2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

# Remarks on the relationship between $\mathscr{L}_p$ stability and internal stability of nonlinear systems

Xu Wang, Håvard Fjær Grip, Ali Saberi, Anton A. Stoorvogel and Ingmar Saberi

***Abstract*— In this paper, we investigate the relationship between $\mathscr{L}_p$ stability and internal stability of nonlinear systems. It is shown that under certain conditions, $\mathscr{L}_p$ stability without finite gain implies attractivity of the equilibrium, and that local $\mathscr{L}_p$ stability with finite gain implies local asymptotic stability of the origin.**

## I. Introduction

In this paper, we study the relationship between $\mathscr{L}_p$ stability and internal stability of nonlinear systems. Specifically, for a nonlinear system that is $\mathscr{L}_p$ stable, we are interested in investigating the internal stability of the autonomous system when the input is zero. The research in this area evolves along two main lines. The first line starts with $\mathscr{L}_p$ stability without finite gain. An important result that emerges in this direction is [1]. It is shown that under a fairly restrictive condition on the structural properties of the system, $\mathscr{L}_p$ stability without finite gain implies global attractivity of the equilibrium. In fact, it turns out that this conclusion can be attained under much weaker conditions than those in [1]. It is shown in this paper that under mild conditions, global $\mathscr{L}_p$ stability without finite gain ensures attractivity of the equilibrium in the absence of input and attractivity of the origin with any $\mathscr{L}_p$ input.

The other line emanates from $\mathscr{L}_p$ stability with finite gain. There is a large body of work in the literature in this direction; see, for instance, [2], [3], [4], [1]. Along this line of research, the objective is to conclude local asymptotic stability of the equilibrium based on $\mathscr{L}_p$ stability with finite gain. It was shown in [2] that under a uniform reachability condition, global $\mathscr{L}_p$ stability with finite gain implies local asymptotic stability of the equilibrium. In [3], the notion of small-signal $\mathscr{L}_p$ stability with finite gain was introduced and its connection to attractivity of the equilibrium was established. This concept of small-signal $\mathscr{L}_p$ stability was extended in [4] by so-called gain-over-set stability, and it was shown that finite-gain $\mathscr{L}_p$ stability over a set in $\mathscr{L}_p$ space yields local asymptotic stability of the equilibrium. In this paper, we prove a result on the relationship between Lyapunov stability and local $\mathscr{L}_p$ stability with finite gain, which further extends, to some level, the result in [4].

Xu Wang, Håvard Fjær Grip, and Ali Saberi are with the School of Electrical Engineering and Computer Science, Washington State University, Pullman, WA 99164-2752, USA. The work of Xu Wang and Ali Saberi is partially supported by NAVY grants ONR KKK777SB001 and ONR KKK760SB0012. The work of Håvard Fjær Grip is supported by the Research Council of Norway. E-mail: xwang@eecs.wsu.edu, grip@ieee.org, saberi@eecs.wsu.edu

Anton A. Stoorvogel is with the Department of Electrical Engineering, Mathematics, and Computing Science, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands. E-mail: A.A.Stoorvogel@utwente.nl

Ingmar Saberi is with the Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton NJ 08544-1000, U.S.A., E-mail: isaberi@princeton.edu

## II. Preliminaries

Consider a nonlinear system

$$\Sigma_1: \quad \dot{x} = f(x,u), \quad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathscr{R}^m$. We assume that for all $x_0 \in \mathbb{R}^n$ and $u \in \mathscr{L}_p$, the system $\Sigma_1$ has a unique solution defined on $[0,\infty)$, which is absolutely continuous on any compact interval. Moreover, we assume that $f(x,u)$ is continuous with respect to $x$. Let $x(t,t_0,u,x_0)$ denote the trajectory of $\Sigma_1$ initialized at time $t_0$ with input $u$ and initial condition $x_0$.

We shall investigate the internal stability of the unforced system

$$\Sigma_2: \quad \dot{x} = f(x,0), \quad x(0) = x_0, \tag{2}$$

under the assumption that $\Sigma_1$ is $\mathscr{L}_p$ stable in some sense.

We formally define the notions of $\mathscr{L}_p$ stability as follows:

*Definition 1:* $\Sigma_1$ is said to be globally $\mathscr{L}_p$ stable without finite gain if for $x_0 = 0$ and any $u \in \mathscr{L}_p$, $x(\cdot,0,u,0) \in \mathscr{L}_p$. $\Sigma_1$ is said to be locally $\mathscr{L}_p$ stable with finite gain if there exists a $\delta$ and $\gamma$ such that for $x_0 = 0$ and any $u$ with $\|u\|_{\mathscr{L}_p} \le \delta$, $\|x(\cdot,0,u,0)\|_{\mathscr{L}_p} \le \gamma \|u\|_{\mathscr{L}_p}$.

The domain of attraction and the notion of an $\mathscr{L}_p$-reachable set are defined as follows:

*Definition 2:* The set

$$\mathscr{A}(\Sigma_2) = \{x_0 \in \mathbb{R}^n \mid x(t,0,0,x_0) \to 0 \text{ as } t \to \infty\} \tag{3}$$

is called the domain of attraction of the system $\Sigma_2$.

