2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

# Stability Conditions for Optimal Filtering over Cognitive Radio System

Xiao Ma, Seddik M. Djouadi and Husheng Li

*Abstract*— **Cognitive radio system is a very popular area in the communication community as it saves money and bandwidth by sensing the available licensed spectrum for unlicensed users. This advantage provides a promising future for the application of cognitive radio in control systems. In this paper, we propose to communicate through a cognitive radio link between the sensor and the estimator. In this way, the state estimator needs to adjust to this new communication link as the link is affected by the interruptions from primary users. We assume the emergence of primary users results in packet losses. The link is assumed to be governed by multiple semi-Markov processes each of which can capture and represent one channel in it. We derive sufficient conditions for the stability of the peak covariance process of the optimal filter. A numerical example is given to demonstrate the theorems.**

## I. INTRODUCTION

Nowadays, the fast development of the communication and networks extend the areas of traditional science. These remote techniques are employed everywhere to facilitate the users located in different areas. However, the widely use of various technologies such as radio, satellite and phone service also increases the need of the spectrum used in transmission. Most of the current spectrum has been licensed to different users to ensure the coexistence of diverse wireless systems [1]. Thus an important question is: *How to save bandwidth in communication without affecting the performance too much*?

Based on Federal Communications Commission's (FCC's) frequency allocation chart [2], it shows that although the majority of the frequency bandwidth has been assigned to different users, large portions of spectrum are frequently unused [3]. Then, **cognitive radio architecture** [4] [5] is proposed, as a communication system, to be used for sensing available spectrum, searching for unutilized spectrum, and, communicating over the unused spectrum with the minimum disturbance to primary users (with license). In cognitive radio system, each secondary user (without license) is able to sense the licensed spectrum and detect the unused spectrum holes. If a frequency channel is not being used by primary users, then the secondary user can access it for communication. Due to the sparse activities of primary users in spectrum, cognitive radio can provide a large amount of spectrum for communications. With cognitive radio system, the question above is answered.

However, cognitive radio suffers from interruptions from primary users since a secondary user must leave the licensed channel when primary users emerge. Hence, the cognitive radio based communication link is not reliable, and can cause significant impact on the control performance since the observations from sensor may not be able to reach destination timely. In this paper, we assume the emergence of primary users can result in packet losses. Then, we will **focus on optimal filtering over cognitive radio and give stability conditions**.

Modern control theory has been increasingly concerned with networks, communication channels, and remote control technology. A lot of research has been performed in the area of control and estimation over communication links under constraints such as packet losses, transmission delays, and bandwidth constraints [6]$\sim$ [14], but minimal research has been performed for cognitive radio architecture. The state estimation of system over a cognitive radio system is first considered in [15], where the cognitive radio link is modeled by a two-switch model with distributed and dynamic spectral activity introduced by [1]. The switching variables are assumed to be Bernoulli variables. Control and estimation of the closed-loop of the system over the same cognitive radio links are discussed in [16]. However, as it is shown in [17], through theory and experiments, that a semi-Markov process captures the stochastic behavior of each channel in cognitive radio system more accurately. **Here, we use a semi-Markov model to represent the behavior of the cognitive radio link.**

The remainder of the paper is organized as follows: In section II, the model for cognitive radio is discussed and the problem is formulated; In section III, the optimal filter is given. In section IV, some preliminaries of semi-Markov processes are presented. The main result is contained in section V and simulation results are given in section VI.

## II. SYSTEM MODEL

### A. Model for Cognitive Radio

Fig. 1 gives an example of a cognitive radio system: There are $N$ ($N > 1$) independent licensed channels that can be sensed named as $f_1$, $f_2$,..., $f_N$, respectively; each channel is divided into parts by vertical lines and each part represents the channel status in one time slot; the marked slot represents that the channel is utilized by primary users and the secondary users can not use it at that time while the blank one means that it is free to be used by other users.

[17] shows that each channel is governed by a semi-Markov process: In each channel, there are two states (busy and idle). The times that the channel stays in one state are i.i.d random variables following some density function, which may depend on the two states between which the move

X. Ma, S. M. Djouadi Husheng Li are with Department of Electrical Engineering and Computer Science, University of Tennessee, Knoxville, TN 37996 USA (email: xma4@utk.edu, djouadi@eecs.utk.edu, husheng@ece.utk.edu)

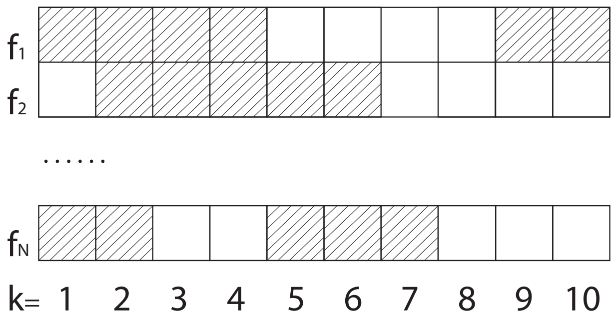


Fig. 1: Bandwidth status in cognitive radio

is made. The cognitive radio structure considered in [15] [16] employs i.i.d Bernoulli variables to represent the switch between idle and busy states. In fact, Bernoulli distribution is a special case of the Markov process and thus a special case of the semi-Markov process. In this work, a homogeneous semi-Markov process is used to model each channel.

