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Simultaneous global external and internal stabilization of linear time-invariant discrete-time systems subject to actuator saturation

Xu Wang[1] Ali Saberi[1] Anton A. Stoorvogel[2] Peddapullaiah Sannuti[4]

***Abstract*— Simultaneous external and internal stabilization in global framework of linear time-invariant discrete-time systems subject to actuator saturation is considered. Internal stabilization is in the sense of Lyapunov while external stabilization is in the sense of ℓ_p stability with different variations, e.g. with or without finite gain, with fixed or arbitrary initial conditions, with or without bias. Several simultaneous external and internal stabilization problems all in global framework are studied in depth, and appropriate adaptive low-and-high gain feedback controllers that achieve the intended simultaneous external and internal stabilization are constructed whenever such problems are solvable.**

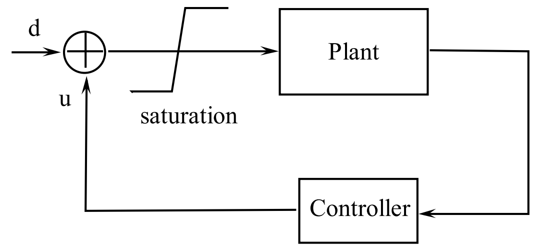

Fig. 1. A linear system subject to actuator saturation

I. Introduction

Most nonlinear systems encountered in practice consist of linear systems and static nonlinear elements. One class of such systems is the class of linear systems subject to actuator saturation as depicted in Figure 1 along with a feedback controller, where u denotes the control input and d is an external input. Since saturation is an ubiquitous non-linearity, during the last two decades, systems as depicted in Figure 1 received intense focus. Many control theoretic issues were studied. Internal stabilization of such systems both in global and semi-global sense was explored by many researchers, and by now there exist several classical results (see [9] and the references therein). Internal stabilization by itself does not in general imply external stabilization. As such, simultaneous external and internal stabilization was initiated in [8] and is also explored in [1], [3], [2], [11]. The picture that emerges in this regard is that, for the case when external input is additive to the control input, all the issues associated with simultaneous external and internal stabilization are more or less resolved, but only for continuous-time systems.

Our focus in this paper is on discrete-time linear systems subject to actuator saturation. For *continuous-time* systems, a key result is given in [8]. This work, while pointing out all the complexities involved in simultaneous *global* external and *global* internal stabilization, resolves all such issues and develops certain scheduled low-and-high gain design methodologies to achieve the required simultaneous global-global stabilization. Analogous results for discrete-time systems do not exist so far in the literature. Discrete-time has its own peculiarities. High-gain cannot be as freely used as in continuous-time, but also almost disturbance decoupling is not possible in general for discrete-time case, while it is so for continuous-time. For discrete-time case, we develop here the sufficient conditions for simultaneous *global* external and *global* internal stabilization, and furthermore develop also the required design methodologies to accomplish such a stabilization whenever it is feasible. The proofs of certain results are very lengthy and hence are omitted. The full version can be found at `www.eecs.wsu.edu/~xwang`.

[1]School of Electrical Engineering and Computer Science, Washington State University, Pullman, WA 99164-2752, U.S.A. E-mail: {xwang,saberi}@eecs.wsu.edu. The work of Ali Saberi and Xu Wang is partially supported by National Science Foundation grants ECS-0528882, NAVY grants ONR KKK777SB001 and ONR KKK760SB0012

[2]Department of Electrical Engineering, Mathematics, and Computing Science, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands. E-mail: A.A.Stoorvogel@utwente.nl

[4]Department of Electrical and Computer Engineering, Rutgers University, 94 Brett Road, Piscataway, NJ 08854-8058, U.S.A., E-mail: sannuti@ece.rutgers.edu

II. Preliminary notations and problem formulation

For $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm and x' denotes the transpose of x. For $X \in \mathbb{R}^{n\times m}$, $\|X\|$ denotes its induced 2-norm and X' denotes the transpose of X. $\operatorname{trace}(X)$ denotes the trace of X. If X is symmetric, $\lambda_{\min}X$ and $\lambda_{\max}X$ denote the smallest and largest eigenvalues of X respectively. For a subset $\mathcal{X} \subset \mathbb{R}^n$, $\mathcal{X}^c$ denotes the complement of $\mathcal{X}$. For $k_1, k_2 \in \mathbb{Z}$ such that $k_1 \le k_2$, $\overline{k_1, k_2}$ denotes the integer set $\{k_1, k_1+1, \dots, k_2\}$.

