system given by (4), (5). For the switched stochastic dynamical system (2), (3), the state jumps and mode switchings occur independently. As a consequence, it can be shown that a state jump and a mode switching do not occur at the same time, with probability one. We introduce the comparison system as a new switched system of $\bar{M} = 2M$ modes, where the first M modes and the second M modes share the same dynamics. Specifically, ith and $(i+M)$th mode of this comparison system has the dynamics of ith mode of the system (2), (3). This comparison system is also subject to state jumps, which only occur at mode switching instances.

The transition between the modes of this comparison system can be represented by a special graph structure of $2M$ nodes. In this graph structure, the nodes are placed in two layers. The nodes in the first layer are numbered as $\{1, 2, \ldots, M\}$, and we number the nodes in the second layer as $\{M+1, \ldots, 2M\}$. Mode switchings within the layers of the comparison system (4), (5) correspond to mode switchings in the original system (2), (3), on the other hand, mode switchings between layers in the comparison system correspond to state jumps in the original system. For example, consider the switched stochastic dynamical system (2), (3) with $M = 3$ modes. The transition rates between each mode of the system are shown on the graph presented in Fig. 1. The nodes of this graph represent the modes of the system, and the arrowed edges represent possible transitions between these modes. The transition diagram for the comparison system of this example is shown in the graph presented in Fig. 2. In this graph, an edge that connects a

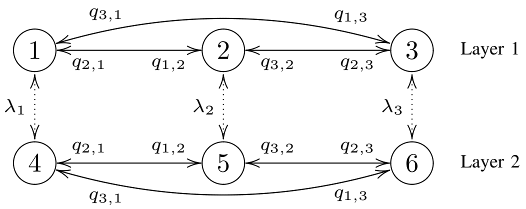

Fig. 2. Transition diagram of a Markov chain of 6 states with a special structure

node in the first layer to a node in the second layer indicates a state jump in the system (2), (3). Specifically, the edges between ith node and $(i+M)$th node represent a state jump while the ith mode of the system (2), (3) is active. Since ith and $(i+M)$th mode share the same dynamics in the comparison system, a transition between these modes does not mean a change in the dynamics of the original system. In addition, the edges between the nodes of the first layer and the edges between the nodes of the second layer represent mode transitions of the system (2), (3).

We now state our main result for the almost sure asymptotic stability of the switched stochastic dynamical system with state jumps (2), (3) based on the stability analysis for the comparison system (4), (5) stated in Theorem 3.1.

Theorem 3.2: Consider the nonlinear stochastic switched system with state jumps (2), (3). If there exist a twice continuously differentiable, positive definite, and proper function $V : \mathbb{R}^n \to \mathbb{R}$, scalars $\zeta_i \in \mathbb{R}$, $\alpha_i \geq 0$, $\beta_i \geq 0$, $i \in I$ and positive definite functions $\mu_i : \mathbb{R} \to \mathbb{R}$, $i \in I$ such that

$$\mathrm{L}_i V(x) \leq \zeta_i V(x), \quad i \in I, \tag{23}$$

$$\alpha_i V(x) \leq \|\nabla V(x) G_i(x)\| \leq \beta_i V(x), \quad i \in I, \tag{24}$$

$$V(\nu J_i(x)) \leq \mu_i(\nu) V(x), \quad i \in I, \tag{25}$$

$$\sum_{i \in I} \pi_i \Big(\lambda_i \gamma_i + \zeta_i - \frac{\alpha_i^2}{2} \Big) < 0, \tag{26}$$

where $\gamma_i \triangleq \int_{-\infty}^{\infty} \ln \mu_i(\nu) \mathrm{d}F_i(\nu)$, $i \in I$, then the zero solution $x(t) \equiv 0$ of the system (2), (3) is asymptotically stable almost surely.

Proof: Consider the Markov modulated Poisson process $\{N_{\mathrm{J}}(t) \in \mathbb{N}\}_{t \geq 0}$, which counts the number of state jumps in the time interval $(0, t]$. We define the piecewise-constant stochastic process $\{Y_{\mathrm{J}}(t)\}_{t \geq 0}$ based on $\{N_{\mathrm{J}}(t) \in \mathbb{N}\}_{t \geq 0}$ as

$$Y_{\mathrm{J}}(t) \triangleq \begin{cases} 1, & N_{\mathrm{J}}(t) = 2k,\, k \in \mathbb{N}_0, \\ 2, & N_{\mathrm{J}}(t) = 2k+1,\, k \in \mathbb{N}_0. \end{cases} \tag{27}$$

