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Obstacle Avoidance in Multi-Vehicle Coordinated Motion via Stabilization of Time-Varying Sets

Parham Ghorbanian, Sergey G. Nersesov, and Hashem Ashrafiuon

Abstract— **In this paper, we review the recent results on stability and control for time-varying sets of nonlinear time-varying dynamical systems and utilize them for the problem of multi-vehicle coordinated motion in the context of obstacle avoidance where obstacles are approximated and enclosed by elliptic shapes. Specifically, we design distributed controllers for individual vehicles moving in a specified formation in the presence of such obstacles. The obstacle avoidance algorithm that we propose is based on transitional trajectories which are defined by a set of ordinary differential equations that exhibit a stable elliptical limit cycle. The control framework is implemented on the system of double integrators and is shown to globally exponentially stabilize moving formation of the agents in pursuit of a leader while ensuring obstacle avoidance.**

I. INTRODUCTION

Multi-vehicle systems present a class of interconnected dynamical systems where vehicles are often coupled through the common task that they need to accomplish, but otherwise dynamically decoupled, meaning that the motion of one does not directly affect the others. The complexity of multi-vehicle cooperative manoeuvres as well as environmental conditions often necessitate the design of feedback control algorithms that use information about current position and velocity of each vehicle to steer them while maintaining a specified formation. For example, for mobile agents operating in foggy environment or located far from each other, open-loop visual control for coordinated motion becomes impractical. In this case, feedback control algorithms are required for individual vehicle steering which determine how a given vehicle maneuvers based on positions and velocities of nearby vehicles and/or on those of a formation leader.

Analysis and control design for networks of mobile agents has received considerable attention in the literature. A number of recent papers propose rigorous mathematical techniques for the analysis of networks of agents [1], [2], [3], [4], [5]. Distributed control of robotic networks has been extensively studied in [6], [5] where the authors develop a variety of control algorithms for network consensus. A survey of recent research results in cooperative control of multi-vehicle systems was performed in [7].

It was shown in [8] that a specified formation of multiple vehicles can be characterized by a time-varying set in the state space, and hence, the problem of control design for multi-vehicle coordinated motion is equivalent to design of stabilizing controllers for time-varying sets of nonlinear dynamical systems. Authors in [8] developed stability analysis and control design framework for time-varying sets of nonlinear time-varying dynamical systems using vector Lyapunov functions. Specifically, distributed control algorithms were designed for multi-vehicle coordination and are shown to globally exponentially stabilize multi-vehicle formations.

Some obstacle avoidance strategies for multi-vehicle problems include decentralized control approaches where local control laws are defined for each agent based on local information [9], [10] and behavior-based methods presented in [11], [12]. Perhaps, the most promising approach to obstacle avoidance is the potential field method which has been extensively utilized for mobile robots with static and dynamic obstacles implemented in real time experiments [13], [14], [15] and applied with robust controllers such as sliding mode control law [16].

Another more recent, rarely employed approach to obstacle avoidance is the limit cycle based method. Authors in [17] use limit cycles to generate trajectories for robot manipulators while avoiding obstacles. They define unstable limit cycles as objects of finite size and shape as a way of modeling complex obstacles to be avoided. The use of stable limit cycles as a navigation method has been introduced for obstacle avoidance of mobile robots in [18], [19]. The approach only considers circular limit cycles for mobile robots which are suitable for shapes with approximately the same length and width.

In this paper, we present an obstacle avoidance strategy that involves transitional trajectories defined as solutions to a set of ordinary differential equations possessing a stable limit cycle of elliptical shape. The obstacle is encircled by such ellipse and once detected, the trajectory of an agent is replanned in such a way so that the new trajectory follows a solution to the above system of ODEs. As soon as the obstacle is cleared, the trajectory of the agent is set back to the original trajectory the agent was following before encountering an obstacle. We combine this obstacle avoidance strategy with the distributed control algorithms for multi-vehicle coordinated motion developed in [8] to achieve stable coordination in the presence of obstacles.

II. STABILITY AND STABILIZATION OF TIME-VARYING SETS

In this section, we review the recent results developed in [8] on stability and stabilization of time-varying sets for time-varying nonlinear dynamical systems using vector Lyapunov functions [20], [21], [22]. To elucidate this, consider the time-varying nonlinear dynamical system given by

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad t \geq t_0, \tag{1}$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \geq t_0$, is the solution to (1), $t_0 \in [0, \infty)$, $\mathcal{D}$ is an open set, $0 \in \mathcal{D}$, $f(t, 0) = 0$, $t \geq t_0$, and $f(\cdot, \cdot)$ is Lipschitz continuous on $[0, \infty) \times \mathcal{D}$.

The following definition introduces several types of stability for time-varying sets of nonlinear time-varying dynamical systems. For this definition, $\mathcal{D}_0^t \triangleq \mathcal{D}_0(t)$, $t \geq t_0$, is a time-varying set such that, at each instant of time $t \geq t_0$, $\mathcal{D}_0(t)$ is a compact set and $\mathcal{O}_\varepsilon(\mathcal{D}_0(t)) \triangleq \{x \in \mathcal{D} : \mathrm{dist}(x, \mathcal{D}_0(t)) < \varepsilon\}$, $t \geq t_0$, defines the ε-neighborhood of

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P. Ghorbanian, S.G. Nersesov, and H. Ahrafiuon are with the Department of Mechanical Engineering, Villanova University, Villanova, PA 19085-1681, USA (parham.ghorbanian@villanova.edu; sergey.nersesov@villanova.edu; hashem.ashrafiuon@villanova.edu).

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$\mathcal{D}_0(t)$ at each instant of time $t \geq t_0$, where $\mathrm{dist}(x, \mathcal{D}_0(t)) \triangleq \inf_{y \in \mathcal{D}_0(t)} \|y - x\|$, $t \geq t_0$.