Assume the sensor in cognitive radio infrastructure senses only one channel at each time step (this avoids costly and a complicated sensor which can sense multiple channels). Every time the sensor chooses one channel to sense according to some sensing policy, if the channel is idle, transmits the signal through it; otherwise, stop transmission (no signal transmitted at this time) to avoid collision.

Denote the signal sent at time $t$ as $y_t$, then the received signal can be written as:

$$\tilde{y}_t = \gamma_t y_t + \omega_t \tag{1}$$

where $\gamma_t$ is governed by $N$ semi-Markov processes each of which represents the behavior of one channel. $\gamma_t = 1$ if a unutilized channel is sensed and the signal is transmitted to the receiver and $\gamma_t = 0$ if a busy channel is sensed and no information is delivered. $\omega_t$ denotes the Gaussian noise with zero mean and variance $R$. Assume $\gamma_t$ is known at the receiver here.

### B. Problem Formulation

The linear discrete time system can be written as follows:

$$x_{t+1} = Ax_t + v_t \tag{2}$$

$$y_t = Cx_t \tag{3}$$

where $x_t \in \mathbb{R}^{r\times 1}$ is the state vector at time $t$, $A \in \mathbb{R}^{r\times r}, C \in \mathbb{R}^{m\times r}$ are system parameters and assume the system is unstable, $(A, C)$ is observable, $v_t$ is Gaussian noise with mean 0 and variance $Q$, $y_t \in \mathbb{R}^{m\times 1}$ is the system output at time $t$. The measurements received through a cognitive radio system discussed above is thus written as:

$$y_t = \gamma_t C x_t + \omega_t \tag{4}$$

Let $\gamma_t^l$ denote the status of the $l$th channel at time $t$ and $\{\gamma_t^l\}_k$ is the $l$th semi-Markov process. $\gamma_t^l = 1$ expresses the $l$th channel is idle at time $t$ otherwise it is busy.

In the following sections, we will discuss about the stability of the optimal filter of the system (2,4). Note that the problem can be viewed as a packet loss problem which has been considered in many works [8]∼ [12], all of which consider that the packet losses are either Bernoulli random variables or Markov processes. However, in our model, $\gamma_t$ **is governed by semi-Markov processes which has not been considered elsewhere**.

## III. Optimal Filter

The optimal state estimator for system (2,4) is well-known. In this case, the problem becomes a standard state estimation of a linear time varying system subject to Gaussian noise. The optimal estimator is the standard Kalman Filter given as follows:

*Priori state estimate and error covariance:*

$$\hat{x}_{t|t-1} = \hat{x}_{t-1|t-1} \tag{5}$$

$$P_{t|t-1} = AP_{t-1|t-1}A^T + Q \tag{6}$$

*Posteriori state estimate and error covariance:*

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + \gamma_t K_t(y_t - C\hat{x}_{t|t-1}) \tag{7}$$

$$K_t = P_{t|t-1}C^T(CP_{t|t-1}C^T + R)^{-1} \tag{8}$$

$$P_{t|t} = P_{t|t-1} - \gamma_t K_t P_{t|t-1} \tag{9}$$

where $\hat{x}_{t|t-1}$ is the prior state estimate at time $t$; $\hat{x}_{t|t}$ is the posterior state estimate at time $t$; $P_{t|t-1}$ is the error covariance of $x_t - \hat{x}_{t|t-1}$;$P_{t|t}$ is the error covariance of $x_t - \hat{x}_{t|t}$; $K_t$ is the Kalman gain.

To characterize the prediction error covariance, one can easily derive the following Riccati equation:

$$P_{t+1} = AP_tA^T + Q - \gamma_t AP_tC^T(CP_{t|t-1}C^T + R)^{-1}CP_tA^T \tag{10}$$

where $P_{t+1} = P_{t+1|t}$. The initial condition of (10) is $P_1 = AP_0A^T + Q$.

