

An Identification Algorithm for Hammerstein Systems Using Subspace Method

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Abstract—This paper describes a new algorithm for the identification of single-input single-output Hammerstein systems using the *multivariable output error state space* (MOESP) class of subspace identification algorithms. The algorithm consists of three main steps. First, the MOESP algorithm is used to determine the system order and estimate two of the state space model matrices. Second, a least squares problem is solved to minimize the prediction error. Finally, the global search optimization is needed to be used to estimate optimal values for the remaining parameters. Performance of the model was evaluated by simulating a model of ankle joint reflex stiffness, a well known Hammerstein system. The results demonstrate that the algorithm estimated the model parameters very accurately in the presence of additive, output noise.

I. INTRODUCTION

The Hammerstein structure consists of a static nonlinearity followed by a linear system [1] as illustrated in Fig. 1. Many physical systems can be modeled using the Hammerstein structure including the reflex stiffness in human ankle joint [2] and neural integrator model of the human VOR [3]. Consequently, the problem of identifying models of Hammerstein structure is an active research area. A variety of methods have been proposed for this purpose including the stochastic method, the *separable least square* method (SLS), the subspace method, etc [4], [5], [6], [7], [8]. Some of these methods are available in a MATLAB toolbox [9].

MOESP is a class of *subspace model identification* (SMI) methods that estimates state space models of linear systems using only input and output measurements. The methods require no *a priori* knowledge of the order of the system, are computationally efficient, and can be extended to identify systems with different types of noises [10], [11]. The MOESP algorithm was proposed in [12] and [13] and is available as part of MATLAB SMI toolbox [14]. The linear MOESP was extended to deal with multiple-input multiple-output (MIMO) Hammerstein systems [7]. The method works by transforming the SISO nonlinear system into a MISO linear system. It is important to note that the algorithm estimates the order of each linear pathway as well as their parameters. Consequently, there is no need to know the order a priori as is the cases with other parametric methods such as those based on ARMA models.

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The resulting Hammerstein models using subspace algorithm have excellent predictive capabilities but its parameters are difficult to relate to the nonlinearity and/or linear dynamics of the original nonlinear SISO system. This is because the transformed MISO system has many more parameters than the SISO system and each MISO parameter depends on both the nonlinearity and the linear dynamics. Thus, when the method described in [7] was used in [15], [16] to estimate a state space model for joint stiffness, a number of additional steps were required to determine the underlying nonlinearity and linear dynamics. In particular, the estimated state space model was simulated using the experimental input to predict the noise-free outputs and then time-domain approaches were used to estimate the linear dynamics and shape of the nonlinearity [15], [16].

In the present paper, a MOESP algorithm similar to that presented in [7] was selected due to its potential of extension to closed-loop or time-varying system [12], [13], [17]. Two state space matrices that relate the states to the derivative of the states (A in Fig. 1) and the one that relates the states to the output (C in Fig. 1) are derived using the approach provided in [7]. At the next step, an algorithm proposed to estimate the nonlinearity as well as the state space model matrices that relate the input to the states (B in Fig. 1) and the input to the output of the system (D in Fig. 1). This approach uses the mathematical framework in [18], proposed for identification of sandwich systems comprising two static nonlinear elements surrounding a linear dynamic block. The optimality of the algorithm is then investigated by imposing certain conditions on the input signal.

There are some significant differences between the algorithm presented here and that proposed in [18]. The linear component of the Hammerstein system is modeled by its state space in this paper whereas in [18] it was modeled as *impulse response function* (IRF). Therefore, using the MOESP algorithm reduce the number of unknown parameters dramatically. Thus, the unknown parameters in [18] were the total IRF samples as well as the coefficient of nonlinearities whereas the unknown parameters in our method are part of the state space matrix elements as well as the coefficient of nonlinearity. This should result in better estimates in the presence of noise.