Definition 2.1 ([8]): Consider the nonlinear time-varying dynamical system (1). Let $\mathcal{D}_0^t$ be a time-varying set such that $\mathcal{D}_0^t$ is positively invariant with respect to (1) and at each instant of time $t \in [t_0, \infty)$, $\mathcal{D}_0(t)$ is a compact set.

i) $\mathcal{D}_0^t$ is *uniformly Lyapunov stable* if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $x_0 \in \mathcal{O}_\delta(\mathcal{D}_0(t_0))$ implies that $x(t) \in \mathcal{O}_\varepsilon(\mathcal{D}_0(t))$ for all $t \geq t_0$ and for all $t_0 \in [0, \infty)$.

ii) $\mathcal{D}_0^t$ is *uniformly asymptotically stable* if it is uniformly Lyapunov stable and there exists $\delta > 0$ such that $x_0 \in \mathcal{O}_\delta(\mathcal{D}_0(t_0))$ implies that $\lim_{t\to\infty} \mathrm{dist}(x(t), \mathcal{D}_0(t)) = 0$ uniformly in t_0 and x_0 for all $t_0 \in [0, \infty)$.

iii) $\mathcal{D}_0^t$ is *globally uniformly asymptotically stable* if it is uniformly Lyapunov stable and $\lim_{t\to\infty} \mathrm{dist}(x(t), \mathcal{D}_0(t)) = 0$ uniformly in t_0 and x_0 for all $x_0 \in \mathbb{R}^n$ and $t_0 \in [0, \infty)$.

iv) $\mathcal{D}_0^t$ is *uniformly exponentially stable* if there exist scalars $\alpha > 0$, $\beta > 0$, $\delta > 0$ such that $x_0 \in \mathcal{O}_\delta(\mathcal{D}_0(t_0))$ implies that $\mathrm{dist}(x(t), \mathcal{D}_0(t)) \leq \alpha \, \mathrm{dist}(x_0, \mathcal{D}_0(t_0)) e^{-\beta(t-t_0)}$, $t \geq t_0$, for all $t_0 \in [0, \infty)$.

v) $\mathcal{D}_0^t$ is *globally uniformly exponentially stable* if there exist scalars $\alpha > 0$, $\beta > 0$ such that $\mathrm{dist}(x(t), \mathcal{D}_0(t)) \leq \alpha \, \mathrm{dist}(x_0, \mathcal{D}_0(t_0)) e^{-\beta(t-t_0)}$, $t \geq t_0$, for all $x_0 \in \mathbb{R}^n$ and $t_0 \in [0, \infty)$.

The following definition introduces the notion of class $\mathcal{W}$ functions involving *quasimonotone increasing* functions.

Definition 2.2 ([22]): A function $w = [w_1, ..., w_q]^{\mathrm{T}} : [0, \infty) \times \mathbb{R}^q \to \mathbb{R}^q$ is of *class* $\mathcal{W}$ if, for every fixed $t \in [0, \infty)$, each component $w_i(\cdot, \cdot)$, $i = 1, ..., q$, of $w(\cdot, \cdot)$ satisfies $w_i(t, z') \leq w_i(t, z'')$ for all $z', z'' \in \mathbb{R}^q$ such that $z'_j \leq z''_j$, $j = 1, ..., q$, $j \neq i$, and $z'_i = z''_i$, where z_i denotes the ith component of z.

Theorem 2.1 ([8]): Consider the nonlinear time-varying dynamical system (1). Assume there exists a continuously differentiable vector function $V(t, x) = [V_1, \ldots, V_q]^{\mathrm{T}} : [0, \infty) \times \mathcal{D} \to \mathcal{Q} \cap \overline{\mathbb{R}}_+^q$, where $\mathcal{Q} \subset \mathbb{R}^q$ and $0 \in \mathcal{Q}$; a continuous function $w = [w_1, \ldots, w_q]^{\mathrm{T}} : [0, \infty) \times \mathcal{Q} \to \mathbb{R}^q$; and class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$ such that $V_i(t, x) = 0$, $x \in \mathcal{D}_i(t)$, $t \geq t_0$, where $\mathcal{D}_i(t) \subset \mathcal{D}$, $t \geq t_0$; $V_i(t, x) > 0$, $x \in \mathcal{D} \backslash \mathcal{D}_i(t)$, $t \geq t_0$, $i = 1, \ldots, q$; $\mathcal{D}_0^t = \mathcal{D}_0(t) \triangleq \cap_{i=1}^q \mathcal{D}_i(t) \neq \emptyset$ is a positively invariant time-varying set with respect to (1) which is compact at each instant of time $t \geq t_0$; $w(t, \cdot) \in \mathcal{W}$; $w(t, 0) = 0$, $t \geq 0$;

$$\alpha(\mathrm{dist}(x, \mathcal{D}_0(t))) \leq \mathbf{e}^{\mathrm{T}} V(t, x) \leq \beta(\mathrm{dist}(x, \mathcal{D}_0(t))), \quad (x, t) \in \mathcal{D} \times [0, \infty), \quad (2)$$

and

$$\frac{\partial V_i(t,x)}{\partial t} + V_i'(t, x) f(t, x) \leq w_i(t, V(t, x)), \quad (x, t) \in \mathcal{D} \times [0, \infty), \quad i = 1, \ldots, q. \quad (3)$$

In addition, assume that the vector comparison system

$$\dot{z}(t) = w(t, z(t)), \quad z(0) = z_0, \quad t \geq t_0, \quad (4)$$

has a unique solution $z(t)$, $t \geq t_0$, forward in time. Then the following statements hold:

i) If the zero solution to (4) is uniformly Lyapunov stable, then $\mathcal{D}_0^t$ is uniformly Lyapunov stable with respect to (1).

ii) If the zero solution to (4) is uniformly asymptotically stable, then $\mathcal{D}_0^t$ is uniformly asymptotically stable with respect to (1).

iii) If $\mathcal{D} = \mathbb{R}^n$, $\mathcal{Q} = \mathbb{R}^q$, $\alpha(\cdot)$ and $\beta(\cdot)$ are class $\mathcal{K}_\infty$ functions, and the zero solution to (4) is globally uniformly asymptotically stable, then $\mathcal{D}_0^t$ is globally uniformly asymptotically stable with respect to (1).

iv) If there exist constants $\nu \geq 1$, $\alpha > 0$, $\beta > 0$ such that, instead of (2), the following inequality holds

$$\alpha \left[\mathrm{dist}(x, \mathcal{D}_0(t))\right]^\nu \leq \mathbf{e}^{\mathrm{T}} V(t, x) \leq \beta \left[\mathrm{dist}(x, \mathcal{D}_0(t))\right]^\nu, \quad (x, t) \in \mathcal{D} \times [0, \infty), \quad (5)$$

and the zero solution to (4) is uniformly exponentially stable, then $\mathcal{D}_0^t$ is uniformly exponentially stable with respect to (1).

v) If $\mathcal{D} = \mathbb{R}^n$, $\mathcal{Q} = \mathbb{R}^q$ and there exist constants $\nu \geq 1$, $\alpha > 0$, $\beta > 0$ such that (5) holds and the zero solution to (4) is globally uniformly exponentially stable, then $\mathcal{D}_0^t$ is globally uniformly exponentially stable with respect to (1).