The process $\gamma_t$ will experience a consecutive sequence of 1, then followed by a consecutive sequence of 0. Thus, starting from a nonnegative definite real matrix $P_1$, when $\gamma_t = 1$, $P_{t+1} = AP_tA^T + Q - AP_tC^T(CP_{t|t-1}C^T + R)^{-1}CP_tA^T$ converges according to Kalman Filtering theory; when $\gamma_t = 0$, $P_{t+1} = AP_tA^T + Q$ diverges as $A$ is unstable. So the covariance will go through a "stable process" (when $\gamma_t = 1$) and then a "unstable process" (when $\gamma_t = 0$). To better illustrate the stability of covariance, we employ peak covariance process introduced by [11].

Let $\beta_k$ denote the time of the $k$th jump of $\gamma_t$ from 0 to 1 (see section V for more details). Labeling a subsequence of the covariance process $P_k$ by the sequence of times $\beta_k$, denote

$$M_k = P_{\beta_k}$$

$M_k$ denotes the value of the covariance $P_{\beta_k} = P_{\beta_k|\beta_k-1}$ computed by $P_{t+1} = AP_tA^T + Q$ at $t = \beta_k - 1$ and $\{M_k\}_k$ is called the peak covariance process. The peak covariance process thus consists of a sequence of covariances which are computed at $t = \beta_k - 1$ before $\gamma_t$ jumping into state $\gamma_{\beta_k} = 1$.

*Definition 1:* We say the peak covariance sequence $\{M_k\}$ is stable if $\sup_{k\geq 1} E \parallel M_k \parallel < \infty$. Accordingly, we say the system satisfies peak covariance stability [11].

The analysis of the stability of this peak covariance process is important and useful for analyzing the filtering performance in that it provides an insight that due to successive packet losses, how "bad" the covariance process may be.

Consider a series of systems:

$$\begin{aligned} x_{t+1} &= Ax_t + v_t \\ y_t &= \gamma_t^l C x_t + \omega_t \end{aligned} \tag{11}$$

where $l = 1, ..., N$. Note $\gamma_t$ in (4) is replaced by $\gamma_t^l$ (defined in section II.B) in (11) and the original problem (2, 4) has been divided into $N$ independent problems, each of which is a packet loss problem governed by a semi-Markov process. The optimal filters for these systems can be derived similarly from (5) to (9). Use $\{P_t^l\}_t$ to denote the covariance process of each optimal filter and use $\{M_k^l\}_k$ to denote the peak covariance process of the $l$th system. The following assumption is made:

*Assumption 1:* Assume there is at least one channel $d$ of $N$ satisfying:

$$\sup_{k\geq 1} E \parallel M_k \parallel \leq \sup_{k\geq 1} E \parallel M_k^d \parallel$$

This assumption is reasonable as the sensor is employed in the cognitive radio system to help the secondary users to search a better way for transmission. If no channel satisfies assumption 1, then the peak covariance $M_k$ is "worse" than the peak covariance $M_k^l$ for each channel, which makes the sensor useless.

Based on the statements above, one can easily reach the following lemma which is useful for stability conditions of optimal filtering over cognitive radio.

*Lemma 1:* Under the assumption 1, the peak covariance process $\{M_k\}_k$ of the optimal filter of the original system (2, 4) is **stable** if $\{M_k^l\}_k$ is **stable** for each $l$.

*Proof:* From the statement of the lemma, $\{M_k^l\}$ is stable for each $l$ of $N$, thus we have $\sup_{k\geq 1} E \parallel M_k^d \parallel < \infty$ which further leads to $\sup_{k\geq 1} E \parallel M_k \parallel < \infty$. ■

The argument for each $l$ is necessary as in practice, the information about which channel satisfies the assumption 1 is known.

## IV. Preliminary of Semi-Markov Process

In this section, we introduce some preliminaries of semi-Markov process that will be useful in the next section.

A semi-Markov chain is characterized by an imbedded Markov chain and a set of sojourn time probability densities. When the process enters state $i$, the next state $j$ is chosen based on imbedded Markovian transition probabilities, and the time after which the jump takes place is obtained from the sojourn time density function.

The associated homogeneous semi-Markov kernel $Q$ is defined by [18]:

$$Q_{ij}(\tau) = P\{\gamma_{n+1} = j, t_{n+1} - t_n \leq \tau \mid \gamma_n = i\}, \tag{12}$$

where $t_{n+1}$ is the time for the $n+1$th jump and $t_n$ for the $n$th jump of the process , and $i, j = 0, 1$. And as is well known [19],

$$p_{ij} = \lim_{\tau\to\infty} Q_{ij}(\tau) = P\{\gamma_{n+1} = j \mid \gamma_n = i\}, \tag{13}$$

where $P = [p_{ij}]$ is the transition probability matrix of the imbedded Markov chain. Now define the following probability density function:

$$S_{ij}(\tau) = P\{t_{n+1} - t_n = \tau \mid \gamma_{n+1} = j, \gamma_n = i\}. \tag{14}$$

It is easy to see that $\sum_{\tau=1}^{\infty} S_{ij}(\tau) = 1$ for both $i, j = 0, 1$ [20].

