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Robustness Analysis of Slow Learning in Iterative Learning Control Systems

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Abstract— **This paper examines robust stability and robust transient growth in Iterative Learning Control (ILC). It is well known that small perturbations in system dynamics can result in very large transient growth of some ILC systems. Even larger perturbations can result in instability. One ad hoc technique commonly employed to improve robustness is to slow the learning rate by reducing the learning filter gain or lowpass filtering the error signal. Here, pseudospectra analysis is used to analyze the robustness of ILC algorithms with slow learning. It is found that robustness bounds can be increased and transient growth decreased with decreasing learning gain. This result provides a new theoretical foundation for tuning approaches for improving robustness.**

I. Introduction

Iterative learning control (ILC) [1-3] is used to improve the performance of systems that repeat the same operation many times. ILC uses the tracking errors from previous iterations of the repeated motion to generate a feedforward control signal for subsequent iterations. Convergence of the learning process results in a feedforward control signal that is customized for the repeated motion, yielding very low or zero tracking error.

ILC is a performance-improving control algorithm, rather than a stabilizing algorithm, and thus the emphasis of much of the ILC literature focuses on behavior at convergence. Of course, convergence of the algorithm is typically demonstrated, but comparatively little attention is given to the nature of the convergence. The transient behavior of the learning process, however, is critically important in many practical applications. For example, in robotics and manufacturing applications, slow convergence leads to delays in process startup and possibly costly material waste. Perhaps of greater concern to the ILC designer is the problem of large transient growth [4], whereby the error may grow rapidly and with little warning, potentially damaging hardware.

The problem of large transient growth has been studied extensively by Longman and colleagues [4-8]. These and other works [9-11] use norm-based tools for analysis and design. The norm-based tools are limited in that they are unable to distinguish between small and large transient growth. Thus, designs based on these tools always result in monotonic convergence (no transient growth). Although this is a desired property, it is an artificial constraint for many applications, and thus may result in sub-optimal performance.

Recently, the authors proposed the use of pseudospectra analysis as an alternative to the norm-based approaches [12]. The pseudospectra is used to estimate transient growth, and thus may provide a suitable framework for ILC design with "softer" transient constraints.

In practice, it is common to reduce learning rate to improve the robustness and transient growth. However, there is little rigorous theoretical work to support this approach. In this paper we use the pseudospectra tools to analyze the robust stability and transient growth behavior of a class of ILC algorithms with slow learning rate. Notably, we find that this approach does have theoretical foundations and further that the recursive filtering of the control signal may play a central role.

The remainder of this paper is organized as follows. In Section II we set up the problem of transient growth in ILC and introduce the pseudospectra analysis tools. Section III develops robust analysis for stability and transient growth of slow-learning ILC algorithms. Finally, concluding remarks are given in Section V.

II. Background

Consider the general description for the finite-time response of a linear time-varying (LTV), multi-input, multi-output (MIMO), discrete-time (DT) servo system,

$$\mathbf{e} = \mathbf{P}\mathbf{u} + \mathbf{e}_0, \tag{1}$$

where,

$$\mathbf{e} = \begin{bmatrix} e(m) & e(m+1) & \cdots & e(m+N-1) \end{bmatrix}^T,$$
$$\mathbf{u} = \begin{bmatrix} u(0) & u(1) & \cdots & u(N-1) \end{bmatrix}^T,$$
$$\mathbf{e}_0 = \begin{bmatrix} e_0(m) & e_0(m+1) & \cdots & e_0(m+N-1) \end{bmatrix}^T.$$

are the vector descriptions of the tracking error $e(k)$ at time k, the ILC input $u(k)$, and the nominal tracking error $e(k)$, respectively, and m is the system delay. The matrix $\mathbf{P}$ is the convolution matrix relating the ILC input to the error, and is given by,

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$$\mathbf{P} = \begin{bmatrix} p_{m,0} & 0 & 0 & \cdots & 0 \\ p_{m+1,0} & p_{m+1,1} & 0 & \ddots & \vdots \\ p_{m+2,0} & p_{m+2,1} & p_{m+2,2} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ p_{m+N-1,0} & p_{m+N-1,1} & \cdots & p_{m+N-1,N-2} & p_{m+N-1,N-1} \end{bmatrix}, \quad (2)$$

The model (1) is sufficiently general to represent a variety of control system configurations [3].