Next, we use the result of Theorem 2.1 to design stabilizing controllers for time-varying sets of multi-agent dynamical systems composed of q agents whose dynamics are given by

$$\dot{x}_i(t) = f_i(t, x(t)) + G_i(t, x(t)) u_i(t), \quad t \geq t_0, \quad i = 1, \ldots, q, \quad (6)$$

where $x(t) = [x_1^{\mathrm{T}}(t) \ldots, x_q^{\mathrm{T}}(t)]^{\mathrm{T}}$, $x_i(t) \in \mathbb{R}^{n_i}$, $t \geq 0$, $f_i : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n_i}$ and $G_i : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n_i \times m_i}$ are continuous functions for all $i = 1, \ldots, q$. Consider the time-varying sets given by $\mathcal{D}_i(t) \triangleq \{x \in \mathbb{R}^n : \mathcal{X}_i(t, x_i) = 0\}$, $t \geq t_0$, where $\mathcal{X}_i : [0, \infty) \times \mathbb{R}^{n_i} \to \mathbb{R}^{s_i}$ are continuous functions for all $i = 1, \ldots, q$. Define the motion of the ith agent on the set $\mathcal{D}_i(\cdot)$ as $x_{\mathrm{e}i}(t)$, $t \geq t_0$, and note that $\mathcal{X}_i(t, x_{\mathrm{e}i}(t)) \equiv 0$. Assume there exist vector functions $u_{\mathrm{e}i}(t)$, $t \geq t_0$, $i = 1, \ldots, q$, such that

$$G_i(t, x_{\mathrm{e}}(t)) u_{\mathrm{e}i}(t) = \dot{x}_{\mathrm{e}i}(t) - f_i(t, x_{\mathrm{e}}(t)), \quad t \geq t_0, \quad i = 1, \ldots, q, \quad (7)$$

where $x_{\mathrm{e}}(t) \triangleq \left[x_{\mathrm{e}1}^{\mathrm{T}}(t), \ldots, x_{\mathrm{e}q}^{\mathrm{T}}(t)\right]^{\mathrm{T}}$, $t \geq t_0$.

The next result presents a controller design that guarantees stabilization of a time-varying set for the time-varying nonlinear dynamical system (6) using vector Lyapunov functions.

Theorem 2.2 ([8]): Consider the multi-agent dynamical system given by (6). Assume there exist a continuously differentiable, component decoupled vector function $V : [0, \infty) \times \mathbb{R}^n \to \overline{\mathbb{R}}_+^q$, that is, $V(t, x) = [V_1(t, x_1), \ldots, V_q(t, x_q)]^{\mathrm{T}}$, $(x, t) \in \mathbb{R}^n \times [0, \infty)$; a continuous function $w = [w_1, \ldots, w_q]^{\mathrm{T}} : [0, \infty) \times \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$, and class $\mathcal{K}$ functions $\alpha : [0, \infty) \to [0, \infty)$ and $\beta : [0, \infty) \to [0, \infty)$ such that $V_i(t, x_i) = 0$, $x \in \mathcal{D}_i(t) \subset \mathbb{R}^n$, $t \geq t_0$, $i = 1, \ldots, q$; $V_i(t, x_i) > 0$, $x \in \mathbb{R}^n \backslash \mathcal{D}_i(t)$, $t \geq t_0$, $i = 1, \ldots, q$; $\mathcal{D}_0^t = \mathcal{D}_0(t) \triangleq \cap_{i=1}^q \mathcal{D}_i(t) \neq \emptyset$, $t \geq t_0$, is a compact set at each $t \geq t_0$; $w(t, \cdot) \in \mathcal{W}$; $w(t, 0) = 0$, $t \geq 0$;

$$\alpha(\mathrm{dist}(x, \mathcal{D}_0(t))) \leq \mathbf{e}^{\mathrm{T}} V(t, x) \leq \beta(\mathrm{dist}(x, \mathcal{D}_0(t))), \quad (x, t) \in \mathbb{R}^n \times [0, \infty), \quad (8)$$

and, for all $i = 1, \ldots, q$,

$$\frac{\partial V_i(t, x_i)}{\partial t} + V_i'(t, x_i) f_i(t, x) \leq w_i(t, V(t, x)), \quad (x, t) \in \mathcal{R}_i, \quad (9)$$

where $\mathcal{R}_i \triangleq \{(x,t) \in \mathbb{R}^n \times [0,\infty) : V_i'(t,x_i)G_i(t,x) = 0\}$, $i = 1,\ldots,q$. In addition, assume that the zero solution $z(t) \equiv 0$ to

$$\dot{z}(t) = w(t,z(t)), \quad z(0) = z_0, \quad t \geq t_0, \tag{10}$$

is uniformly asymptotically stable. Then $\mathcal{D}_0^t$ is uniformly asymptotically stable with respect to the nonlinear dynamical system (6) with the feedback control law $u = \phi(t,x) = [\phi_1^{\rm T}(t,x),\ldots,\phi_q^{\rm T}(t,x)]^{\rm T}$, $x \in \mathbb{R}^n$, $t \in [0,\infty)$, given by

$$\phi_i(t,x) = \begin{cases} u_{{\rm e}i}(t) - \left(c_{0i} + \frac{\mu_i(t,x)+\sqrt{\lambda_i(t,x)}}{\sigma_i^{\rm T}(t,x)\sigma_i(t,x)}\right)\sigma_i(t,x), \\ \qquad \text{if } \sigma_i(t,x) \neq 0; \\ u_{{\rm e}i}(t), \ \text{if } \sigma_i(t,x) = 0, \end{cases} \tag{11}$$

where $u_{{\rm e}i}(t)$, $t \geq t_0$, satisfies (7), $\lambda_i(t,x) \triangleq \mu_i^2(t,x) + (\sigma_i^{\rm T}(t,x)\sigma_i(t,x))^2$, $\mu_i(t,x) \triangleq \rho_i(t,x) - w_i(t,V(t,x)) + \frac{\partial V_i(t,x_i)}{\partial t} + \sigma_i^{\rm T}(t,x)u_{{\rm e}i}(t)$, $\rho_i(t,x) \triangleq V_i'(t,x_i)f_i(t,x)$, $\sigma_i(t,x) \triangleq G_i^{\rm T}(t,x)V_i'^{\rm T}(t,x_i)$, and $c_{0i} > 0$, $i = 1,\ldots,q$. If, in addition, $\alpha(\cdot)$ and $\beta(\cdot)$ are class $\mathcal{K}_\infty$ functions and the zero solution $z(t) \equiv 0$ to (10) is globally uniformly asymptotically stable, then $\mathcal{D}_0^t$ is globally uniformly asymptotically stable with respect to (6) with the feedback control law $u = \phi(t,x)$ given by (11). Furthermore, if there exist constants $\nu \geq 1$, $\alpha > 0$ and $\beta > 0$ such that, instead of (8), the following inequality holds