Clearly, $\{Y_{\mathrm{J}}(t)\}_{t \geq 0}$ takes values from the finite set $\{1, 2\}$, and $Y_{\mathrm{J}}(0) = 1$. Now consider the bivariate process $\{\tilde{r}(t)\}_{t \geq 0} \triangleq \{(Y_{\mathrm{J}}(t), r(t))\}_{t \geq 0}$, which is a Markov chain with $2M$ states given by $\{(1,1), (1,2), \ldots, (1,M), (2,1), (2,2), \ldots, (2,M)\}$. We enumerate the states in this order as $\{1, 2, \ldots, 2M\}$. The generator of the Markov chain $\{\tilde{r}(t)\}_{t \geq 0}$ is given by

$$\tilde{Q} = \begin{bmatrix} Q - \Lambda & \Lambda \\ \Lambda & Q - \Lambda \end{bmatrix}, \tag{28}$$

where $\Lambda \in \mathbb{R}^{M \times M}$ is the diagonal matrix with the diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_M$. The Markov chain $\{\tilde{r}(t)\}_{t \geq 0}$ is irreducible, and has the stationary probability distributions $\tilde{\pi}_i = \tilde{\pi}_{i+M} = \frac{1}{2}\pi_i$, $i \in I$.

Consider the comparison system (4), (5) for $t \geq 0$ with initial conditions $v(0) = x(0) = x_0$, $\bar{r}(0) = \tilde{r}(0)$, where $\bar{r}(t) = \tilde{r}(t)$, $\bar{f}_i = \bar{f}_{i+M} = f_i$, $\bar{G}_i = \bar{G}_{i+M} = G_i$, $i \in I$,

$$\bar{J}_{i,j}(x) = \begin{cases} x, & i, j \in \{1, 2, \ldots, M\}, \\ x, & i, j \in \{M+1, M+2, \ldots, 2M\}, \\ J_i(x), & j = i + M,\, i \in \{1, 2, \ldots, M\}, \\ J_j(x), & i = j + M,\, j \in \{1, 2, \ldots, M\}, \end{cases}$$

and

$$\bar{F}_{i,j}(y) = \begin{cases} \int_{-\infty}^{y} \delta(\tau - 1) \mathrm{d}\tau, & i, j \in \{1, 2, \ldots, M\}, \\ \int_{-\infty}^{y} \delta(\tau - 1) \mathrm{d}\tau, & i, j \in \{M+1, \ldots, 2M\}, \\ F_i(y), & j = i + M,\, i \in \{1, 2, \ldots, M\}, \\ F_j(y), & i = j + M,\, j \in \{1, 2, \ldots, M\}, \end{cases}$$

where $\delta(\cdot)$ denotes Dirac's delta function. The state variable of the comparison system (4), (5), is equal to the state variable of the original system (2), (3), that is, $v(t) = x(t)$, $t \geq 0$, almost surely. Consequently, almost sure asymptotic stability of the zero solution of the comparison system (4), (5) implies almost sure asymptotic stability of the zero solution of the original system (2), (3). Thus, the result follows from Theorem 3.1 with $\bar{V} = V$, $\bar{\zeta}_i = \bar{\zeta}_{i+M} = \zeta_i$, $\bar{\alpha}_i = \bar{\alpha}_{i+M} = \alpha_i$, $\bar{\beta}_i = \bar{\beta}_{i+M} = \beta_i$, $i \in I$, and

$$\bar{\mu}_{i,j}(y) = \begin{cases} \|y\|, & i, j \in \{1, 2, \ldots, M\}, \\ \|y\|, & i, j \in \{M+1, \ldots, 2M\}, \\ \mu_i(y), & j = i + M,\, i \in \{1, 2, \ldots, M\}, \\ \mu_j(y), & i = j + M,\, j \in \{1, 2, \ldots, M\}. \end{cases}$$

□

We have obtained the stability conditions (23)–(26) for the system (2), (3), by adapting the stability conditions for the comparison system (4), (5).

IV. Illustrative Numerical Example

In this section, we present a numerical example to demonstrate the effectiveness of our results. Consider the 2-dimensional switched stochastic dynamical system (2), (3) with $M = 3$ modes characterized by the functions

$$f_1(x) = \begin{bmatrix} x_1 + x_1 \sin^2 x_2 \\ x_2 \sin^2 x_1 + x_2 \end{bmatrix},\ G_1(x) = \begin{bmatrix} 0.5x_1 \\ 0.5x_2 \end{bmatrix},$$

$$f_2(x) = \begin{bmatrix} \sin(x_1)\cos(x_2)x_1 \\ x_2 \end{bmatrix},\ G_2(x) = \begin{bmatrix} 0.7x_1 \\ 0.5x_2 \end{bmatrix},$$

$$f_3(x) = \begin{bmatrix} -2x_1 + x_2 \sin^2(x_1) \\ -x_1 \sin^2(x_1) - 2x_2 \end{bmatrix},\ G_3(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

and $J_1(x) = [2x_1,\, 3x_2]^{\mathrm{T}}$, $J_2(x) = [0.5x_1,\, 0.5x_2]^{\mathrm{T}}$, $J_3(x) = [x_1,\, x_2]^{\mathrm{T}}$. The mode signal $\{r(t) \in I \triangleq \{1, 2, 3\}\}_{t \geq 0}$ of the system is assumed to be a 3-state Markov