One common configuration is the so-called "plug-in ILC" whereby the ILC input is added to the control signal of an existing feedback controller, as illustrated in Figure 1. In this case, e_0 is the tracking error achieved by the feedback controller, which may include initial condition response and disturbances. The elements of the convolution matrix **P** are $p_{i,l} = h_{i-l}$, where $h_0, h_1, h_2, \ldots$ is the impulse response of the transfer function between u and e, $-G(z)\left[I + G(z)C(z)\right]^{-1}$.

In the ILC setting, we consider repetitions of the tracking process,

$$\mathbf{e}_j = \mathbf{P}\mathbf{u}_j + \mathbf{e}_0, \quad (3)$$

where j is the iteration index. It is assumed that $\mathbf{e}_0$ is iteration-invariant (and thus the reference, disturbances, and initial conditions are iteration-invariant). A commonly used ILC algorithm for this process is the first-order algorithm,

$$\mathbf{u}_{j+1} = \mathbf{L}_\mathbf{u}\mathbf{u}_j + \mathbf{L}_\mathbf{e}\mathbf{e}_j, \quad (4)$$

where $\mathbf{L}_\mathbf{u}$ and $\mathbf{L}_\mathbf{e}$ are NxN matrices.

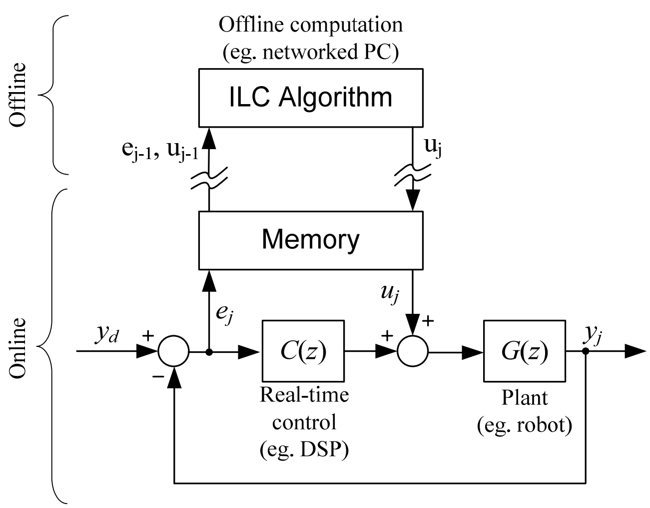

Figure 1. Plug-in ILC configuration.

A. Stability and Transient Analysis

Combining (3), (4), closed-loop dynamics in the iteration-domain are given by,

$$\mathbf{u}_{j+1} = \mathbf{T}\mathbf{u}_j + \mathbf{f}_0, \quad (5)$$

where $\mathbf{T} = \mathbf{L}_\mathbf{u} - \mathbf{L}_\mathbf{e}\mathbf{P}$ and $\mathbf{f}_0 = \mathbf{L}_\mathbf{e}\mathbf{e}_0$. It follows that the ILC system is stable if $\rho(\mathbf{T}) < 1$, where $\rho(\mathbf{T})$ is the spectral radius, or largest absolute value of the eigenvalues, of **T**. If the system is stable, define $\mathbf{u}_\infty = \lim_{j\to\infty} \mathbf{u}_j$, and rewrite (5) as,

$$\mathbf{u}_\infty - \mathbf{u}_{j+1} = \mathbf{T}\left(\mathbf{u}_\infty - \mathbf{u}_j\right),$$

or equivalently,

$$\mathbf{u}_\infty - \mathbf{u}_j = \mathbf{T}^j\left(\mathbf{u}_\infty - \mathbf{u}_0\right), \quad (6)$$

Thus, $\left\|\mathbf{u}_\infty - \mathbf{u}_j\right\| \le \left\|\mathbf{T}^j\right\| \cdot \left\|\mathbf{u}_\infty - \mathbf{u}_0\right\|$, where the norm is the standard 2-norm. Therefore, the transient response of the learning process is bounded by the sequence,

$$\left\|\mathbf{T}\right\|, \left\|\mathbf{T}^2\right\|, \left\|\mathbf{T}^3\right\|, \ldots, \left\|\mathbf{T}^j\right\|, \ldots. \quad (7)$$

If **T** is known, one may numerically compute the sequence (7), at least for some finite number of iterations. However, such an approach does not provide meaningful design insight. Furthermore, it is numerically expensive when N is large and many iterations need to be calculated to determine the behavior of (7). One approach to analyzing the transient response is the pseudospectra, given by the following definition.