$$\alpha\left[{\rm dist}(x,\mathcal{D}_0(t))\right]^\nu \leq \mathbf{e}^{\rm T}V(t,x) \leq \beta\left[{\rm dist}(x,\mathcal{D}_0(t))\right]^\nu, \quad (x,t) \in \mathbb{R}^n \times [0,\infty), \tag{12}$$

and the zero solution to (10) is uniformly exponentially stable, then $\mathcal{D}_0^t$ is uniformly exponentially stable with respect to (6) with the feedback control law (11). Finally, if (12) holds and the zero solution to (10) is globally uniformly exponentially stable, then $\mathcal{D}_0^t$ is globally uniformly exponentially stable with respect to (6) with the feedback control law (11).

III. OBSTACLE AVOIDANCE STRATEGY

In this section, we consider planar motion of the agents. Our obstacle avoidance strategy is based on approximating obstacles as continuously differentiable shapes that can be represented as the limit cycle solution of the planar system of ordinary differential equations. Specifically, we encircle an obstacle by an ellipse which serves as a limit cycle orbit of a certain two dimensional dynamical system. As soon as the obstacle is detected, the trajectory of an agent is replanned so as to follow a solution of the above system until the obstacle is cleared. To elucidate this approach, introduce the state variables of the transitional trajectory as

$$\begin{cases} \tilde{x} = x - x_{\rm c}, \\ \tilde{y} = y - y_{\rm c}, \end{cases} \tag{13}$$

where x, y denote horizontal and vertical displacements of an agent and $x_{\rm c}$, $y_{\rm c}$ denote the location of the limit cycle origin. We consider limit cycles of elliptical form given by the zero level set of the following function

$$l(\tilde{x},\tilde{y}) = \left[\frac{\tilde{x}\cos\phi + \tilde{y}\sin\phi}{a}\right]^2 + \left[\frac{-\tilde{x}\sin\phi + \tilde{y}\cos\phi}{b}\right]^2 - 1, \tag{14}$$

where $\tilde{x}$ and $\tilde{y}$ are defined in (13), a and b are semi-major and semi-minor axes, respectively, and ϕ is the angle representing the orientation of the ellipse's semi-major axis relative to the horizontal axis. Thus,

$$l(\tilde{x},\tilde{y}) \equiv 0 \tag{15}$$

defines an ellipse centered at $(x_{\rm c}, y_{\rm c})$ with semi-major and semi-minor axes a and b, respectively, and with the semi-major axis forming angle ϕ relative to the horizontal axis.

Next, we consider a planar dynamical system that exhibits a limit cycle of the form (15) and that is given by

$$\begin{cases} \dot{\tilde{x}} = h_1(\tilde{x},\tilde{y}) - \tilde{x}l(\tilde{x},\tilde{y}), \\ \dot{\tilde{y}} = h_2(\tilde{x},\tilde{y}) - \tilde{y}l(\tilde{x},\tilde{y}), \end{cases} \tag{16}$$

where $h_1(\tilde{x},\tilde{y})$ and $h_2(\tilde{x},\tilde{y})$ represent the agent dynamics on the limit cycle, that is, when $l(\tilde{x},\tilde{y}) = 0$. The dynamics of (16) must ensure that a trajectory starting from any point outside of the limit cycle, that is, when $l(\tilde{x},\tilde{y}) > 0$, will converge to the limit cycle without crossing it.

The motion of a particle along the ellipse given by (15) with the angular speed ω is given by

$$\begin{cases} \tilde{x} = a\cos\phi\cos\omega t - b\sin\phi\sin\omega t, \\ \tilde{y} = a\sin\phi\cos\omega t + b\cos\phi\sin\omega t. \end{cases} \tag{17}$$

Thus, the time derivative of (17) can be written as

$$\begin{cases} \dot{\tilde{x}} = -\omega(a\cos\phi\sin\omega t - b\sin\phi\cos\omega t), \\ \dot{\tilde{y}} = \omega(-a\sin\phi\sin\omega t + b\cos\phi\cos\omega t). \end{cases} \tag{18}$$

Note that, $\omega > 0$ and $\omega < 0$ represent counterclockwise (CCW) and clockwise (CW) rotation of the particle, respectively. Eliminating $\cos\omega t$ and $\sin\omega t$ from (17) and (18), we obtain

$$\begin{cases} \dot{\tilde{x}} = \frac{\omega}{ab}(h_{e11}\tilde{x} - h_{e12}\tilde{y}), \\ \dot{\tilde{y}} = \frac{\omega}{ab}(h_{e21}\tilde{x} - h_{e11}\tilde{y}), \end{cases} \tag{19}$$

which represents particle dynamics on the limit cycle, where

$$\begin{cases} h_{e11} = (a^2 - b^2)\sin\phi\cos\phi, \\ h_{e12} = a^2\cos^2\phi + b^2\sin^2\phi, \\ h_{e21} = b^2\cos^2\phi + a^2\sin^2\phi. \end{cases}$$

Thus, $h_1(\tilde{x},\tilde{y})$ and $h_2(\tilde{x},\tilde{y})$ in (16) are given by

$$\begin{cases} h_1(\tilde{x},\tilde{y}) = \frac{\omega}{ab}(h_{e11}\tilde{x} - h_{e12}\tilde{y}), \\ h_2(\tilde{x},\tilde{y}) = \frac{\omega}{ab}(h_{e21}\tilde{x} - h_{e11}\tilde{y}). \end{cases} \tag{20}$$

Lemma 3.1: The trajectories defined by (16) asymptotically converge to the elliptical limit cycle given by (15) for all initial conditions in $\mathcal{N} = \{(\tilde{x},\tilde{y}) \in \mathbb{R}^2 : l(\tilde{x},\tilde{y}) > 0\}$.