A continuous function $\phi(\cdot) : [0,\infty) \to [0,\infty)$ is said to belong to class $\mathcal{K}$ if (1) $\phi(0) = 0$ and (2) ϕ is strictly increasing. The ℓ_p space with $p \in [1,\infty)$ consists of all vector-valued discrete-time signals $y(\cdot)$ from $\mathbb{Z}^+ \cup \{0\}$ to $\mathbb{R}^n$ for which

$$\sum_{k=0}^{\infty} \|y(k)\|^p < \infty.$$

For a signal $y \in \ell_p$, the ℓ_p norm of y is defined as

$$\|y\|_p = \left(\sum_{k=0}^{\infty} \|y(k)\|^p\right)^{\frac{1}{p}}.$$

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The ℓ_∞ space consists of all vector-valued discrete-time signals $y(\cdot)$ from $\mathbb{Z}^+ \cup \{0\}$ to $\mathbb{R}^n$ for which

$$\sup_{k\geq 0} \|y(k)\| < \infty.$$

For a signal $y \in \ell_p$, the ℓ_∞ norm of y is defined as

$$\|y\|_\infty = \sup_{k\geq 0} \|y(k)\|.$$

The following relationship holds for all ℓ_p spaces: for $1 < p < q < \infty$

$$\ell_1 \subset \ell_p \subset \ell_q \subset \ell_\infty.$$

Moreover, for any $y \in \ell_p$ with $p \in [1, \infty)$, the following properties hold:

1) $\|y\|_\infty \leq \|y\|_p$;
2) $y(k) \to 0$ as $k \to \infty$.

Next we recall the definitions of external stability. Consider a system

$$\Sigma : \begin{cases} x(k+1) = f(x(k), d(k)), & x(0) = x_0 \\ \quad y(k) \;\; = g(x(k), d(k)) \end{cases}$$

with $x(k) \in \mathbb{R}^n$ and $d(k) \in \mathbb{R}^m$. The two classical ℓ_p stabilities are defined as follows:

Definition 1: For any $p \in [1, \infty]$, the system Σ is said to be **ℓ_p stable with fixed initial condition and without finite gain** if for $x(0) = 0$ and any $d \in \ell_p$, we have $y \in \ell_p$.

Definition 2: For any $p \in [1, \infty]$, the system Σ is said to be **ℓ_p stable with fixed initial condition and with finite gain** if for $x(0) = 0$ and any $d \in \ell_p$, we have $y \in \ell_p$ and there exists a γ_p such that for any $d \in \ell_p$,

$$\|y\|_p \leq \gamma_p \|d\|_p.$$

The infimum over all γ_p with this property is called the ℓ_p gain of the system Σ.

As observed in [10], the initial condition plays a dominant role in whether achieving ℓ_p stability is possible or not. Hence any definition of external stability must take into account the effect of initial condition. The notion of external stability with arbitrary initial condition was introduced in [10]. We recall these definitions below:

Definition 3: For any $p \in [1, \infty]$, the system Σ is said to be **ℓ_p stable with arbitrary initial condition and without finite gain** if for any $x_0 \in \mathbb{R}^n$ and any $d \in \ell_p$, we have $y \in \ell_p$.

Definition 4: For any $p \in [1, \infty]$, the system Σ is said to be **ℓ_p stable with arbitrary initial condition with finite gain and with bias** if for any $x_0 \in \mathbb{R}^n$ and any $d \in \ell_p$, we have $y \in \ell_p$ and there exists a γ_p and a class $\mathcal{K}$ function $\phi(\cdot)$ such that for any $d \in \ell_p$

$$\|y\|_p \leq \gamma_p \|d\|_p + \phi(\|x_0\|).$$

The infimum over all γ_p with this property is called the ℓ_p gain of the system Σ.