Definition 1 [13]**:** The ε-pseudospectra of a matrix **T** is the set $\sigma_\varepsilon(\mathbf{T})$ in the complex plane consisting of all points $z \in \mathbb{C}$ such that z is an eigenvalue of **T**+**E** for some $\mathbf{E} \in \mathbb{C}^{N\times N}$ with $\|\mathbf{E}\| < \varepsilon$. Equivalently, the pseudospectra is the set where the resolvent matrix $(z\mathbf{I} - \mathbf{T})^{-1}$ is large:

$$\sigma_\varepsilon(\mathbf{T}) = \{z \in \mathbb{C} : \left\|(z\mathbf{I} - \mathbf{T})^{-1}\right\| > \varepsilon^{-1}\}.$$

The pseudospectra can be used to generate a number of bounds on the transient response [13]. One such bound is given by,

$$\left\|\mathbf{T}^k\right\| \le \left[\rho_\varepsilon(\mathbf{T})\right]^{k+1} / \varepsilon, \quad (8)$$

where,

$$\rho_\varepsilon(\mathbf{T}) = \left\{\max|z| : z \in \sigma_\varepsilon(\mathbf{T})\right\}, \quad (9)$$

is referred to as the ε-pseudospectra radius. Numerical tools for efficiently calculating the pseudospectra for large matrices are developed in [13] and implemented in [14].

III. ROBUSTNESS ANALYSIS OF SLOW LEARNING SYSTEMS

Methods of slowing the learning rate to improve robustness are commonly employed in ILC. Here, we consider a general a class of these algorithms, given by,

$$\mathbf{u}_{j+1} = \mathbf{L}_\mathbf{u}\mathbf{u}_j + \mathbf{\Phi}(\phi)\mathbf{L}_\mathbf{e}\mathbf{e}_j, \quad (10)$$

where $\mathbf{\Phi}(\phi)$ is an NxN matrix and ϕ is a scalar. Several examples of such algorithms reported in the literature are:

- Uniform scaling: $\mathbf{\Phi}(\phi) = \phi$, $0 \le \phi \le 1$
- Lowpass filtering [8]: $\mathbf{\Phi}(\phi)$ is a lowpass filter with bandwidth ϕ
- Exponential weighting [15]: $\mathbf{\Phi}(\phi) = diag\left\{\phi, \phi^2, \ldots, \phi^N\right\}$, $0 \le \phi \le 1$

We make several assumptions regarding the slowing filter.

1. $\left\|\mathbf{\Phi}(\phi_1)\right\| < \left\|\mathbf{\Phi}(\phi_2)\right\|$ if $\phi_1 < \phi_2$.
2. $\lim_{\phi \to 0} \left\|\mathbf{\Phi}(\phi)\right\| = 0$.

A. Robust Stability

Substituting the slow learning ILC algorithm, (10), for the first-order algorithm, (4), stability and transient analysis follows identically to the analysis presented in Section II.A. The transition matrix for the slow learning algorithm is given by,

$$\mathbf{T}_s = \mathbf{L_u} - \mathbf{\Phi}(\phi)\mathbf{L_e P}. \tag{11}$$

It follows from Assumption 2 above, that the dynamics of the slow-learning system approach the dynamics of the $\mathbf{L_u}$ filter as the learning rate slows, or,

$$\lim_{\phi \to 0} \mathbf{T}_s = \mathbf{L_u}. \tag{12}$$

Thus, in the limit, the dynamics of the ILC system are independent of the plant, $\mathbf{P}$. This fact provides the basis for the following theorem, in which ϕ can be used to provide robustness to arbitrarily large plant perturbations.

Theorem 1: Let $\mathbf{L_e}$ and $\mathbf{P}$ be any bounded matrices. If $\mathbf{L_u}$ is strictly stable, $\rho(\mathbf{L_u}) < 1$, then there exists a $\bar{\phi}$ such that the slow-learning ILC system is stable for all $0 < \phi \le \bar{\phi}$.

Proof: From (12), we have that $\mathbf{T_s}$ converges to $\mathbf{L_u}$ in norm. This norm convergence and eigenvalue perturbation theory [17] gives that the eigenvalues of $\mathbf{T_s}$ converge to the eigenvalues of $\mathbf{L_u}$ as $\phi \to 0$. This proves the result.

Remark 1: Note that in the eigenvalues of $\mathbf{L_u}$ must be strictly inside of the unit disk to achieve the robustness properties described in Theorem 1. Conversely, it is well known that $\mathbf{L_u}$=$\mathbf{I}$ is necessary for convergence to zero error [3]. Thus, the robustness properties of Theorem 1 do not apply to zero-error convergence algorithms.

The parameter $\bar{\phi}$ can be estimated using pseudospectra analysis as shown in the following corollary.