Proof. Consider the following Lyapunov function candidate given by

$$V(\tilde{x},\tilde{y}) = \frac{a^2b^2}{2}l(\tilde{x},\tilde{y}), \quad (\tilde{x},\tilde{y}) \in \mathcal{N}. \tag{21}$$

Note that $V(\tilde{x},\tilde{y}) > 0$ for all $(\tilde{x},\tilde{y}) \in \mathcal{N}$. Furthermore, the Lyapunov derivative along trajectories of (16) is given by

$$\dot{V}(\tilde{x},\tilde{y}) = -l(\tilde{x},\tilde{y})[a^2(-\tilde{x}\sin\phi + \tilde{y}\cos\phi)^2 + b^2(\tilde{x}\cos\phi + \tilde{y}\sin\phi)^2] < 0, \quad (\tilde{x},\tilde{y}) \in \mathcal{N}, \tag{22}$$

which implies asymptotic convergence of the trajectories of (16) to the ellipse (15) for all initial conditions in $\mathcal{N}$. □

Note that an agent is only required to remain on the trajectory converging to a limit cycle for a finite period of time until an obstacle is cleared. As soon as the obstacle is cleared the agent returns back to its original trajectory.

IV. CONTROL DESIGN FOR MULTI-VEHICLE COORDINATED MOTION

In this section, we apply the results of Sections II and III to a problem of coordinated motion of multiple vehicles in pursuit of a (virtual) leader while the vehicles and the leader encounter obstacles. Since a specified formation of multiple vehicles can be characterized by a time-varying set in the state space, the problem of control design for multi-vehicle coordinated motion is equivalent to design of stabilizing controllers for time-varying sets of nonlinear dynamical system. Thus, using the stability and control results developed for time-varying sets, we design distributed control algorithms for stabilization of a multi-vehicle formation. Specifically, we design a distributed feedback control law that drives individual vehicles to a configuration with specified distance and orientation with respect to a leader while maintaining this configuration throughout the motion of the leader. To elucidate the control design, consider planar motion of q agents with the individual agent dynamics given by

$$\ddot{x}_i(t) = u_{xi}(t), \quad x_i(0)=x_{i0}, \quad \dot{x}_i(0)=\dot{x}_{i0}, \quad t\geq 0, \tag{23}$$

$$\ddot{y}_i(t) = u_{yi}(t), \quad y_i(0)=y_{i0}, \quad \dot{y}_i(0)=\dot{y}_{i0}, \tag{24}$$

where $x_i : [0,\infty) \to \mathbb{R}$ and $y_i : [0,\infty) \to \mathbb{R}$, $i = 1,\ldots,q$, are the displacements of the ith agent in the horizontal and vertical directions, respectively, and u_{xi} and u_{yi} are the control forces acting on the ith agent in the horizontal and vertical directions, respectively. Next, define $\eta_i \triangleq [x_i, y_i, \dot{x}_i, \dot{y}_i]^{\mathrm{T}}$, $i = 1,\ldots,q$, and $\eta \triangleq [\eta_1^{\mathrm{T}},\ldots,\eta_q^{\mathrm{T}}]^{\mathrm{T}}$. Then the generalized dynamics (23), (24) for q agents can be written in the state space form as

$$\dot{\eta}(t) = (I_q\otimes A)\eta(t) + (I_q\otimes B)u(t), \quad \eta(0)=\eta_0, \quad t\geq 0, \tag{25}$$

where $\eta_0 = [\eta_{10}^{\mathrm{T}},\ldots,\eta_{q0}^{\mathrm{T}}]^{\mathrm{T}}$, $\eta_{i0} = [x_{i0}, y_{i0}, \dot{x}_{i0}, \dot{y}_{i0}]^{\mathrm{T}}$, $u \triangleq [u_1^{\mathrm{T}},\ldots,u_q^{\mathrm{T}}]^{\mathrm{T}}$, $u_i \triangleq [u_{xi}, u_{yi}]^{\mathrm{T}}$, "$\otimes$" is the Kronecker product, $I_q \in \mathbb{R}^{q\times q}$ is the identity matrix, and A, B are given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{26}$$

Furthermore, we define the time-varying sets

$$\mathcal{D}_i(t) \triangleq \{\eta \in \mathbb{R}^{4q} : \eta_i - p_i(t) = 0\}, \quad t\geq 0, \quad i=1,\ldots,q, \tag{27}$$

where

$$p_i(t) \triangleq \begin{bmatrix} x_{\mathrm{L}}(t) + l_{xi\mathrm{L}} \\ y_{\mathrm{L}}(t) + l_{yi\mathrm{L}} \\ \dot{x}_{\mathrm{L}}(t) \\ \dot{y}_{\mathrm{L}}(t) \end{bmatrix}, \quad t\geq 0, \quad i=1,\ldots,q, \tag{28}$$

$x_{\mathrm{L}}(t)$, $y_{\mathrm{L}}(t)$, $t \geq 0$, are, respectively, horizontal and vertical positions of the leader, $\dot{x}_{\mathrm{L}}(t)$, $\dot{y}_{\mathrm{L}}(t)$, $t \geq 0$, are, respectively, horizontal and vertical velocities of the leader, and $l_{xi\mathrm{L}}, l_{yi\mathrm{L}} \in \mathbb{R}$ are, respectively, desired horizontal and vertical distances between the ith agent and the leader. Note that each set $\mathcal{D}_i(t)$, $t \geq 0$, $i = 1,\ldots,q$, defines relative position and velocity of the ith agent with respect to the leader. In order to construct the set $\mathcal{D}_i(t)$, $t\geq 0$, $i = 1,\ldots,q$, only the local information about the relative position and velocity of the ith agent with respect to the leader is needed. Furthermore, the intersection of the sets (27) given by

$$\mathcal{D}_0^t = \mathcal{D}_0(t) \triangleq \bigcap_{i=1,\ldots,q} \mathcal{D}_i(t), \quad t\geq 0, \tag{29}$$

characterizes the desired formation of agents with respect to the leader where all agents maintain specified distances and velocities with respect to the leader.

Next, we define the component decoupled vector function $V : [0,\infty)\times\mathbb{R}^{4q} \to \mathbb{R}^q$ such that $V(t,\eta) = [V_1(t,\eta_1),\ldots,V_q(t,\eta_q)]^{\mathrm{T}}$, where

$$V_i(t,\eta_i) = (\eta_i - p_i(t))^{\mathrm{T}} P(\eta_i - p_i(t)), \quad \eta_i \in \mathbb{R}^4, \quad t\geq 0, \quad i=1,\ldots,q, \tag{30}$$