Now we are ready to formulate the control problems studied in this paper. Consider a linear discrete-time system subject to actuator saturation,

$$x(k+1) = Ax(k) + B\sigma(u(k) + d(k)), \tag{1}$$

where state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, and external input $d \in \mathbb{R}^m$. Here $\sigma(\cdot)$ denotes the standard saturation function defined as

$$\sigma(u) = [\sigma_1(u_1), \ldots, \sigma_1(u_m)]$$

where $\sigma_1(s) = \text{sgn}(s) \min\{|s|, \Delta\}$ for some $\Delta > 0$.

The simultaneous global external and internal stabilization problems studied in this paper are formulated as follows:

Problem 1: For any $p \in [1, \infty]$, the system (1) is said to be simultaneously globally ℓ_p stabilizable with fixed initial condition and without finite gain and globally asymptotically stabilizable via static time invariant state feedback, which we refer to as (G_p/G), if there exists a static state feedback controller $u = f(x)$ such that the following properties hold:

1) the closed-loop system is ℓ_p stable with fixed initial condition and without finite gain;
2) In the absence of external input d, the equilibrium $x = 0$ is globally asymptotically stable.

Problem 2: For any $p \in [1, \infty]$, the system (1) is said to be simultaneously globally ℓ_p stabilizable with fixed initial condition and with finite gain and globally asymptotically stabilizable via state feedback, which we refer to as $(G_p/G)_{fg}$, if there exists a static time invariant state feedback controller $u = f(x)$ such that the following properties hold:

1) the closed-loop system is finite gain ℓ_p stable with fixed initial condition with finite gain ;
2) In the absence of external input d, the equilibrium $x = 0$ is globally asymptotically stable.

Note that the notion of global ℓ_p stability with arbitrary initial condition embeds in it the internal stability in some sense. We also formulate below additional external stabilization problems with arbitrary initial conditions.

Problem 3: For any $p \in [1, \infty]$, the system (1) is said to be globally ℓ_p stabilizable with arbitrary initial condition and without finite gain via static time invariant state feedback, if there exists a static state feedback controller $u = f(x)$ such that the closed-loop system is ℓ_p stable with arbitrary initial condition and without finite gain.

Problem 4: For any $p \in [1, \infty]$, the system (1) is said to be globally ℓ_p stabilizable with arbitrary initial condition with finite gain and with bias via state feedback, if there exists a static time invariant state feedback controller $u = f(x)$ such that the closed-loop system is finite gain ℓ_p stable with arbitrary initial condition with finite gain and with bias.

Without loss of generality, the following assumption is made throughout the paper:

Assumption 1: The pair (A, B) is controllable, and the matrix A has all its eigenvalues on the unit circle.

III. Controller design

The controller design in this paper is based on the classical low-gain and low-and-high-gain feedback design methodologies. The low-gain feedback can be constructed using different approaches such as direct eigenstructure assignment [4], H_2 and H_∞ algebraic Riccati equation based methods [7], [12] and parametric Lyapunov equation based method

[13], [14]. In this paper, we choose parametric Lyapunov equation method to build the low-gain feedback because of its special properties; as will become clear later on, it greatly simplifies the expressions for our controllers and the subsequent analysis.

Since the low-gain feedback, as indicated by its name, does not allow complete utilization of control capacities, the low-and-high-gain feedback was developed to rectify this drawback and was intended to achieve control objectives beyond stability, such as performance enhancement, robustness and disturbances rejection. The low-and-high gain feedback is composed of a low-gain and a high-gain feedback. As shown in [2], the solvability of simultaneous global external and internal stabilization problem critically relies on a proper choice of high-gain. In this section, we shall first recall the low-gain feedback design and propose a new high-gain design methodology.