Denote $S_{ij}^l(\tau)$ as the probability function of the sojourn time of the $l$th channel (the $l$th semi-Markov process). In practical situation, the stochastic properties of each channel can be observed through a period of time.

## V. Stability Analysis

Based on Lemma 1 and due to the independence of each system in (11), the stability problem for the optimal filter over cognitive radio system is reduced to the stability problem for each system in (11). The system (11) is rewritten by suppressing the superscript $l$ as follows:

$$\begin{aligned} x_{t+1} &= Ax_t + m_t \\ y_t &= \gamma_t C x_t + n_t \end{aligned} \tag{15}$$

where the packet indicator $\gamma_t$ is governed by a semi-Markov process different from $N$ semi-Markov processes in the original problem. We are now in the position to derive the stability conditions for the peak covariance process $\{M_k\}$ (after suppressing the superscript $l$) in (15).

For a given initial condition $\gamma_1 = 1$, the following two stopping times are introduced [11]:

$$\begin{aligned} \tau_1 &= inf\{t : t > 1, \gamma_t = 0\}. \\ \beta_1 &= inf\{t : t > \tau_1, \gamma_t = 1\}. \end{aligned}$$

Thus $\tau_1$ is the first time when primary users occur and $\beta_1$ is the first time the channel becomes idle again. The above procedure then generates two sequences:

$$\begin{aligned} &\tau_1,\ \tau_2, ...,\ \tau_k, ... \\ &\beta_1,\ \beta_2, ...,\ \beta_k, ... \end{aligned}$$

where for $i > 1$:

$$\begin{aligned} \tau_i &= inf\{t : t > \beta_{i-1}, \gamma_t = 0\}. \\ \beta_i &= inf\{t : t > \tau_i, \gamma_t = 1\}. \end{aligned}$$

*Lemma 2:* The two sequences $\{\tau_i, i \geq 1\}$ and $\{\beta_i, i \geq 1\}$ have finite values for each of their entries [11].

Define:

$$\begin{aligned} \tau_i^* &= \tau_i - \beta_{i-1} \\ \beta_i^* &= \beta_i - \tau_i \end{aligned}$$

where $\beta_0 = 1$. Here $\tau_i^*$ and $\beta_i^*$ denote the sojourn times at state 1 and state 0, respectively.

*Lemma 3:* The following hold

(i) The random variables $\{\tau_i^*, i \geq 1\}$ are i.i.d., and $P(\tau_i^* =$

$k) = S_{10}(k)p_{10}, k \geq 0$.
(ii) The random variables $\{\beta_i^*, i \geq 1\}$ are i.i.d., and $P(\beta_i^* = k) = S_{01}(k)p_{01}, k \geq 0$.
(iii) The random variables $\{\tau_i^*, \beta_i^*, i \geq 1\}$ are independent of each other.

*Proof:* We only give the proof of (i) here, the proof of (ii) and (iii) can be obtained similarly. By the assumption in section II-A, the sojourn time $\{\tau_i^*, i \geq 1\}$ are i.i.d.

By definition:

$$\begin{aligned} P(\tau_i^* = k) &= P(t_{2i-1} - t_{2i-2} = k, \gamma_{2i-1} = 0 | \gamma_{2i-2} = 1) \\ &= P(t_{2i-1} - t_{2i-2} = k | \gamma_{2i-1} = 0, \gamma_{2i-2} = 1) \\ &* \ P(\gamma_{2i-1} = 0 | \gamma_{2i-2} = 1) \\ &= S_{10}(k)p_{10} \end{aligned} \tag{16}$$

■

Definition 2 and Lemma 4 from [11] are useful in deriving the main theorem, we simply present them below.

Let $S^r$ denote the set of all $r \times r$ nonnegative definite real matrices. Define the map $F(\cdot)$: $S^r \to S^r$ by

$$F(P) = APA' + Q - APC'(CPC' + R)^{-1}CPA'$$

where $P \in S^r$. It is obvious that for any $P \in S^r$, $F(P) \geq F(0) = Q$ and therefore $F(P) \in S^r$.

*Definition 2:* For the observable linear system $[A, C]$, the observability index is the smallest integer $I_0$ such that $[C', A'C', ..., (A^{I_0-1})'C']$ has rank $r$, where $C'$ and $A'$ denote the transpose of $C$ and $A$, respectively.

Define $S_0^r := \{P : 0 \leq P \leq A\tilde{P}A' + Q$, for some $\tilde{P} \geq 0\}$. Note that $S_0^r$ is a convex subset of $S^r$.