where

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} > 0. \tag{31}$$

Note that $V_i(t,\eta_i) = 0$, $\eta \in \mathcal{D}_i(t)$, $t \geq 0$, and $V_i(t,\eta_i) > 0$, $\eta \in \mathbb{R}^{4q}\setminus\mathcal{D}_i(t)$, $t\geq 0$, $i = 1,\ldots,q$. In addition, since $\lambda_{\min}(P) = \lambda_{\max}(P) = 1$, condition (12) is satisfied with $\alpha = \frac{1}{2}$, $\beta = 2$, $\nu = 2$, $\mathrm{dist}(\eta,\mathcal{D}_0(t)) \triangleq \left[(\eta - p(t))^{\mathrm{T}}(\eta-p(t))\right]^{\frac{1}{2}}$, $\eta\in\mathbb{R}^{4q}$, $t\geq 0$, where $p(t) \triangleq [p_1^{\mathrm{T}}(t),\ldots,p_q^{\mathrm{T}}(t)]^{\mathrm{T}}$. Furthermore, it can be shown that, for $\mathcal{R}_i \triangleq \{(\eta,t)\in\mathbb{R}^{4q}\times[0,\infty) : V_i'(t,\eta_i)B = 0\}$, $i = 1,\ldots,q$, condition (9) is satisfied with

$$\frac{\partial V_i(t,\eta_i(t))}{\partial t} + V_i'(t,\eta_i(t))A\eta_i(t) \leq -\gamma_i V_i(t,\eta_i(t)), \quad (\eta,t)\in\mathcal{R}_i, \quad i=1,\ldots,q, \tag{32}$$

for $\gamma_i \in (0,1]$, $i = 1,\ldots,q$. In this case, the zero solution to (10) is globally exponentially stable with

$$w(z) = [-\gamma_1 z_1,\ldots,-\gamma_q z_q]^{\mathrm{T}}. \tag{33}$$

Hence, it follows from Theorem 2.2 that the time-varying set $\mathcal{D}_0^t$ defined by (29) is globally uniformly exponentially stable with respect to (25) with the feedback control law $u_i = \phi_i(t,\eta_i)$, $i = 1,\ldots,q$, given by (11) with $\mu_i(t,\eta_i) \triangleq \rho_i(t,\eta_i) - w_i(V_i(t,\eta_i)) + \frac{\partial V_i(t,\eta_i)}{\partial t} + \sigma_i^{\mathrm{T}}(t,\eta_i)u_{\mathrm{e}i}(t)$, $\rho_i(t,\eta_i) \triangleq V_i'(t,\eta_i)A\eta_i$, $\sigma_i(t,\eta_i) \triangleq B^{\mathrm{T}}V_i'^{\mathrm{T}}(t,\eta_i)$, $u_{\mathrm{e}i}(t) = [\ddot{x}_{\mathrm{L}}(t), \ddot{y}_{\mathrm{L}}(t)]^{\mathrm{T}}$, and $w(V(t,\eta))$ given by (33). Note that the feedback control law $u_i = \phi_i(t,\eta_i)$, $i = 1,\ldots,q$, is a distributed control algorithm [5], [6] which uses only local information about relative position and velocity of the ith agent with respect to the leader. This allows to reproduce this controller without changing its structure as many times as the number of agents in order to steer individual agent while maintaining a specified formation with respect to the leader.

Now, according to our obstacle avoidance strategy, when obstacles are detected, they are approximated by ellipses as shown in Section III. In this case, the dynamics of the ith agent is forced to obey

$$\begin{cases} \dot{\tilde{x}}_i = h_1(\tilde{x}_i,\tilde{y}_i) - \tilde{x}_i l(\tilde{x}_i,\tilde{y}_i), \\ \dot{\tilde{y}}_i = h_2(\tilde{x}_i,\tilde{y}_i) - \tilde{y}_i l(\tilde{x}_i,\tilde{y}_i), \end{cases} \tag{34}$$

where $\tilde{x}_i = x_i - x_{\mathrm{c}i}$, $\tilde{y}_i = y_i - y_{\mathrm{c}i}$, $(x_{\mathrm{c}i}, y_{\mathrm{c}i})$ is the position of the center of an ellipse, and $h_1(\cdot,\cdot)$, $h_2(\cdot,\cdot)$, and $l(\cdot,\cdot)$ are given by (20) and (14).

Next, consider the case when an obstacle is detected on the way of a formation leader. In this case, in addition to the elliptical limit cycle that encircles an obstacle, we define an elliptical region surrounding the obstacle as a safety measure. This region includes all points in $\mathbb{R}^2$ satisfying $l(\tilde{x}_{\rm L}, \tilde{y}_{\rm L}) > k$, where k is a safety factor which specifies the size of the elliptical region. Introduction of such elliptical region as a safety zone ensures that the leader will not collide with the obstacle and will have enough time to change its trajectory and converge to the elliptical limit cycle. When an obstacle is detected in the elliptical region, the leader is scheduled to change its path from the original one to the one that is given by a solution to (34). In order to make this transition smooth, we define an intermediate path for the leader given by the fifth-order polynomial. Specifically, as soon as the leader reaches the boundary of the elliptical region given by $l(\tilde{x}_{\rm L}, \tilde{y}_{\rm L}) = k$ at time $t = t_0$, the trajectory of the leader is forced to follow

$$\begin{aligned} x(t) &= a_5\Delta t^5 + a_4\Delta t^4 + a_3\Delta t^3 + a_2\Delta t^2 + a_1\Delta t + a_0, \\ y(t) &= b_5\Delta t^5 + b_4\Delta t^4 + b_3\Delta t^3 + b_2\Delta t^2 + b_1\Delta t + b_0. \end{aligned} \tag{35}$$

where $\Delta t \triangleq t - t_0$ and coefficients a_i, b_i, $i = 1, \ldots, 5$, are determined from the boundary conditions for the position, velocity, and acceleration of the leader at times t_0 and t_1 with t_1 being the end time of the transitional path (35). After the transitional phase given by (35), the motion of the leader switches to obey the dynamics given by (34). The control algorithm developed in Section II guarantees that while the leader is bypassing the obstacle on its new path, the agents will follow the leader in a specified formation due to exponential stability of the time-varying set describing the formation. As soon as the obstacle is cleared, the formation leader will switch its trajectory back to the original one. In order to find the proper point of diverging from the motion given by (34) to the motion on the original path, a line-drawing method has been used. In this method, a line is drawn between the current leader's position and its (virtual) position as if there were no obstacle. As long as there is an intersection between this line and the ellipse $l(\tilde{x}, \tilde{y}) = 0$, the leader's dynamics will remain obeying (34) and as soon as there is no intersection between the line and the ellipse, the leader will switch back to its original path and the obstacle avoidance is guaranteed. The transition phase when the leader is diverging from the motion according to (34) to its original trajectory is again described by the fifth-order polynomials given by (35).