The major advantage of the proposed algorithm over the previously developed MOESP-based algorithm [5], [15], [16] is that it does not over parameterize the system model. Consequently, it gives explicit information on the coefficient of nonlinearity and the state space model of the linear component. In other words, using the approach presented here,

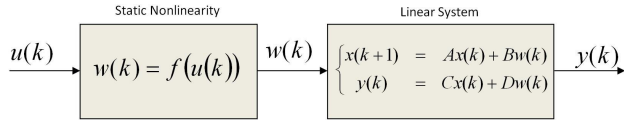


Fig. 1. Hammerstein system model.

the continuous-time representation of the nonlinearity and the (IRF) of the linear component can be easily computed. The other advantage of this algorithm over stochastic and correlation-based algorithms is that it has the potential of extension to closed-loop systems. This is due to the fact that MOESP algorithm and in particular, *error in variable* (EIV-MOESP) provides unbiased result in closed-loop case. The proposed algorithm is simple and inexpensive in computation since it is not iterative.

The paper is organized as follows. Section II formulates the Hammerstein system model. Section III presents the new identification algorithm, Section IV presents simulation results that demonstrate the efficiency of the method. Finally, Section V concludes the paper with some summary remarks.

II. PROBLEM FORMULATION

This section formulates the Hammerstein model used for the identification algorithm. In this model, the nonlinearity is expressed using a basis function expansion and the linear subsystem is described using a state space model.

Consider a SISO Hammerstein discrete system as shown in Fig. 1. This system comprises a static nonlinear block followed by a dynamic linear system. The nonlinearity is assumed to be a static nonlinear function given by:

$$w(k) = f(u(k)) = \sum_{i=1}^n \alpha_i g_i(u(k)) \quad (1)$$

where, $g_i(\cdot)$ is the i^{th} basis function expansion of the nonlinearity which can be power polynomial, Tchebyshev, Hermite, etc and α_i is the corresponding i^{th} coefficient. Assume also that N samples are recorded, i.e., $k \in \{0, \dots, N-1\}$ for the input, $u(k)$, and output $y(k)$ of the Hammerstein system. The signal $w(k)$ is an intermediate signal that is not available and cannot be recorded. Further, assume that the linear component can be described by the state space model:

$$\begin{cases} x(k+1) = Ax(k) + Bw(k) \\ y(k) = Cx(k) + Dw(k) \end{cases} \quad (2)$$

where, $x(k)$ is the state of the linear subsystem and assumed to be a $m \times 1$ vector. Moreover, $A_{m \times m}$, $B_{m \times 1}$, $C_{1 \times m}$ and $D_{1 \times 1}$ are state space model matrices and $y(k)$ is the output of the system. Assume the elements of matrices B and D are represented as:

$$\begin{aligned} B &= [b_1, \dots, b_m]^T \\ D &= [d] \end{aligned} \quad (3)$$

The measured output $\tilde{y}(k)$ is contaminated with additive noise, $n(k)$:

$$\tilde{y}(k) = y(k) + n(k) \quad (4)$$

Define the following vectors:

$$\alpha = [\alpha_1, \dots, \alpha_n]^T \quad (5)$$

$$U(k) = [g_1(u(k)), \dots, g_n(u(k))]^T \quad (6)$$

Substituting (5) and (6) into (2) with (1) yields:

$$\begin{cases} x(k+1) = Ax(k) + B_\alpha U(k) \\ y(k) = Cx(k) + D_\alpha U(k) \end{cases} \quad (7)$$

where B_α and D_α are given by:

$$B_\alpha = \begin{bmatrix} b_1 \alpha_1 & \dots & b_1 \alpha_n \\ \vdots & \ddots & \vdots \\ b_m \alpha_1 & \dots & b_m \alpha_n \end{bmatrix} \quad (8)$$

$$D_\alpha = [d \alpha_1 \quad \dots \quad d \alpha_n] \quad (9)$$

Note that this parametrization of the system is not unique. For any arbitrary scalar β , the vectors βB , βD and $\beta^{-1} \alpha$ represent the same matrix B_α and D_α . This means that the identification algorithm cannot distinguish between these two sets. The following constraint will be used in the rest of the paper to obtain a unique parameterization.