Corollary 1: Let $\bar{\varepsilon}$ be the (unique) ε-pseudospectra radius such that $\rho_{\bar{\varepsilon}}(\mathbf{L_u}) = 1$ and select $\bar{\phi}$ such that $\left\|\mathbf{\Phi}(\bar{\phi})\right\| < \bar{\varepsilon} / \left\|\mathbf{L_e P}\right\|$. Then, the slow-learning ILC system is stable for all $0 < \phi \le \bar{\phi}$.

Proof: Let $\sigma(\mathbf{T_s})$ be the spectra, or set of eigenvalues, of $\mathbf{T_s}$. Define $\hat{\varepsilon}(\phi) = \left\|\mathbf{\Phi}(\phi)\right\| \cdot \left\|\mathbf{L_e P}\right\|$. Then, from Definition 1,

$$\sigma(\mathbf{T_s}) = \sigma\left(\mathbf{L_u} - \mathbf{\Phi}(\phi)\mathbf{L_e P}\right) \subset \sigma_{\hat{\varepsilon}(\phi)}(\mathbf{L_u}),$$

and likewise,

$$\rho(\mathbf{T_s}) = \rho\left(\mathbf{L_u} - \mathbf{\Phi}(\phi)\mathbf{L_e P}\right) \le \rho_{\hat{\varepsilon}(\phi)}(\mathbf{L_u}).$$

Note that $\hat{\varepsilon}(\bar{\phi}) < \bar{\varepsilon}$, and because $\rho_{\bar{\varepsilon}}(\mathbf{L_u}) = 1$, it follows that $\rho_{\hat{\varepsilon}(\bar{\phi})}(\mathbf{L_u}) < 1$. Furthermore,

$$\rho(\mathbf{T_s}) \le \rho_{\hat{\varepsilon}(\phi)}(\mathbf{L_u}) \le \rho_{\hat{\varepsilon}(\bar{\phi})}(\mathbf{L_u}) < 1, \text{ for } 0 < \phi \le \bar{\phi}. \qquad \blacksquare$$

B. Robust Transients

As discussed in Section II, the transient growth in an ILC system is related to the pseudospectra of the transition matrix. The following theorem provides a relationship between the $\mathbf{L_u}$ pseudospectra and the $\mathbf{T_s}$ pseudospectra. As the learning gain $\left\|\mathbf{\Phi}(\phi)\right\|$ approaches zero, the pseudospectra converge.

Theorem 2: The ε-pseudospectra of $\mathbf{T_s}$ is bounded by $\varepsilon + \hat{\varepsilon}$-pseudospectra of $\mathbf{L_u}$,

$$\sigma_{\varepsilon}(\mathbf{T_s}) \subset \sigma_{\varepsilon + \hat{\varepsilon}}(\mathbf{L_u}),$$

where $\hat{\varepsilon} = \left\|\mathbf{\Phi}(\phi)\right\| \cdot \left\|\mathbf{L_e P}\right\|$.

Proof: By definition, the ε-pseudospectra of $\mathbf{T_s}$ is the set $\sigma(\mathbf{T_s} + \mathbf{E})$, $\left\|\mathbf{E}\right\| \le \varepsilon$. Then,

$$\begin{aligned}
\sigma(\mathbf{T_s} + \mathbf{E}) &= \sigma\left(\mathbf{L_u} - \mathbf{\Phi}(\phi)\mathbf{L_e P} + \mathbf{E}\right), \left\|\mathbf{E}\right\| < \varepsilon \\
&\subset \sigma\left(\mathbf{L_u} + \hat{\mathbf{E}}\right), \left\|\hat{\mathbf{E}}\right\| < \varepsilon + \underbrace{\left\|\mathbf{\Phi}(\phi)\right\| \cdot \left\|\mathbf{L_e P}\right\|}_{\hat{\varepsilon}} \\
&= \sigma_{\varepsilon + \hat{\varepsilon}}(\mathbf{L_u}),
\end{aligned}$$

which completes the proof.

Since the pseudospectra is related to transient growth, the above result gives that the transient response of $\mathbf{T_s}$ will be close to the transient response of $\mathbf{L_u}$ if $\left\|\mathbf{\Phi}(\phi)\right\|$ is small enough.

C. Implications for L_u Design

As evident from (12), the ILC system dynamics approach the $\mathbf{L_u}$ dynamics for slow learning. Corollary 1 indicates that the design of $\mathbf{L_u}$ plays an important role in the tradeoff between the slow learning gain and the system robustness. Specifically, it is desirable that ε-pseudospectra sets of $\mathbf{L_u}$ lie inside of the unit circle for large ε. Thus, particular $\mathbf{L_u}$ designs may have robustness advantages over other designs.