Now, consider the case when the kth agent in the formation encounters an obstacle on its way. Recall that we design a stabilizing controller for the time-varying set (27) for each agent to ensure that the agent will be on the specified formation while following the formation leader. Now, as soon as the kth agent detects an obstacle on its way, the time-varying set (27) for this agent is redefined to be

$$\tilde{\mathcal{D}}_k(t) \triangleq \{\eta \in \mathbb{R}^{4q} : \eta_k - \tilde{p}_k(t) = 0\}, \quad t \geq 0, \tag{36}$$

where

$$\tilde{p}_k(t) \triangleq \begin{bmatrix} \tilde{x}_k(t) + x_{{\rm c}k} \\ \tilde{y}_k(t) + y_{{\rm c}k} \\ \dot{\tilde{x}}_k(t) \\ \dot{\tilde{y}}_k(t) \end{bmatrix}, \quad t \geq 0, \tag{37}$$

where $\tilde{x}_k(t)$, $\tilde{y}_k(t)$, $t \geq 0$, are solutions to (34). Furthermore, the intersection of sets (27) and (36) given by

$$\tilde{\mathcal{D}}_0^t = \tilde{\mathcal{D}}_0(t) \triangleq \bigcap_{i=1,\ldots,q,\, i\neq k} \mathcal{D}_i(t) \bigcap \tilde{\mathcal{D}}_k(t), \quad t \geq 0, \tag{38}$$

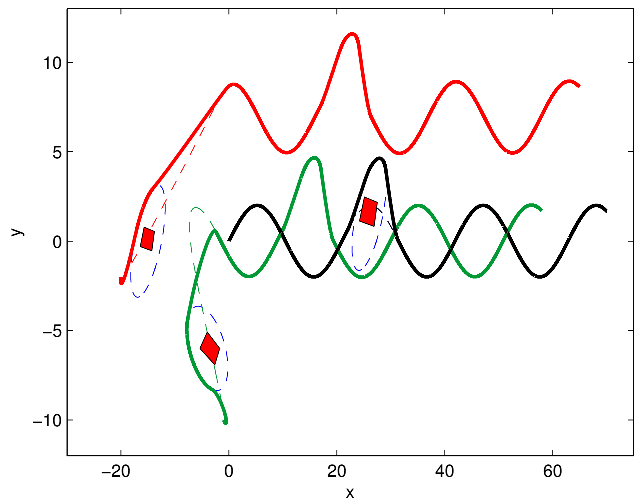

Fig. 1. Position phase portrait of two agents following the leader. Black solid line represents trajectory of the leader.

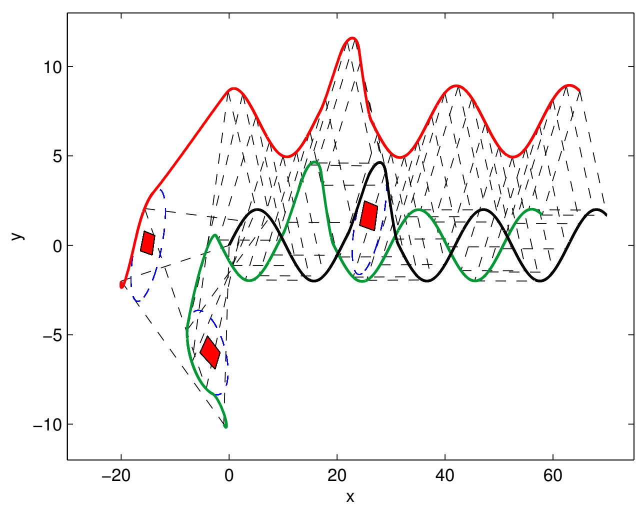

Fig. 2. Position phase portrait of two agents following the leader and the moving formation. Black solid line represents trajectory of the leader.

characterizes the temporary desired formation of agents with respect to the leader until the obstacle is cleared by the kth agent. Specifically, when the kth agent encounters an obstacle on its path, the time-varying set describing the desired formation will switch from (27) to (36). Then, as soon as the obstacle is cleared, the time-varying set describing the desired formation will switch back to the original set (27). By switching between the above time-varying sets, we guarantee the obstacle avoidance for each agent while maintaining the desired formation at the steady state.

In the following simulation, we consider two agents pursuing a leader in a triangular formation, while there is an obstacle on the way of each agent as well as the leader. For this, we set $l_{x1{\rm L}} = -5$, $l_{y1{\rm L}} = 7$, $l_{x2{\rm L}} = -12$, $l_{y2{\rm L}} = 0$, $c_{0i} = 0.2$, $i = 1, 2$, $\gamma_i = \frac{1}{5}$, $i = 1, 2$, $\eta_{10} = [-20, -2, -1, -2]^{\rm T}$, and $\eta_{20} = [-1, -10, 3, -2]^{\rm T}$.

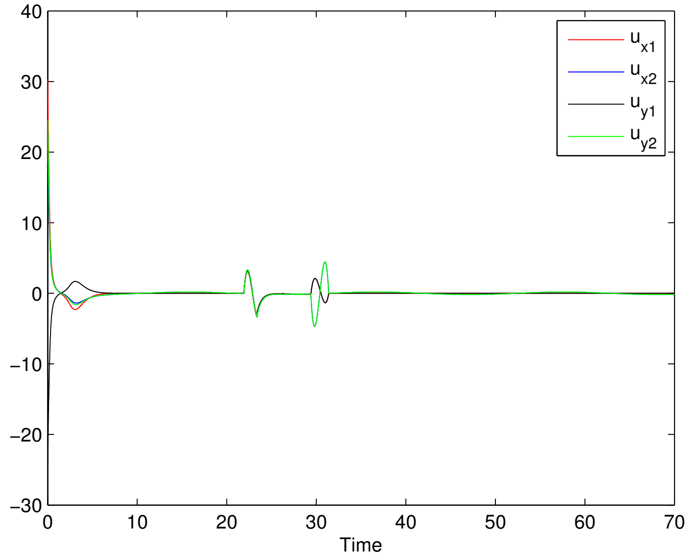

Fig. 3. Control forces in horizontal and vertical directions versus time.