Assumption 1: Let $\|[\alpha_1, \dots, \alpha_n]^T\| = 1$, where $\|\cdot\|$ is the two norm. Moreover, α_1 must be positive.

Note that in (7), the total Hammerstein system is modeled as a multi-input single-output (MISO) system, so that the new input to the new system is $U(k)$ which is a $n \times 1$ vector. On the other hand, the matrices estimated with the identification algorithm have the following structure:

$$\begin{cases} \hat{x}_T(k+1) = \hat{A}_T \hat{x}(k) + \hat{B}_T U(k) \\ \hat{y}(k) = \hat{C}_T \hat{x}_T(k) + \hat{D}_T U(k) \end{cases} \quad (10)$$

where, the subscript T in (10) is due to the fact that the identification is achieved up to a similarity transform with the transformation matrix T . The hat symbol is used to indicate that the estimates may have associated errors. Based on the similarity transform, the following can be shown:

$$\hat{B}_T \simeq T^{-1} B_\alpha = \begin{bmatrix} b'_1 \alpha_1 & \dots & b'_1 \alpha_n \\ \vdots & \ddots & \vdots \\ b'_m \alpha_1 & \dots & b'_m \alpha_n \end{bmatrix} \quad (11)$$

$$\hat{D}_T \simeq D_\alpha = [d \alpha_1 \quad \dots \quad d \alpha_n] \quad (12)$$

The vector $B' = [b'_1, \dots, b'_m]^T$ represents the effect of similarity transform i.e., $B' = T^{-1} B$.

III. IDENTIFICATION ALGORITHM

First, the MOESP algorithm is applied to the constructed input (6), and the measured noisy output $\hat{y}(k)$, to determine the system order at the first step, and then estimate the system matrices \hat{A}_T and \hat{C}_T . This estimation is described in [12], [13], [10] and [7] and is not repeated here. The objective now is to estimate the matrices \hat{B}_T , \hat{D}_T and the coefficient of basis expansion of the nonlinear block α .

The output of a system like (10) can be expressed using the state space representation matrices at each discrete time [10]:

$$\hat{y}(k) = \sum_{\tau=0}^{k-1} \hat{C}_T \hat{A}_T^{k-1-\tau} \hat{B}_T U(\tau) + \hat{D}_T U(k) \quad (13)$$

Definition 1: The Kronecker product of two matrices $F \in \mathbb{R}_{p \times q}$ and $G \in \mathbb{R}_{r \times s}$ is denoted by $F \otimes G \in \mathbb{R}_{pr \times qs}$ and is given by [19]:

$$F \otimes G = \begin{bmatrix} F_{11}G & \cdots & F_{1q}G \\ \vdots & \ddots & \vdots \\ F_{p1}G & \cdots & F_{pq}G \end{bmatrix} \quad (14)$$

One of the properties of Kronecker product is:

$$\text{vec}(FGH) = (H^T \otimes F) \text{vec}(G) \quad (15)$$

where F , G and H are matrices with arbitrary dimension and the function $\text{vec}(\cdot)$ stacks the columns of a matrix on top of each other in a tall vector.