A. Low gain state feedback

In this subsection, we review the low-gain feedback design methodology recently introduced in [13], [14] which is based on the solution of a parametric Lyapunov equation. The following lemma is adapted from [14]:

Lemma 1: Assume that (A, B) is controllable and A has all its eigenvalues on the unit circle. For any $\varepsilon \in (0, 1)$, the Parametric Algebraic Riccati Equation,

$$(1-\varepsilon)P_\varepsilon = A'P_\varepsilon A - A'P_\varepsilon B(I+B'P_\varepsilon B)^{-1}B'P_\varepsilon A, \quad (2)$$

has a unique positive definite solution $P_\varepsilon = W_\varepsilon^{-1}$ where W_ε is the solution of

$$W_\varepsilon - \frac{1}{1-\varepsilon}AW_\varepsilon A' = -BB'.$$

Moreover, the following properties hold:

1) $A_c(\varepsilon) = A - B(I + B'P_\varepsilon B)^{-1}B'P_\varepsilon A$ is Schur stable for any $\varepsilon \in (0, 1)$;
2) $\frac{\mathrm{d}P_\varepsilon}{\mathrm{d}\varepsilon} > 0$ for any $\varepsilon \in (0, 1)$;
3) $\lim_{\varepsilon \to 0^+} P_\varepsilon = 0$;
4) There exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$,
$$\|[P_\varepsilon^{\frac{1}{2}} A P_\varepsilon^{-\frac{1}{2}}\| \leq \sqrt{2};$$
5) Let ε^* be given by property 4. There exists a M_{ε^*} such that $\|\frac{P_\varepsilon}{\varepsilon}\| \leq M_{\varepsilon^*}$ for all $\varepsilon \in (0, \varepsilon^*]$.

We define the low-gain state feedback which is a family of parameterized state feedback laws given by

$$u_L(x) = F_L x = -(B'P_\varepsilon B + I)^{-1}B'P_\varepsilon Ax, \quad (3)$$

where P_ε is the solution of (2). Here, as usual, ε is called the low-gain parameter. From the properties given by Lemma 1, it can be seen that the magnitude of the control input can be made arbitrarily small by choosing ε sufficiently small so that the input never saturates for any, a priori given, set of initial conditions.

B. Low-and-high-gain feedback

The low-and-high-gain state feedback is composed of a low-gain state feedback and a high-gain state feedback as

$$u_{LH}(x) = F_{LH}x = F_L x + F_H x \quad (4)$$

where $F_L x$ is given by (3). The high-gain feedback is of the form, $F_H x = \rho F_L x$ where ρ is called the high-gain parameter.

For continuous-time systems, the high gain parameter ρ can be any positive real number. However, this is not the case for discrete-time systems. In order to preserve local asymptotic stability, this high gain has to be bounded at least near the origin. The existing results in literature on the choice of high-gain parameter for discrete-time system are really sparse. To the best of our knowledge, the only available result is in [5], [6] where the high-gain parameter is a nonlinear function of x. To solve the global external and internal stabilization problem, we need to schedule the high-gain parameter with respect to x. However, this nonlinear high-gain parameter is not suitable for adaptation since it will make the analysis extremely complicated. Instead, we need a constant high-gain parameter so that the controller (4) remains linear. A suitable choice of such a high-gain parameter satisfies

$$\rho \in [0, \tfrac{2}{\|B'P_\varepsilon B\|}] \quad (5)$$

where P_ε is the solution of parametric Lyapunov equation (2). To justify this, we consider the local stabilization of system (1) over a set $\mathcal{X}$. Suppose Assumption 1 holds. Let P_ε be the solution of (2). The low-and-high-gain feedback is given by

$$u_{LH} = -(1 + \rho)(I + B'P_\varepsilon B)^{-1}B'P_\varepsilon Ax$$

with ρ satisfying (5). Define $u_L = -(I+B'P_\varepsilon B)^{-1}B'P_\varepsilon Ax$. Let c be such that

$$c = \sup_{\substack{\varepsilon \in (0, \varepsilon^*] \\ x \in \mathcal{X}}} x'P_\varepsilon x.$$

Define a Lyapunov function $V(x) = x'P_\varepsilon x$ and a level set $\mathcal{V}(c) = \{x \mid V(x) \leq c\}$. We have $\mathcal{X} \subset \mathcal{V}_c$. From Lemma 1, there exists an ε_1 such that for any $\varepsilon \in (0, \varepsilon_1]$ and $x \in \mathcal{V}_c$,

$$\|(I + B'P_\varepsilon B)^{-1}B'P_\varepsilon Ax\| \leq \Delta.$$

Define $\mu = \|B'P_\varepsilon B\|$. We evaluate $V(k+1)-V(k)$ along the trajectories as