With this choice of the parameters $l_{xi\mathrm{L}}$ and $l_{yi\mathrm{L}}$, $i = 1, 2$, the agents will form a configuration of an equilateral triangle with respect to the leader. Furthermore, the leader is set to be moving on a sinusoidal path according to $x_\mathrm{L}(t) = t$, $y_\mathrm{L}(t) = 2\sin(0.3t)$, $t \geq 0$. We define an obstacle for the first agent encircled by an ellipse with the parameters $x_\mathrm{c} = -15$, $y_\mathrm{c} = 0$, $a = 2$, $b = 4$, $\phi = -0.8(\mathrm{rad})$, and $\omega = -0.75(\mathrm{rad/s})$ and we define an obstacle for the second agent encircled by an ellipse with the parameters $x_\mathrm{c} = -4$, $y_\mathrm{c} = -6$, $a = 2$, $b = 4$, $\phi = 1.2(\mathrm{rad})$, and $\omega = -0.32(\mathrm{rad/s})$. When the leader gets close to the obstacle, it switches its path to the solution of (34) which possesses an elliptical limit cycle with the parameters $x_\mathrm{c} = 26$, $y_\mathrm{c} = 1.5$, $a = 2$, $b = 4$, $\phi = -0.8(\mathrm{rad})$, and $\omega = -0.22(\mathrm{rad/s})$. Thus, the new $x_\mathrm{L}(t)$ and $y_\mathrm{L}(t)$ are determined by a solution to (34). As soon as the obstacle is cleared, the leader switches its path back to the original sinusoidal path.

For the feedback controller (11), Figure 1 shows position phase portrait of two agents following the leader while each agent and the leader avoid obstacles and Figure 2 shows that the agents eventually converge to the desired triangular formation after all obstacles are cleared. Finally, Figure 3 shows the time history of the control forces acting on each agent.

V. CONCLUSION

In this paper, we reviewed stability analysis and control design framework for time-varying sets of nonlinear time-varying dynamical systems developed in [8] and introduced an obstacle avoidance planning strategy based on approximating obstacles as ellipses in order to plan and track transitional trajectories around them until the obstacle is cleared. The transitional trajectories are defined as solutions to a set of planar ODE's that exhibit stable elliptical limit cycle approximating the obstacles. It was shown that for a system of planar double integrators with a specified moving formation with respect to the leader and with an obstacle on the way of each agent as well as the leader, the developed distributed control algorithm globally exponentially stabilizes the moving formation, while obstacle collision is avoided for leader as well as the follower agents. Finally, it should be noted that the stability results for time-varying sets of nonlinear dynamical systems developed in this paper can be used to design various other control algorithms to achieve stable coordinated motion of multi-vehicle systems including obstacle avoidance.

REFERENCES

[1] J. P. Desai, J. P. Ostrowski, and V. Kumar, "Modeling and control of formations of nonholonomic mobile robots," *IEEE Trans. Robot. Automat.*, vol. 17, pp. 905–908, 2001.
[2] P. Örgen, M. Egerstedt, and X. Hu, "A control Lyapunov function approach to multiagent coordination," *IEEE Trans. Robot. Automat.*, vol. 18, no. 5, pp. 847–851, 2002.
[3] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Trans. Autom. Contr.*, vol. 49, pp. 1465–1476, 2004.
[4] R. Olfati-Saber, "Flocking for multi-agent dynamic systems: Algorithms and theory," *IEEE Trans. Autom. Contr.*, vol. 51, pp. 401–420, 2006.
[5] W. Ren and R. W. Beard, *Distributed Consensus in Multi-Vehicle Cooperative Control*. London, UK: Springer-Verlag, 2008.
[6] F. Bullo, J. Cortéz, and S. Martinéz, *Distributed Control of Robotic Networks*. Princeton, NJ: Princeton University Press, 2009.
[7] R. M. Murray, "Recent research in cooperative control of multi-vehicle systems," *J. Dyn. Syst. Meas. Contr.*, vol. 129, pp. 571–583, 2007.
[8] S. G. Nersesov, P. Ghorbanian, and A. G. Aghdam, "Stabilization of sets with application to multi-vehicle coordinated motion," *Automatica*, vol. 46, pp. 1419–1427, 2010.
[9] N. Aggarwal and K. Fujimura, "Motion planning amidst planar moving obstacles," in *IEEE Inter. Conf. Robot. Autom.*, San Diego, CA, 1994, pp. 2153–2158.
[10] W. Xiaohua, V. Yadav, and B. S. N., "Cooperative uav formation flying with obstacle/collision avoidance," *IEEE Transactions on Control Systems Technology*, vol. 15, no. 4, pp. 672–679, 2007.
[11] P. G. Zavlangas and S. G. Tzafestas, "Motion control for mobile robot obstacle avoidance and navigation: A fuzzy logic-based approach," *Systems Analysis Modelling Simulation*, vol. 43, pp. 1625–1637, 2003.
[12] A. R. Dieguez, R. Sanz, and J. Lopez, "Deliberative on-line local path planning for autonomous mobile robots," *Journal of Intelligent and Robotic Systems: Theory and Applications*, vol. 37, no. 1, pp. 1–19, 2003.
[13] K. Pathak and S. K. Agrawal, "An integrated path-planning and control approach for nonholonomic unicycles using switched local potentials," *IEEE Transactions on Robotics*, vol. 21, no. 6, pp. 1201–1208, 2005.
[14] S. S. Ge and Y. J. Sui, "Dynamic motion planning for mobile robots using potential field method," *Autonomous Robots*, vol. 1, pp. 207–222, 2002.
[15] F. A. Cosio and M. P. Castaneda, "Autonomous robot navigation using adaptive potential fields," *Mathematical and Computer Modelling*, vol. 49, no. 9, pp. 1141–1156, 2004.
[16] F. Fahimi, C. Nataraj, and H. Ashrafiuon, "Real-time obstacle avoidance for multiple mobile robots," *Robotica*, vol. 27, no. 2, pp. 189–198, 2009.
[17] L. P. Ellekilde and J. W. Perram, "Tool center trajectory planning for industrial robot manipulators using dynamical systems," *The International Journal of Robotics Research*, vol. 24, no. 5, pp. 385–396, 2005.
[18] D. H. Kim and J. H. Kim, "A real-time limit-cycle navigation method for fast mobile robots and its application to robot soccer," *Robotics and Autonomous Systems*, vol. 42, no. 1, pp. 17–30, 2003.
[19] D. H. Kim and P. Chongkug, "Limit cycle navigation method for mobile robot," in *27th Chinese Control Conference*, Kunming, Yunnan, China, 2008, pp. 320–324.
[20] R. Bellman, "Vector Lyapunov functions," *SIAM Journal of Control*, vol. 1, pp. 32–34, 1962.
[21] V. M. Matrosov, "Method of vector Liapunov functions of interconnected systems with distributed parameters (Survey)," *Avtomatika i Telemekhanika*, vol. 33, pp. 63–75, 1972, (in Russian).
[22] S. G. Nersesov and W. M. Haddad, "On the stability and control of nonlinear dynamical systems via vector Lyapunov functions," *IEEE Trans. Autom. Contr.*, vol. 51, no. 2, pp. 203–215, 2006.