Now using the Kronecker product expressed in (15), the output of the system given in (13) can be expressed as:

$$\hat{y}(k) = \left[\sum_{\tau=0}^{k-1} U^T(\tau) \otimes \hat{C}_T \hat{A}_T^{k-1-\tau} \right] \text{vec}(\hat{B}_T) + [U^T(k) \otimes I_l] \text{vec}(\hat{D}_T) \quad (16)$$

where I_l is identity matrix with dimension $l = 1$. Define the following matrices:

$$\begin{aligned} Y_{0,N,1} &= [\tilde{y}(0), \dots, \tilde{y}(N-1)]^T \\ \Delta_{0,N,1} &= [n(0), \dots, n(N-1)]^T \\ \Gamma_N &= \left[0, \dots, \sum_{\tau=0}^{N-2} U^T(\tau) \otimes C A^{N-2-\tau} \right]^T \\ \Phi_N &= [U^T(0) \otimes I_l, \dots, U^T(N-1) \otimes I_l] \\ \bar{B} &= \text{vec}(\hat{B}_T) \\ \bar{D} &= \text{vec}(\hat{D}_T) \\ \theta_{\alpha b' d} &= \begin{bmatrix} b'_1 \alpha_1, & \cdots, & b'_m \alpha_1, & d \alpha_1 \\ \vdots & \ddots & \vdots & \vdots \\ b'_1 \alpha_n, & \cdots, & b'_m \alpha_n, & d \alpha_n \end{bmatrix} \end{aligned} \quad (17)$$

Now, (16) can be rewritten as a matrix equation:

$$Y_{0,N,1} = \Psi \theta + \Delta_{0,N,1} \quad (18)$$

where the matrix Ψ is a $N \times n(m+1)$ data matrix:

$$\Psi = [\Gamma_N, \Phi_N] \quad (19)$$

and the vector θ contains the unknown parameters stacked in a single vector:

$$\theta = \begin{bmatrix} \bar{B} \\ \bar{D} \end{bmatrix} = \begin{bmatrix} \text{vec}(\hat{B}_T) \\ \text{vec}(\hat{D}_T) \end{bmatrix} \quad (20)$$

Algorithm 1: This algorithm estimates the unknown parameters $b'_1, \dots, b'_m, d, \alpha_1, \dots, \alpha_n$ as follows.

(a) Initialization:

Construct the matrix:

$$\Xi = [\Psi_1, \dots, \Psi_m, \Psi_{nm+1}, \Psi_{m+1}, \dots, \Psi_{2m}, \Psi_{nm+2}, \Psi_{2m+1}, \dots, \Psi_{n(m+1)}]^T \quad (21)$$

where, Ψ_i is the i^{th} column of Ψ , the data matrix defined in (19). This rearranges the data matrix Ψ to be consistent with the structure of a new unknown vector. In this unknown vector, the first $m+1$ parameters correspond to α_1 , the second $m+1$ parameters correspond to α_2 , etc. In other words, the new unknown vector is $[\underbrace{b'_1 \alpha_1, \dots, b'_m \alpha_1, d \alpha_1}_{m+1}, \dots, \underbrace{b'_1 \alpha_n, \dots, b'_m \alpha_n, d \alpha_n}_{m+1}]^T$

(b) Main algorithm:

– Perform the following least square estimation:

$$\begin{aligned} \hat{\xi} &= (\Xi^T \Xi)^{-1} \Xi^T Y_{0,N,1} \\ &= [\hat{\xi}_1, \dots, \hat{\xi}_{n(m+1)}]^T \end{aligned} \quad (22)$$

– Construct the following matrix:

$$\hat{\theta}_{\alpha b' d} = \begin{bmatrix} \hat{\xi}_1 & \cdots & \hat{\xi}_{m+1} \\ \vdots & \ddots & \vdots \\ \hat{\xi}_{m(n-1)-1} & \cdots & \hat{\xi}_{n(m+1)} \end{bmatrix} \quad (23)$$

– Perform the singular value decomposition (SVD):

$$\hat{\theta}_{\alpha b' d} = U_{\alpha b' d} \Sigma_{\alpha b' d} V_{\alpha b' d}^T \quad (24)$$

– Let u_1 be the first column of $U_{\alpha b' d}$ and v_1 be the first column of $V_{\alpha b' d}$. Let also s be the sign of the first element of u_1 .

$$s = \text{sgn}(u_1(1)) \quad (25)$$

– The unknown system parameters can now be estimated using the following:

$$\begin{aligned} [\hat{\alpha}_1, \dots, \hat{\alpha}_n] &= s u_1 \\ [\hat{b}'_1, \dots, \hat{b}'_m, \hat{d}] &= s \Sigma_{\alpha b' d}(1) v_1 \end{aligned} \quad (26)$$

where, $\Sigma_{\alpha b' d}(1)$ is the first element of $\Sigma_{\alpha b' d}$.