*Lemma 4:* For the map $F(P)$ defined above, there exists a constant $K > 0$ such that:
(i) For any $\bar{P} \in S_0^r$, $F^k(\bar{P}) \leq KI$ for all $k \geq I_0$;
(ii) For any $\bar{P} \in S^r$, $F^{k+1}(\bar{P}) \leq KI$ for all $k \geq I_0$;
(iii) For $1 \leq i \leq (I_0 - 1) \vee 1$, there exist positive constants $d_i^{(0)}$ and $d_i^{(1)}$ satisfy the following inequality:

$$\| F^i(P) \| \leq d_i^{(1)} \| P \| + d_i^{(0)}, \quad \forall P \in S_0^r \tag{17}$$

where $I$ is the $r \times r$ identity matrix; $(I_0-1) \vee 1 = max\{(I_0-1), 1\}$;$\| \cdot \|$ denotes the induced norm for matrices. For the case $I_0 = 1$, $d_1^{(1)} = 0$ and $d_i^{(0)} > 0$.

Now, we are going to present the main theorem of this paper.

*Theorem 1:* The peak covariance process of (15) is stable if the following three conditions hold:

$$(i) \qquad \limsup_{k \to \infty} \Big(1 - \frac{S_{01}(k+1)}{1 - \sum_{j=1}^{j=k} S_{01}(j)}\Big) < \frac{1}{|\lambda_A|^2}$$

$$(ii) \qquad \limsup_{k \to \infty} \Big(\frac{S_{01}(k+1)}{S_{01}(k)}\Big) < \frac{1}{|\lambda_A|^2}$$

$$\begin{aligned} (iii) \qquad & p_{01}p_{10}d_1^{(1)}[S_{10}(1) + \sum_{i=1}^{I_0-1} d_i^{(1)} S_{10}(i+1)] \sum_{j=1}^{\infty} \\ & \| A^j \|^2 S_{01}(j) < 1 \end{aligned}$$

where $\lambda_A$ is an eigenvalue of the largest magnitude for matrix $A$. Moreover, if $C$ is invertible, then condition (iii) above vanishes and the peak covariance stability holds under condition (i) and (ii).

*Proof:* The expectation of $\| P_{\beta_{k+1}+1} \|$ conditioned on $P_{\beta_k+1} = P \geq 0$ is calculated as:
$E[\| P_{\beta_{k+1}+1} \| | P_{\beta_k+1} = P]$

$$\begin{aligned} &= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} E[\| P_{\beta_{k+1}+1} \| \\ &\quad \times 1_{\tau_{k+1} - \beta_k = i, \beta_{k+1} - \tau_{k+1} = j} | P_{\beta_k+1} = P] \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \| F[A^j F^{i-1}(P)(A')^j + A^{j-1}Q(A')^{j-1} \\ &\quad + \cdots + AQA' + Q] \| \times S_{10}(i)p_{10}S_{01}(j)p_{01} \\ &\leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} d_1^{(1)} \| A^j F^{i-1}(P)(A')^j + A^{j-1}Q(A')^{j-1} \\ &\quad + \cdots + AQA' + Q \| \times S_{10}(i)p_{10}S_{01}(j)p_{01} + d_1^{(0)} \\ &\leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} d_1^{(1)} \| A^{j-1}Q(A')^{j-1} + \cdots + AQA' \\ &\quad + Q \| \times S_{10}(i)p_{10}S_{01}(j)p_{01} + \sum_{j=1}^{\infty} \sum_{i=I_0+1}^{\infty} d_1^{(1)} \\ &\quad \| A^j F^{i-1}(P)(A')^j \| \times S_{10}(i)p_{10}S_{01}(j)p_{01} \\ &\quad + \sum_{j=1}^{\infty} \sum_{i=1}^{I_0} d_1^{(1)} \| A^j F^{i-1}(P)(A')^j \| \\ &\quad \times S_{10}(i)p_{10}S_{01}(j)p_{01} + d_1^{(0)} \\ &= \Gamma_1 + \Gamma_2 + \Gamma_3 + d_1^{(0)} p_{10}p_{01} \end{aligned} \tag{18}$$

Then,

$$\begin{aligned} \Gamma_1 &= \sum_{j=1}^{\infty} d_1^{(1)} \sum_{i=1}^{\infty} S_{10}(i) \| \sum_{k=0}^{j-1} A^k Q(A')^k \| \\ &\qquad \times S_{01}(j)p_{01}p_{10} \\ &\leq \sum_{j=1}^{\infty} d_1^{(1)} \sum_{k=0}^{j-1} \| A^k \|^2 \| Q \| S_{01}(j)p_{01}p_{10} \\ &= p_{01}p_{10}d_1^{(1)} \| Q \| \sum_{k=0}^{\infty} \| A^k \|^2 \sum_{j=k+1}^{\infty} S_{01}(j) < \infty \end{aligned} \tag{19}$$

where by positive series property, the series converges if:

$$\limsup_{k \to \infty} \frac{\| A^{k+1} \|^2 \sum_{j=k+2}^{\infty} S_{01}(j)}{\| A^k \|^2 \sum_{j=k+1}^{\infty} S_{01}(j)} < 1 \tag{20}$$

Thus we have condition (i) from (20) by the fact that $\sum_{j=1}^{\infty} S_{01}(j) = 1$.