$$\begin{aligned} & V(k+1) - V(k) \\ =& -\varepsilon V(k) - \sigma(u_{LH}(k))'\sigma(u_{LH}(k)) \\ & \quad + [\sigma(u_{LH}(k)) - u_L(k)]'(I + B'PB) \\ & \quad [\sigma(u_{LH}(k)) - u_L(k)] \\ \leq& -\varepsilon V(k) - \sigma(u_{LH}(k))'\sigma(u_{LH}(k)) \\ & \quad + (1+\mu)\|\,\sigma(u_{LH}(k)) - u_L(k)\,\|^2 \\ =& -\varepsilon V(k) - \tfrac{1+\mu}{\mu}\|u_L(k)\|^2 \\ & \quad + \mu\|\sigma(u_{LH}(k)) - \tfrac{1+\mu}{\mu}u_L(k)\|^2. \end{aligned}$$

Since $\|u_L(k)\| \leq \Delta$ and ρ satisfies (5), we have

$$\|u_L(k)\| \leq \|\sigma(u_{LH}(k))\| \leq (1 + \tfrac{2}{\mu})\|u_L(k)\|.$$

This implies that

$$\|\sigma(u_{LH}(k)) - \tfrac{1+\mu}{\mu}u_L(k)\| \leq \tfrac{1}{\mu}\|u_L(k)\|.$$

and thus,

$$\mu\|\sigma(u_{LH}(k)) - \tfrac{1+\mu}{\mu}u_L(k)\|^2 - \tfrac{1}{\mu}\|u_L(k)\|^2 \leq 0.$$

Finally, we get for any $x(k) \in \mathcal{V}(c)$,

$$V(k+1) - V(k) \leq -\varepsilon V(k).$$

We conclude local asymptotic stability of the origin with a domain of attraction containing $\mathcal{X}$.

C. Scheduling of low-gain parameter

In the semi-global framework, with controller (3), the domain of attraction of the closed-loop system is determined by the low-gain parameter ε. In order to solve the global stabilization problem, this ε can be scheduled with respect to the state. This has been done in the literature, see for instance [2].

We are looking for an associated scheduled parameter satisfying the following properties:

1) $\varepsilon(x) : \mathbb{R}^n \to (0, \varepsilon^*]$ is continuous and piecewise continuously differentiable where ε^* is a design parameter.
2) There exists an open neighborhood $\mathcal{O}$ of the origin such that $\varepsilon(x) = \varepsilon^*$ for all $x \in \mathcal{O}$.
3) For any $x \in \mathbb{R}^n$, we have $\|F_{\varepsilon(x)}x\| \leq \delta$.
4) $\varepsilon(x) \to 0$ as $\|x\| \to \infty$.
5) $\{\ x \in \mathbb{R}^n \mid x'P_{\varepsilon(x)}x \leq c\ \}$ is a bounded set for all $c > 0$.

A particular choice of scheduling satisfying the above conditions is given in [2],

$$\varepsilon(x) = \max\{r \in (0, \varepsilon^*] \,|\, (x'P_r x)\operatorname{trace}(P_r) \leq \tfrac{\Delta^2}{b}\} \tag{6}$$

where $\varepsilon^* \in (0,1)$ is any a priori given constant and $b = 2\operatorname{trace}(BB')$ while P_r is the unique positive definite solution of Lyapunov equation (2) with $\varepsilon = r$.

A family of scheduled low-gain feedback controllers for global stabilization is given by

$$u_L(x) = F_{\varepsilon(x)}x = -(B'P_{\varepsilon(x)}B + I)^{-1}B'P_{\varepsilon(x)}Ax. \tag{7}$$

Here $P_{\varepsilon(x)}$ is the solution of (2) with ε replaced by $\varepsilon(x)$.