*Definition 3:* A point $\xi \in \mathbb{R}^n$ is an $\mathscr{L}_p$-reachable point of system $\Sigma_1$ if there exist finite $T$, $M$ and a measurable input $u:[0,T] \to \mathbb{R}^m$ such that $x(T,0,u,0) = \xi$ and

$$\int_0^T \|u(t)\|^p \mathrm{d}t \le M. \tag{4}$$

The set of all $\mathscr{L}_p$-reachable points of $\Sigma_1$ is called the $\mathscr{L}_p$-reachable set of $\Sigma_1$, which is denoted as $\mathscr{R}_p(\Sigma_1)$.

The following definition of small-signal local $\mathscr{L}_p$-reachability is adapted from [4]:

*Definition 4:* The system $\Sigma_1$ is said to be small-signal locally $\mathscr{L}_p$-reachable if for any $\varepsilon > 0$, there exists $\delta$ such that for any $\xi \in \mathbb{R}^n$ with $\|\xi\| \le \delta$, we can find a finite time $T$ and a measurable input $u:[0,T] \to \mathbb{R}^m$ such that $x(T,0,u,0) = \xi$ and $\|u\|_{\mathscr{L}_p} \le \varepsilon$.

## III. Main result

*Theorem 1:* Suppose system $\Sigma_1$ is globally $\mathscr{L}_p$ stable without finite gain for some $p \in [1,\infty)$. Then $\mathscr{A}(\Sigma_2) \supseteq \mathscr{R}_p(\Sigma_1)$.

In order to prove Theorem 1, we need the following lemma:

*Lemma 1:* Consider system $\Sigma_2$. If $x(t,0,0,x_0) \in \mathscr{L}_p$ for some $p \in [1,\infty)$, then $x(t,0,0,x_0) \to 0$.

*Proof:* For simplicity, we denote $x(t,0,0,x_0)$ by $x(t)$ and $f(x(t),0)$ by $f(x(t))$ in this proof. Suppose, for the sake of establishing a contradiction, that $x(t) \to 0$ does not hold. Then there exists a $\delta > 0$ such that, for any arbitrarily large $T \geq 0$, there is a $\tau \geq T$ such that $\|x(\tau)\| \geq 2\delta$. Let $m$ be a bound on $\|f(x,0)\|$ on the closed ball $B(2\delta)$. This bound exists due to continuity of $f(x,0)$ with respect to $x$.

For some $\tau$ such that $\|x(\tau)\| \geq 2\delta$, let $t_2 > \tau$ be the smallest value such that $\|x(t_2)\| = \delta$, and let $t_1$ be the largest value such that $t_1 < t_2$ and $\|x(t_1)| = 2\delta$. Such $t_1$ and $t_2$ exist because $x \in \mathscr{L}_p$. Since $\|x(t)\| \in B(2\delta)$ for all $t \in [t_1,t_2]$, we have, due to the absolute continuity of the solution,

$$\|x(t_1)\| - \|x(t_2)\| \leq \|x(t_2) - x(t_1)\| = \left\| \int_{t_1}^{t_2} f(x(\tau))\,\mathrm{d}\tau \right\| \leq \int_{t_1}^{t_2} \|f(x(\tau))\|\,\mathrm{d}\tau \leq (t_2 - t_1)m.$$

Hence, $t_2 - t_1 \geq (\|x(t_1)\| - \|x(t_2)\|)/m = \delta/m$. Clearly $\|x(t)\| \geq \delta$ for all $t \in [\tau,t_2]$, and furthermore $t_2 - \tau \geq t_2 - t_1 \geq \delta/m$. It follows that for each $\tau$ such that $\|x(\tau)\| \geq 2\delta$, we have $\|x(t)\| \geq \delta$ for all $t \in [\tau, \tau + \delta/m]$.

Let $T$ be chosen large enough that

$$\int_T^\infty \|x(t)\|^p\,\mathrm{d}\tau < \frac{\delta^{p+1}}{m}. \tag{5}$$

Such a $T$ must exist, since $x(t) \in L_p$. Let $\tau \geq T$ be chosen such that $\|x(\tau)\| \geq 2\delta$. We have

$$\int_T^\infty \|x(t)\|^p\,\mathrm{d}\tau \geq \int_\tau^{\tau+\delta/m} \|x(t)\|^p\,\mathrm{d}\tau \geq \frac{\delta^{p+1}}{m}.$$