Similarly,

$$\Gamma_2 \quad \leq \quad K d_1^{(1)} \sum_{i=I_0+1}^{\infty} S_{10}(i) \sum_{j=1}^{\infty} \| A^j \|^2 S_{01}(j) \tag{21}$$

where the positive series converges if:

$$\limsup_{j \to \infty} \frac{\| A^{j+1} \|^2 S_{01}(j+1)}{\| A^j \|^2 S_{01}(j)} = \lim_{j \to \infty} |\lambda_A^2| \frac{S_{01}(j+1)}{S_{01}(j)} < 1 \tag{22}$$

Thus condition (ii) is obtained from (22). At last, we have:

$$\begin{aligned}\Gamma_3 &\leq \sum_{j=1}^{\infty} d_1^{(1)} \parallel A^j \parallel^2 S_{01}(j)[S_{10}(1) \parallel P \parallel \\ &\quad + \sum_{i=1}^{I_0-1}(d_i^{(1)} \parallel P \parallel + d_i^{(0)}) S_{10}(i+1)] p_{01}p_{10} \\ &= \{[S_{10}(1) + \sum_{i=1}^{I_0-1}(d_i^{(1)} S_{10}(i+1))] \parallel P \parallel + \sum_{i=1}^{I_0-1} d_i^{(0)} \\ &\quad \times S_{10}(i+1)\} d_1^{(1)} \sum_{j=1}^{\infty} \parallel A^j \parallel^2 S_{01}(j) p_{01}p_{10} \\ &= C_0 \parallel P \parallel + C_1 \end{aligned} \tag{23}$$

By (22), $C_1$ is a positive finite constant. And to guarantee the stability, let

$$\begin{aligned} C_0 &= [S_{10}(1) + \sum_{i=1}^{I_0-1}(d_i^{(1)} S_{10}(i+1))] \\ &\quad \times d_1^{(1)} \sum_{j=1}^{\infty} \parallel A^j \parallel^2 S_{01}(j) p_{01}p_{10} < 1 \end{aligned}$$

Then, by (19), (22) and (23), (18) can be written as:

$$E[\parallel P_{\beta_{k+1}+1} \parallel | \parallel P_{\beta_k+1} = P] = C_0 \parallel P \parallel + C_2 \tag{24}$$

and this implies:

$$E[\parallel P_{\beta_{k+1}+1} \parallel | \parallel P_{\beta_k+1}] = C_0 \parallel P_{\beta_k+1} \parallel + C_2 \tag{25}$$

which leads to

$$E[\parallel P_{\beta_{k+1}+1} \parallel] \leq C_0 \parallel P_{\beta_k+1} \parallel + C_2 \tag{26}$$

which means $\limsup_k E[\parallel P_{\beta_{k+1}+1} \parallel] < \infty$.

Similarly, we estimate $E[\parallel P_{\beta_{k+1}} \parallel]$ starting with $P_{\beta_k+1}$: $E[\parallel P_{\beta_{k+1}} \parallel | P_{\beta_k+1}, \beta_k]$

$$\begin{aligned} &= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \parallel A^j F^{i-1}(P_{\beta_k+1})(A')^j + A^{j-1}Q(A')^{j-1} \\ &\quad + \cdots + AQA' + Q \parallel \times S_{10}(i)p_{10}S_{01}(j)p_{01} \\ &\leq \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \parallel A^j F^{i-1}(P_{\beta_k+1})(A')^j \parallel + \parallel A^{j-1}Q(A')^{j-1} \\ &\quad + \cdots + AQA' + Q \parallel \times S_{10}(i)S_{01}(j) \\ &= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \parallel A^j F^{i-1}(P_{\beta_k+1})(A')^j \parallel S_{10}(i)S_{01}(j) + O(1) \\ &= \sum_{i=1}^{\infty} \parallel F^{i-1}(P_{\beta_k+1}) \parallel + O(1) \\ &= \sum_{i=1}^{I_0} \parallel F^{i-1}(P_{\beta_k+1}) \parallel + O(1) \leq K_1 \parallel P_{\beta_k+1} \parallel + K_2 \end{aligned}$$

where $K_1, K_2$ are positive constants. In the above, the second equality is from condition (i), the third comes from condition (ii), the forth is from lemma 4 and the last inequality is from (17). Then it is easily follows that $\sup_{k\geq 1} E[\parallel P_{\beta_{k+1}} \parallel] < \infty$ and the stability of the peak covariance process is obtained.