Lemma 1 [18]: Let $\Theta \in \mathbb{R}_{n \times (m+1)}$ be nonzero and $\Theta = U \Sigma V^T = \sum_{i=1}^{\min(n, m+1)} \mu_i \sigma_i v_i^T$ be its SVD decomposition, where:

$$\begin{aligned} U &= [\mu_1, \dots, \mu_n] \\ V &= [v_1, \dots, v_{m+1}] \\ \sigma_1 &\geq \dots \geq \sigma_{\min(n, m+1)} \end{aligned} \quad (27)$$

Then, for any $\nu \in \mathbb{R}_n$ and $\eta \in \mathbb{R}_{m+1}$

$$\min \|\Theta - \nu\eta^T\| = \min \sum_{i=2}^{n, (m+1)} \sigma_i^2$$

$$(\mu_1, \sigma_1 v_1) = \operatorname{argmin} \|\Theta - \nu\eta^T\| \quad (28)$$

Theorem 1: Consider system (10) which satisfies Assumption 1. If Ξ is full rank, and the output noise $n(k) = 0$, then for any $N > 0$, Algorithm 1 results in:

$$\begin{aligned} [\hat{\alpha}_1, \dots, \hat{\alpha}_n] &= [\alpha_1, \dots, \alpha_n] \\ [\hat{b}'_1, \dots, \hat{b}'_m, \hat{d}] &= [b'_1, \dots, b'_m, d] \end{aligned} \quad (29)$$

Proof:

The algorithm first solves the least square solution (22) that minimizes the prediction error. The objective is to show that parameters given by (26) are equal to the true values of the parameters [18]. To do so, it suffices to show that the following holds:

$$\left\| [\hat{\alpha}_1, \dots, \hat{\alpha}_n] [\hat{b}'_1, \dots, \hat{b}'_m, \hat{d}]^T - \theta_{\alpha b' d} \right\| = 0 \quad (30)$$

Equation (30) can be expanded as follows:

$$\begin{aligned} &\left\| [\hat{\alpha}_1, \dots, \hat{\alpha}_n] [\hat{b}'_1, \dots, \hat{b}'_m, \hat{d}]^T - \theta_{\alpha b' d} \right\| = \\ &\left\| [\hat{\alpha}_1, \dots, \hat{\alpha}_n] [\hat{b}'_1, \dots, \hat{b}'_m, \hat{d}]^T - \hat{\theta}_{\alpha b' d} + \hat{\theta}_{\alpha b' d} - \theta_{\alpha b' d} \right\| \leq \\ &\left\| [\hat{\alpha}_1, \dots, \hat{\alpha}_n] [\hat{b}'_1, \dots, \hat{b}'_m, \hat{d}]^T - \hat{\theta}_{\alpha b' d} \right\| + \left\| \hat{\theta}_{\alpha b' d} - \theta_{\alpha b' d} \right\| \end{aligned} \quad (31)$$

Since, the output noise is zero i.e., $\Delta_{0, N, 1} = 0$, it can be shown that:

$$\left\| \hat{\theta}_{\alpha b' d} - \theta_{\alpha b' d} \right\| = 0 \quad (32)$$

Moreover, the rank of $\theta_{\alpha b' d}$ is one, and hence, Lemma 1 yields $\sum_{i=2}^{\min n, (m+1)} \sigma_i^2 = 0$ and subsequently, using unknown parameters selected from singular value decomposition (26), the following can be deduced:

$$\left\| [\hat{\alpha}_1, \dots, \hat{\alpha}_n] [\hat{b}'_1, \dots, \hat{b}'_m, \hat{d}]^T - \hat{\theta}_{\alpha b' d} \right\| = 0 \quad (33)$$