Figure 2 shows the pseudospectra for four different lowpass filter designs. All four filters have a digital bandwidth of 0.05 (1/samples) and are described in Table 1. The first two filters are causal Butterworth filters of differing order. The last two filters are noncausal, zero-phase implementations of the first two, using a forward-backward filtering method [16].

The results of the pseudospectra calculations for the four lowpass filter designs show that although the filter bandwidths are the same, the pseudospectra are quite different. Notably, the causal filters, Filter 1 and 2, appear significantly more robust than the noncausal filters, Filter 3 and 4. Interestingly, for the causal filters, a higher order is more robust, but the opposite is true for the noncausal filters.

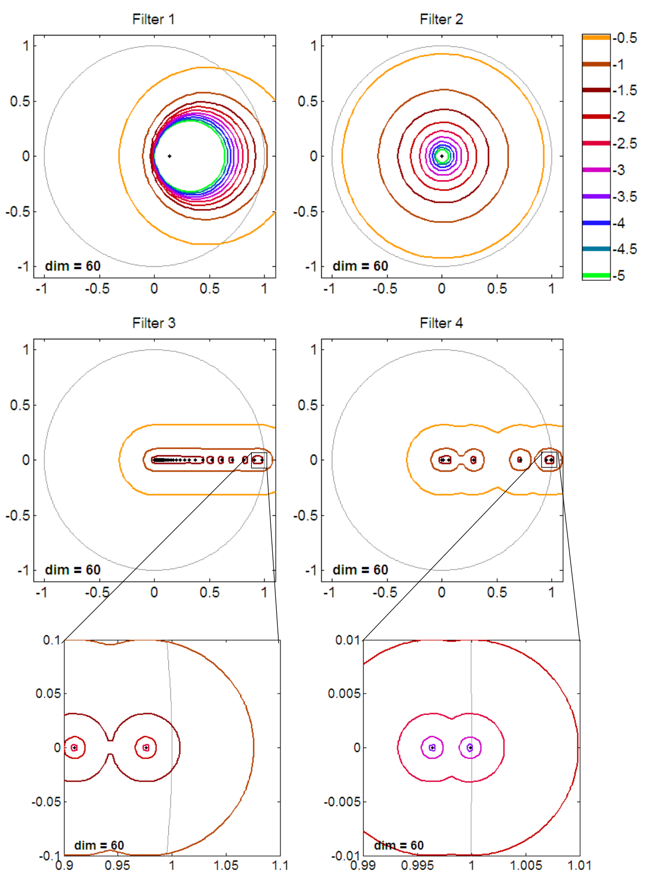

Figure 2. ε-Pseudospectra for the four filters listed in Table 1. The color bar is on a $\log_{10}$ scale so that the values of ε are $10^{-5},\ldots,10^{-0.5}$ from inside to outside.

Table 1. Four designs for lowpass L_u filters.

Filter 1: 1st Order Butterworth	$L_{u1}(z)=\dfrac{0.1367(z+1)}{z-0.7265}$	$\bar{\varepsilon}=10^{-1.5}$
Filter 2: 8th Order Butterworth	$L_{u2}(z)=\dfrac{1.7626\text{e-}007(z+1)}{\prod_{i=1}^{4}(z^2+a_iz+b_i)}$ $a_1=-1.46, b_1=0.5348,$ $a_2=-1.513, b_2=0.5912,$ $a_3=-1.623, b_3=0.7069,$ $a_4=-1.794, b_4=0.8863$	$\bar{\varepsilon}=10^{-0.5}$
Filter 3: Zero-phase 1st Order Butterworth	$L_{u3}(z)=L_{u1}(z)L_{u1}(z^{-1})$	$\bar{\varepsilon}=10^{-2}$
Filter 4: Zero-phase 8th Order Butterworth	$L_{u4}(z)=L_{u2}(z)L_{u2}(z^{-1})$	$\bar{\varepsilon}=10^{-4}$

IV. Conclusions

This work considered robust stability and robust transient growth in slow learning ILC. Although in practice it is common to slow the learning rate to increase stability and yield less transient growth, there has been little theory to support this approach. We applied pseudospectra to this problem and gave a rigorous bound on the learning gain to ensure stability in the slow learning system. Furthermore, we found that decreasing the learning gain will cause the transient growth in the slow learning system to approach the growth in the L_u system. Moreover, pseudospectra can be used in the analysis and design of the L_u system to control transient growth.

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