When $C$ is invertible, then $I_0 = 1$, which means $d_1^{(1)} = 0$. Thus condition (iii) vanishes. ■

The next theorem is a direct result of lemma 1 and theorem 1.

*Theorem 2:* The peak covariance process of the original system (2,4) is stable if each of the channels sensed in the cognitive radio system can be represented by a semi-Markov process that satisfies theorem 1.

***Remark:*** (1) When C is invertible, condition (iii) vanishes in theorem 1, thus an appropriate chosen of $S_{01}(k)$ will stablize the covariance process, and provide a way to design cognitive radio channels to guarantee stability. (2) If $\gamma_k$ is a Markov process which is a special case of the semi-Markov process, conditions in theorem 1 coincide with the two conditions in theorem 6 in [11].

## VI. Numerical Example

In this section, we give an example to illustrate the performance of the theorem. For simplicity, assume there is only one channel: $N = 1$. Due to the independence of each channel, this assumption does not lose any generality. The parameters of the system is given as:
$A = \begin{bmatrix} 1.1 & 0.1 \\ 0 & 1.2 \end{bmatrix}$, $C = [1 \ \ 1]$, $Q = I_{2\times 2}$, $R = 1$

The channel is characterized by a semi-Markov process with transition probability matrix $P = [p_{ij}]$ and sojourn time probability mass function $S_{ij}(\tau)$:

$$P = \begin{bmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{bmatrix}$$

$$\begin{aligned} S_{01}(\tau) &= s_0 \exp(-|\tau|) \\ S_{10}(\tau) &= s_1 \exp(-|\tau - 2|) \end{aligned}$$

with $s_i$ such that $\sum_{\tau=0}^{\infty} S_{ij}(\tau) = 1$.

It is easy to see with the above information, the left hand side of condition (i) and (ii) are both $e^{-1} = 0.3679$ and $|\lambda_A| = 1.2$, thus condition (i) and (ii) are satisfied.

We also have $\parallel F(P) \parallel \leq \parallel AA' \parallel \parallel P \parallel$ and since $AA'$ has two eigenvalues $\lambda_1 = 1.1672$ and $\lambda_2 = 1.4927$. Thus choose $d_1^{(1)} = 1.4928$. By numerical calculation, we have $\sum_{j=1}^{\infty} \parallel A^j \parallel^2 S_{01}(j) \leq 2.1$, and $S_{10}(1) = 0.18868, S_{10}(2) = 0.51286$ gives $S_{10}(1) + d_1^{(1)} S_{10}(2) = 0.95428$. Thus the left hand side of condition (iii) is computed as $p_{01}p_{10}d_1^{(1)}[S_{10}(1) + d_1^{(1)} S_{10}(2)] \sum_{j=1}^{\infty} \parallel A^j \parallel^2 S_{01}(j) \approx 0.9573 < 1$. Thus conditions in theorem 1 are all satisfied. $P_{11}(t)$ and $P_{12}(t)$ are two entries of the covariance matrix $P_t$, from Fig. 2 and Fig. 3, it is obvious they are bounded. Similarly, the other two entries $P_{21}(t)$ and $P_{22}(t)$ are also bounded but the figures are omitted for the sake of space.

## VII. CONCLUSIONS AND FUTURE WORKS

The paper discusses the optimal filtering over the cognitive radio system governed by semi-Markov processes, each of which can represent and capture the behavior of one channel. This new communication link may cause packet losses during the transmission due to the activities of primary

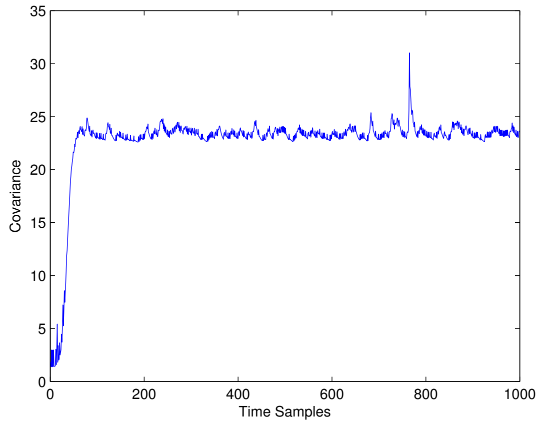


Fig. 2: $P_{11}(t)$ of the covariance.