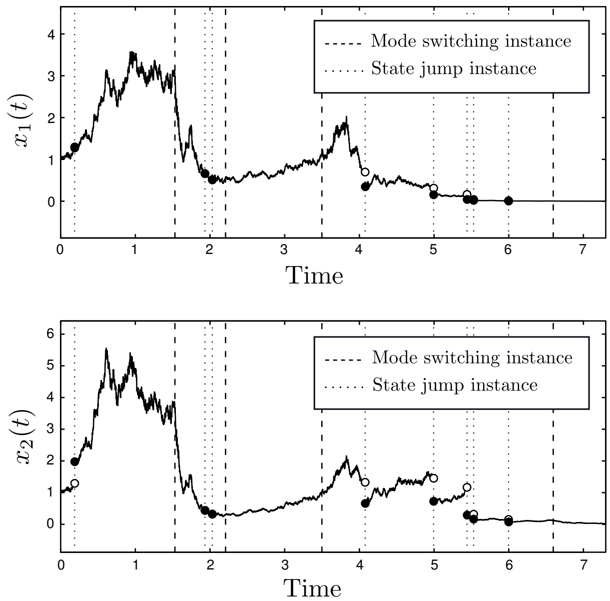

Fig. 3. State trajectory versus time

chain with equal mode transition probabilities. The generator matrix is given by,

$$Q = \begin{bmatrix} -0.5 & 0.25 & 0.25 \\ 0.25 & -0.5 & 0.25 \\ 0.25 & 0.25 & -0.5 \end{bmatrix}.$$

The stationary probability distributions for this Markov chain are $\pi_i = \frac{1}{3}$, $i \in I$. The switched stochastic system is driven by a 1-dimensional Brownian motion and a Markov modulated compound Poisson process with intensities $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 2$. For any state jump instance t_k, the distributions for the state jumps are assumed to be given by the discrete distributions,

$$\begin{aligned} &\mathrm{P}[\xi_1(k) = 0.5] = 0.5, \quad \mathrm{P}[\xi_1(k) = 1] = 0.5, \\ &\mathrm{P}[\xi_2(k) = 0.5] = 0.5, \quad \mathrm{P}[\xi_2(k) = 1] = 0.5, \\ &\mathrm{P}[\xi_3(k) = 1] = 1. \end{aligned}$$

Note that the positive-definite function $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$, the scalars $\alpha_1 = \beta_1 = 1$, $\alpha_2 = 1$, $\beta_2 = 2$, $\alpha_3 = \beta_3 = 2$, $\zeta_1 = 5$, $\zeta_2 = 2$, $\zeta_3 = -1$, and the positive-definite functions $\mu_1(\nu) = 9\nu^2$, $\mu_2(\nu) = \frac{1}{2}\nu^2$, $\mu_3(\nu) = \nu^2$ satisfy the conditions (23)–(26). As a consequence, it follows from Theorem 3.2 that the zero solution $x(t) \equiv 0$ of the system (2), (3) is asymptotically stable almost surely.

With initial conditions $x(0) = [1, 1]^{\mathrm{T}}$ and $r(0) = 1$, we obtain sample paths of $x(t)$ and $r(t)$, which we present in Fig. 3 and Fig. 4 respectively.

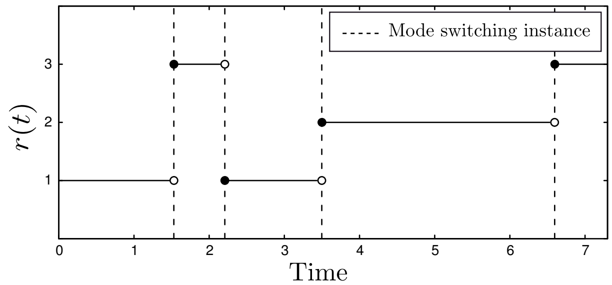

Fig. 4. Mode signal versus time

V. Conclusion

The stability of continuous-time switched stochastic dynamical systems driven by Brownian motion and Markov modulated compound Poisson process has been investigated by employing a comparison approach. Specifically, we first analyzed the stability of a "comparison system", which is another switched stochastic dynamical system with state jumps. The comparison system is special in the sense that the state jumps only occur exactly at mode switching instances. We have shown that the system at hand can be expressed as the comparison system. Furthermore, by analyzing the stability of this comparison system, we obtained sufficient conditions of almost sure asymptotic stability.

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