This contradicts (5), which proves that $x(t) \to 0$. ■

*Proof of Theorem 1:* For any $x_0 \in \mathscr{R}_p(\Sigma_1)$, there exist finite $T$, $M$ and an input $u_0(t)$ for $t \in [0,T]$ such that $x(T,0,u_0,0) = x_0$ and

$$\int_0^T \|u_0(t)\|^p \mathrm{d}t \leq M$$

Define

$$u(t) = \begin{cases} u_0(t), & t \in [0,T] \\ 0, & t > T \end{cases}$$

Clearly, $u \in \mathscr{L}_p$. Since $\Sigma_1$ is globally $\mathscr{L}_p$ stable without finite gain, we have that $x(\cdot,0,u,0) \in \mathscr{L}_p$. On the other hand, $u(t) = 0$ for $t > T$ implies that after $T$ the system $\Sigma_1$ is equivalent with system $\Sigma_2$ initialized at $x_0$, i.e. $x(t,0,u,0) = x(t-T,0,0,x_0)$ with $t > T$. Therefore, $x(t,0,0,x_0) \in \mathscr{L}_p$ over $[0,\infty)$. It follows from Lemma 1 that $x(t,0,0,x_0) \to 0$ as $t \to \infty$. This completes the proof. ■

*Corollary 1:* Suppose system $\Sigma_1$ is globally $\mathscr{L}_p$ stable without finite gain for some $p \in [1,\infty)$. If $\mathscr{R}_p(\Sigma_1) = \mathbb{R}^n$, then the origin of $\Sigma_2$ is globally attractive.

Due to space limitation, the proofs of subsequent theorems are omitted.

The next theorem shows that under a certain condition on the structure of $f(x,u)$, the origin of $\Sigma_1$ is attractive for any input $u \in \mathscr{L}_p$.

*Theorem 2:* Suppose that $\Sigma_1$ is globally $\mathscr{L}_p$ stable without finite gain for some $p \in [1,\infty)$. If there exist $\delta$, $m_1$, $m_2$ and $q \in [0,p]$ such that for any $x$ with $\|x\| \leq \delta$

$$\|f(x,u)\| \leq m_1 + m_2\|u\|^q, \tag{6}$$

then for $x_0 = 0$ and any $u \in \mathscr{L}_p$, $x(t,0,u,0) \to 0$ as $t \to \infty$.

*Remark 1:* In [1], in order to prove the same result as in Theorem 2, the following condition was imposed on $f(x,u)$: there exists $\delta_1$, $K_1$, $K_2$ and $\alpha \in [0,p]$ such that for $x \in \mathbb{R}^n$ with $\|x\| \leq \delta_1$,

$$\|f(x,u)\| \leq K_1(\|x\| + \|u\|) + K_2(\|x\|^\alpha + \|u\|^\alpha)$$

Theorem 2 shows that the restrictions on $x$ in the above condition is not necessary.

An immediate consequence of Theorem 2 is the next theorem.

*Theorem 3:* Suppose that $\Sigma_1$ is globally $\mathscr{L}_p$ stable without finite gain and $\mathscr{R}_p(\Sigma_1) = \mathbb{R}^n$ for some $p \in [1,\infty)$. If there exist $\delta$, $m_1$, $m_2$ and $q \in [0,p]$ such that for any $x$ with $\|x\| \leq \delta$

$$\|f(x,u)\| \leq m_1 + m_2\|u\|^q,$$

then $\Sigma_1$ is globally $\mathscr{L}_p$ stable without finite gain with arbitrary initial condition.[1] Moreover, for any $x_0 \in \mathbb{R}^n$ and any $u \in \mathscr{L}_p$, $x(t,0,u,x_0) \to 0$ as $t \to \infty$.

The following theorem is a slight generalization of results in [4].

*Theorem 4:* Suppose that $\Sigma_1$ is locally $\mathscr{L}_p$ stable with finite gain and small-signal locally $\mathscr{L}_p$-reachable. Then the origin of $\Sigma_2$ is locally asymptotically stable.

*Remark 2:* Compared with the result in [4], Theorem 4 only requires a finite gain within an arbitrary small neighborhood of the origin of $\mathscr{L}_p$ space.

*Remark 3:* We assume in this paper that $f(x,u)$ is continuous with respect to $x$, which covers a large class of dynamical systems. In fact, it can be seen from the proof that we only need continuity of $f(x,u)$ with respect to $x$ at $x = 0$.

## References


[1] W. Liu, Y. Chitour, and E. Sontag, "On finite-gain stabilizability of linear systems subject to input saturation," *SIAM J. Contr. & Opt.*, vol. 34, no. 4, pp. 1190–1219, 1996.

[2] D. Hill and P. Moylan, "Connections between finite-gain and asymptotic stability," *IEEE Trans. Aut. Contr.*, vol. 25, no. 5, pp. 931 – 936, 1980.

[3] M. Vidyasagar and A. Vannelli, "New relationships between input-output and Lyapunov stability," *IEEE Trans. Aut. Contr.*, vol. 27, no. 2, pp. 481–483, 1982.

[4] J. Choi, "Connections between local stability in Lyapunov and input/output senses," *IEEE Trans. Aut. Contr.*, vol. 40, no. 12, pp. 2139 –2143, 1995.


[1] $\Sigma_1$ is said to be global $\mathscr{L}_p$ stable without finite gain with arbitrary initial condition if for any $x_0 \in \mathbb{R}^n$ and $u \in \mathscr{L}_p$, we have $x(\cdot,0,u,x_0) \in \mathscr{L}_p$.