This completes the proof. \blacksquare

Corollary 1: The results of Theorem 1 can be extended to the case where there is output additive noise. If the output noise is Gaussian white, zero mean and has finite variance, and if the input signal is bounded and the matrix Ψ is full rank, then following results as $N \rightarrow \infty$.

$$\begin{aligned} [\hat{\alpha}_1, \dots, \hat{\alpha}_n] &\rightarrow [\alpha_1, \dots, \alpha_n] \\ [\hat{b}'_1, \dots, \hat{b}'_m, \hat{d}] &\rightarrow [b'_1, \dots, b'_m, d] \end{aligned} \quad (34)$$

This is because if the regressor is persistently exciting, then as $N \rightarrow \infty$, the least square estimate yields:

$$\left\| \hat{\theta}_{\alpha b' d} - \theta_{\alpha b' d} \right\| \rightarrow 0 \quad (35)$$

Moreover, since the matrix $\theta_{\alpha b' d}$ is of rank 1. From Lemma 1, it can be deduced that $\sum_{i=2}^{\min n, (m+1)} \sigma_i^2 = 0$ which means:

$$\left\| [\hat{\alpha}_1, \dots, \hat{\alpha}_n] [\hat{b}'_1, \dots, \hat{b}'_m, \hat{d}]^T - \hat{\theta}_{\alpha b' d} \right\| \rightarrow 0 \quad (36)$$

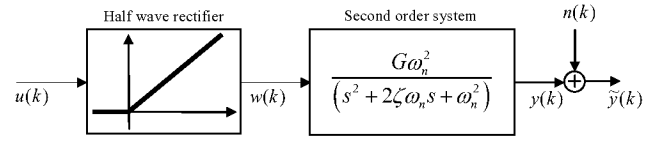


Fig. 2. Hammerstein model of reflex stiffness.

IV. SIMULATION RESULTS

A simulation scenario was constructed using the model of the reflex system of ankle joint stiffness described in [20]. This system was modeled as a Hammerstein system consisting of a half wave rectifier (static nonlinearity) followed by a second-order low-pass filter as illustrated in Fig. 2. The input to this system is angular velocity of the joint and the output is the reflex torque. For simulation purposes, the nonlinear component(i.e. the half wave rectifier) was modeled using the eighth order power polynomial:

$$\begin{aligned} f(u(k)) &= -0.3u^8(k) + 0.7u^6(k) - 0.58u^4(k) \\ &\quad + 0.27u^2(k) + 0.09u(k) + 0.01 \end{aligned} \quad (37)$$

The linear component of the Hammerstein system was described using a second order low pass filter with the transfer function:

$$\frac{Y(s)}{W(s)} = \frac{G\omega_n^2}{s^2 + 2s\zeta\omega_n + \omega_n^2} \quad (38)$$

The parameters of the linear system were set to values that are found experimentally [20]:

$$\begin{cases} G &= -25 \\ \omega_n &= 26 \\ \zeta &= 0.98 \end{cases} \quad (39)$$

The input was as a white Gaussian zero mean signal for identification purpose. The simulation is done in MATLAB Simulink. Thirty seconds of data was sampled with sampling rate of 1 KHz. Independent, white Gaussian noise was added to the output to resemble the additive measured noise to give a signal to noise ratio of 30 dB.