Note that the scheduled low-gain controller (7) with (6) satisfies

$$\|(B'P_{\varepsilon(x)}B + I)^{-1}B'P_{\varepsilon(x)}Ax\| \leq \Delta.$$

To see this, observe that

$$\begin{aligned}
&\|(B'P_{\varepsilon(x)}B + I)^{-1}B'P_{\varepsilon(x)}Ax\|^2\\
&\quad\leq \|B'P_{\varepsilon(x)}Ax\|^2\\
&\quad\leq \|B'\|^2\|P_{\varepsilon(x)}^{\frac{1}{2}}\|^2\|P_{\varepsilon(x)}^{\frac{1}{2}}AP_{\varepsilon(x)}^{-\frac{1}{2}}\|^2\|P_{\varepsilon(x)}^{\frac{1}{2}}x\|^2\\
&\quad\leq 2\|BB'\|\|P_{\varepsilon(x)}\|x'P_{\varepsilon(x)}x\\
&\qquad\text{(where we use property 4 of Lemma 1)}\\
&\quad\leq 2\operatorname{trace}(BB')\operatorname{trace}(P_{\varepsilon(x)})x'P_{\varepsilon(x)}x \leq \Delta^2.
\end{aligned}$$

D. Scheduling of high-gain parameter

As emphasized earlier, the high gain parameter plays a crucial role in dealing with external inputs/disturbances. In order to solve the simultaneous external and internal stabilization problems for continuous-time systems, different schedulings of high-gain parameter have been developed in the literature [2], [3], [8]. Unfortunately, none of them carry over to discrete-time case because the high gain has to be restricted near the origin. In this subsection, we introduce a new scheduling of the high-gain parameter with which we shall solve the (G_p/G) and $(G_p/G)_{fg}$ problems as formulated in Section II.

Our scheduling depends on the specific control objective. If one is not interested in finite gain, the following scheduled high gain suffices to solve (G_p/G) problem,

$$\rho_0(x) = \tfrac{1}{\|B'P_{\varepsilon(x)}B\|}. \tag{8}$$

Clearly, this high gain satisfies the constraints that $\rho_0(x) \leq \frac{2}{\|B'P_{\varepsilon(x)}B\|}$.

We observe that this high-gain parameter is radially unbounded. However, if we further pursue finite gain ℓ_p stabilization, the rate of growth of $\rho(x)$ with respect to $\|x\|$ as given in (8) is not sufficient for us. The scheduled high-gain parameter must rise quickly enough to overwhelm any disturbances in ℓ_p space before the state is steered so large that it actually prevents finite gain. Therefore, we shall introduce a different scheduling of high-gain parameter. In order to do so, we need the following lemma:

Lemma 2: Assume that $p \geq \frac{1}{2}$. For any $\eta > 1$ there exists a $\beta > 0$ such that $(u+v)^p \leq u^p + \eta u^p + \beta v^p$ for all $u, v \geq 0$.

Let ε^* and M_{ε^*} be given by Lemma 1 and let P^* be the solution of (2) with $\varepsilon = \varepsilon^*$. The scheduled high gain parameter is given by:

$$\rho_f(x) = \begin{cases} \rho_0(x) = \frac{1}{\|B'P_{\varepsilon(x)}B\|}, & x'P_\varepsilon(x)x \leq c \\ \frac{8\rho_1(x)}{\varepsilon(x)\lambda_{\min}P_{\varepsilon(x)}}, & \text{otherwise} \end{cases} \tag{9}$$

with

$$\rho_1(x) = \begin{cases} \frac{\lambda_{\max}P_{\varepsilon(x)}}{\lambda_{\min}P_{\varepsilon_1(x)}}, & p = \infty \\ \frac{\rho_p\beta(\varepsilon(x))\lambda_{\max}P_{\varepsilon(x)}}{\lambda_{\min}P_{\varepsilon_1(x)}\left[1-\left(1-\frac{\varepsilon_1(x)}{4(1+L_{\varepsilon_1(x)})}\right)^{p/2}\right]^{2/p}} + 1, & p \in [1,\infty) \end{cases} \tag{10}$$

where ρ_p is a positive constant to be determined later and c, $\varepsilon_1(x)$ and L_s are given by

$$\begin{aligned}
&c = \Delta^2\max\{4M_{\varepsilon^*}b, 4(1 + \|B'P^*B\|)\},\\
&\varepsilon_1(x) = \max\{r \in (0,\varepsilon^*] \mid 2x'P_r x\operatorname{trace}(P_r) \leq \tfrac{\Delta^2}{b}\},\\
&L_s = \tfrac{\operatorname{trace}(P^*)}{\lambda_{\min}P_s},
\end{aligned}$$

and $\beta(\varepsilon) > 1$ is such that Lemma 2 holds with $\eta = \eta(\varepsilon)$ satisfying,

$$\left[1 - \tfrac{\varepsilon}{4(1+L_\varepsilon)}\right]^{p/2} \leq (1 + \eta(\varepsilon))\left[1 - \tfrac{\varepsilon}{2(1+L_\varepsilon)}\right]^{p/2} < 1.$$

IV. MAIN RESULTS

In this section, we shall solve the simultaneous external and internal stabilization problems as formulated in Section II using the proposed low-and-high-gain controller in Section III.