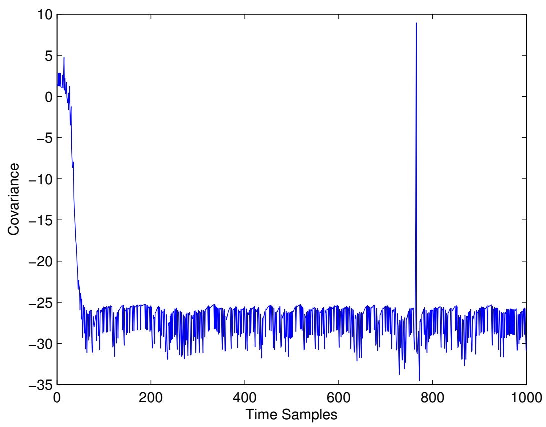


Fig. 3: $P_{12}(t)$ of the covariance.

users. Sufficient stability conditions are derived for the peak covariance process of the optimal filter. An illustrative example is provided and demonstrate the method's viability.

## REFERENCES


[1] S. Srinivasa, and S. A. Jafar, "The throughput potential of cognitive radio: A theoretical perspective," *IEEE Communication Magazine*, May. 2007, pp. 73-79.

[2] NTIA, "FCC Frequency Allocation Chart," 2003; http://www.ntia.doc.gov/osmhome/allochrt.pdf.

[3] FCC Spectrum Policy Task Force, "Report of the Spectrum Efficiency Working Group," *Tech. rep.02-135*, Nov. 2002; http://www.fcc.gov/sptf/files/SEWGFinalReport 1.pdf

[4] J. Mitola, Cognitive radio: An integrated agent architecture for software defined radio, D*octor of Technology, Royal Inst. Technol. (KTH)*, Stockholm, Sweden, 2000.

[5] I. F. Akyildiz, W. Y. Lee, M. C. Vuran and S. Mohanty, "NeXt generation/dynamic spectrum access/cognitive radio wireless networks: A survey," *Computer Networks 50*, pp. 2127-2159, 2006.

[6] J. Nilsson . *Real-time control systems with delays*. Ph.D. dissertation, Department of Automatic Control, Lund Institute of Technology, 1998.

[7] K.J. Rajasekaran, N. Satyanarayana and M.D. Srinath, Optimum linear estimation of stochastic signals in the presence of multiplicative noise. *IEEE Trans. Aerospace and Electronic Systems*, vol. AES-7, pp.462-468, 1971.

[8] M. Hadidi and S.Schwartz, Linear recursive state estimators under uncertain observations. *IEEE Trans. Inform. Theory*, vol. IT-24, pp. 944-948, 1979.

[9] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poola, M. Jordan and S. Satry, Kalman filtering with intermittent observations. *IEEE Trans. Automat. Contr.*, vol. 49(9), pp. 458-463, 2004.

[10] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poola and S. Satry, Foundations of Control and Estimation over Lossy Networks, *Proc. of the IEEE*, vol. 95, pp. 163-187, 2007.

[11] M. Huang and S. Dey, Stability of Kalman filtering with Markovian packet losses. *Automatica*, vol. 43, pp. 598-607, 2007.

[12] V. Gupta, B. Hassibi and R. M. Murray, Optimal LQG control across packet-dropping links. *Systems and Control Letters*, vol. 56, pp. 439-446, 2007.

[13] L. Xiao, A. Hassibi and J. P. How, Control with random Communication delays via a discrete-time jump system approach. *ACC*, Chicago, Illinois, 2000.

[14] L. Xiao, M. Jonathan, H. Hindi, S. Boyd, and A. Goldsmith, Joint optimization of communication rates and linear systems. *IEEE Trans. Automat. Contr.*, vol. 48(1), pp. 148-153, 2003.

[15] X. Ma, S.M. Djouadi, T. Kuruganti, J.J. Nutaro, and H. Li, "Optimal estimation over unreliable communication links with application to cognitive radio," *48th IEEE CDC and 28th CCC*,pp. 4062-4067, Shanghai, China, 2009.

[16] X. Ma, S.M. Djouadi, T. Kuruganti, J.J. Nutaro, and H. Li, "Control and estimation through cognitive radio with distributed and dynamic spectral activity," *American Control Conference*, pp. 289-294, Baltimore, MD, 2010.

[17] S. Geirhofer, L. Tong abd B. M. Sadler, "Dynamic spectrum access in the time domain: Modeling and exploiting whitespace," *IEEE Communications Magazine*, pp 66-72, May 2007.

[18] J. Janssen, A. Blasi, R. Blasi and R. Manca, "Discrete time homogeneous and reliability models," *Universit'a "LaSapienza" Roma*, 2002.

[19] R. Pyke, "Markov renewal processes with finitely many states," *Am. Math. Sta. 32*, pp 1245-1259, 1961.

[20] V. J. Mathews and J. K. Tugnait, "Detection and estimation with fixed lag for abruptly changing systems," *IEEE Trans. Aerospace and Electronic Systems*, vol. AES-19, No. 5, pp 730-739, 1983.