The input matrix $U(k)$ was constructed according to (6) and the SMI toolbox used to estimate state space matrices \hat{A}_T and \hat{C}_T . The past input was used as instrumental variable in the MOESP algorithm (PI-MOESP) to eliminate the effects of noise. The algorithm presented in this paper was then used to identify the coefficient of nonlinearity and the state space model matrices \hat{B}_T and \hat{D}_T . The matrices \hat{A}_T , \hat{B}_T , \hat{C}_T , \hat{D}_T and $\hat{\alpha}$ are estimated as follows:

$$\begin{aligned} \hat{A}_T &= \begin{bmatrix} 0.9965 & -0.1151 \\ 0.0043 & 0.9534 \end{bmatrix} \\ \hat{B}_T &= \begin{bmatrix} -0.8773 \\ 0.3644 \end{bmatrix} \\ \hat{C}_T &= [-0.1855, -0.3738] \\ \hat{D}_T &= [0.0135] \\ \hat{\alpha} &= [0.3023, -0.6992, 0.5835, -0.2669, -0.0894, -0.0067] \end{aligned} \quad (40)$$

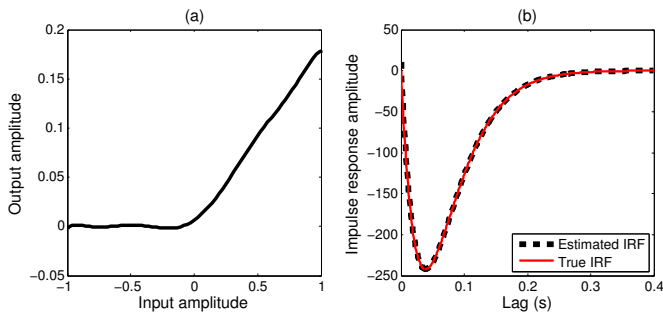


Fig. 3. Identified Hammerstein system: (a) Estimated nonlinearity; (b) Impulse response of the linear component.

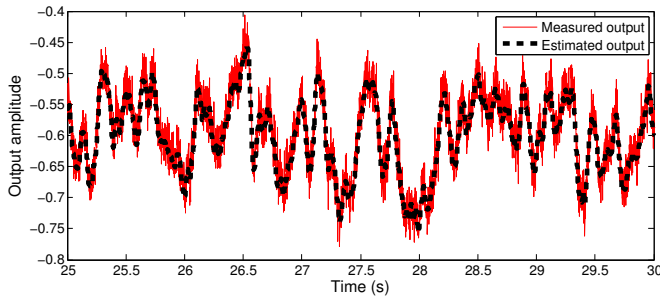


Fig. 4. Measured output along with the estimated output.

It is evident that there is good agreement between coefficients estimated for the nonlinearity ($\hat{\alpha}$) and the one that are originally used in the simulation (40) and (37). This can also be seen in Fig. 3(a) which shows that the estimated nonlinearity closely resembles a half wave rectifier. To test the efficiency of the algorithm on the identification of the linear component of the Hammerstein system, the impulse response of the estimated system was computed and is superimposed on the impulse response of the simulated system in Fig. 3(b). It is evident again that there is an excellent agreement between two IRFs.

Variance accounted for (VAF) was then calculated based between the predicted output and the noise-free, simulated output using the equation:

$$\text{VAF} = \left(1 - \frac{\text{var}(\hat{y} - y)}{\text{var}(y)} \right) \quad (41)$$

where, $\text{var}(\cdot)$ is the variance estimator. The VAF was 99.99% for this identification demonstrating that the model was estimated very accurately despite the presence of noise. Fig. 4 shows a five second segment of the simulated and predicted torques.

V. CONCLUSION

An algorithm for identification of Hammerstein is presented. The algorithm consists of three main steps. First, MOESP algorithm is used to determine the system order and estimate two of state space model matrices. Second, a least square problem is solved to minimize the prediction

error. Finally, global search optimization for estimation of parameters is modeled as a singular value decomposition problem to estimate optimal value for the rest of parameters. Performance of the model was evaluated by simulating a model of ankle joint reflex stiffness, a well known Hammerstein systems. The results demonstrated that the algorithm estimated the model parameters very accurately despite the presence of additive, output noise.

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