The first theorem solves the global ℓ_p stabilization with arbitrary initial condition and without finite gain as formulated in Problem 3.

Theorem 1: Consider the system (1) satisfying Assumption 1. For any $p \in [1, \infty]$, the ℓ_p stabilization with arbitrary initial conditions and without finite gain as formulated in Problem 3 can be solved by the adaptive-low-gain and high-gain controller,

$$u = -(1 + \rho_0(x))(I + B'P_{\varepsilon(x)}B)^{-1}B'P_{\varepsilon(x)}Ax, \qquad (11)$$

where $P_{\varepsilon(x)}$ is the solution of (2), $\varepsilon(x)$ is determined adaptively by the scheduling (6) and $\rho_0(x)$ is determined by (8).

An immediate consequence of Theorem 1 is:

Theorem 2: Consider the system (1) satisfying Assumption 1. For any $p \in [1, \infty]$, the (G_P/G) as formulated in Problem 1 can be solved by the same adaptive-low-gain and high-gain controller (11).

In order to pursue the finite gain ℓ_p stabilization, it is necessary to modify the high gain parameter. We first consider the case $p = \infty$.

Theorem 3: Consider the system (1) satisfying Assumption 1. For $p = \infty$, ℓ_p stabilization with arbitrary initial condition with finite gain and with bias, as formulated in Problem 4, can be achieved by the adaptive-low-gain and high-gain controller,

$$u = -(1 + \rho_f(x))(I + B'P_{\varepsilon(x)}B)^{-1}B'P_{\varepsilon(x)}Ax, \qquad (12)$$

where $P_{\varepsilon(x)}$ is the solution of (2) with $\varepsilon = \varepsilon(x)$, $\varepsilon(x)$ is determined adaptively by (6) and $\rho_f(x)$ is determined by (9) and (10).

Theorem 3 readily yields the following result:

Theorem 4: Consider the system (1) satisfying Assumption 1. For $p = \infty$, the $(G_p/G)_{fg}$ as formulated in Problem 2 can be solved by the same adaptive-low-gain and high-gain controller as (12).

Theorem 5: Consider the system (1) satisfying Assumption 1. For any $p \in [1, \infty)$, the ℓ_p stabilization with arbitrary initial condition with finite gain with bias problem as formulated in Problem 4 can be solved by the adaptive-low-gain and high-gain controller,

$$u = -(1 + \rho_f(x))(I + B'P_{\varepsilon(x)}B)^{-1}B'P_{\varepsilon(x)}Ax, \qquad (13)$$

where $P_{\varepsilon(x)}$ is the solution of (2) with $\varepsilon = \varepsilon(x)$, $\varepsilon(x)$ is determined adaptively by (6) and $\rho_f(x)$ is determined by (9), (10) with ρ_p sufficiently large.

Theorem 5 also produces as a special case the solution to $(G_p/G)_{fg}$. This is stated in the following theorem.

Theorem 6: Consider the system (1) satisfying Assumption 1. For any $p \in [1, \infty)$, the $(G_p/G)_{fg}$ as formulated in Problem 2 can be solved by the adaptive-low-gain and high-gain controller (13).

V. CONCLUSIONS

It is shown in this paper that (G_p/G) and $(G_p/G)_{fg}$ problems for discrete-time linear systems subject to actuator saturation are solvable if the given linear system is controllable and it has all its poles on the unit disc. We also develop here an adaptive-low-gain and high-gain controller design methodology by using a parametric Lyapunov equation. By utilizing the developed methodology, one can explicitly construct the required state feedback controllers that solve the (G_p/G) and $(G_p/G)_{fg}$ problems whenever they are solvable.

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