2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Decomposition of Linear Port-Hamiltonian Systems

K. Höffner and M. Guay

***Abstract*— It is well known that the power conserving interconnection of finite dimensional port-Hamiltonian systems is also a port-Hamiltonian system. Given a linear port-Hamiltonian system, this paper proposes conditions under which the control system can be expressed as a composition of two linear port-Hamiltonian systems. This decomposition of linear port-Hamiltonian systems is based on the inherent interconnection structure and can be applied without knowledge of the physical interconnection structure.**

I. Introduction

One of the important properties of port-Hamiltonian systems is that, by interconnection of simple system, complex networks of port-Hamiltonian systems can be constructed with properties that can be inferred by the properties of the elementary systems. The reversal of this construction is subject of this paper. That is, we consider the decomposition of a complex systems into an interconnection of port-Hamiltonian systems.

One motivation for this research is to establish a framework such that the matching problem, known from the interconnection damping assignment passivity-based control methodology, can be effectively divided into smaller subproblems. The concept of decomposition of port-Hamiltonian systems has received limited attention in the literature, notable exceptions are [7] and [2]. In the latter reference, it is noted that an explicit algorithm for the minimal representation of a complex composed Dirac structure is profitable in the context of modelling interconnected systems. The decomposition results presented in [7] are based on the Lagrangian formulation, but are also applicable to control-affine systems. One limitation is that it requires the knowledge of two distinct Lagrangians *a priori*.

We denote by Σ the linear control system $\dot{x} = Ax + Bu$ with $x \in \mathcal{X} \subset \mathbb{R}^n$, $u \in \mathcal{U} \subset \mathbb{R}^m$ and $\mathcal{X}$, $\mathcal{U}$ are open in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. We are interested in linear port-Hamiltonian system of the form $\dot{x} = JQx + Bu$, where $J = -J^\top$ is called the interconnection structure and $Q = Q^\top$ is the constant Hessian matrix of the Hamiltonian function $H = 1/2x^\top Qx$. We assume throughout this paper that the matrix B has full column rank $m < n$. Denote by $B_\perp$ a full rank $n \times (n-m)$ matrix such that $\mathrm{Im}\, B_\perp$ is a complement of $\mathrm{Im}\, B$, *i.e.,* $\mathrm{Im}\, B \oplus \mathrm{Im}\, B_\perp = \mathbb{R}^n$ and we define $B^\perp = B_\perp^\top$ as a full row rank $(n-m) \times n$ matrix that annihilates B, *i.e.,* $B^\perp B = 0$. The main thrust of this research comes from the concept of linear abstractions and C-related control systems (see [8]).

Kai Höffner and Martin Guay are with the Department of Chemical Engineering, Queen's University, Kingston, ON, Canada, K7L 3N6. martin.guay@chee.queensu.ca

The remainder of this paper is structured as follows; In Section II, we review the concept of Dirac structures and abstractions. In Section III, we show that abstractions of port-Hamiltonian systems can be realized by interconnection of the abstraction with a "virtual controller"-port-Hamiltonian system, where we study two types of interconnection. In the final section we discuss implications and extensions of the proposed framework.

II. Dirac structures and linear abstractions

It is well known that power conserving interconnection of port-Hamiltonian system are again port-Hamiltonian systems, see for example [5]. Hence, under this general type of interconnection, the port-Hamiltonian structure of the control system is preserved. Furthermore, the interconnection enjoys several desirable properties. Its Hamiltonian function is the sum of the individual Hamiltonians and it is passive if the individual systems are passive. It would be interesting to understand if this process of interconnection can be reversed. More precisely, we would like to know if, given a port-Hamiltonian system, one can we write this system as an interconnection of "smaller" port-Hamiltonian systems. We first define what a smaller port-Hamiltonian system is with respect to a given one. For this we consider the notation of abstraction of a linear control system. Then, we develop a framework that allows us to determine when an abstraction of a port-Hamiltonian system is also a port-Hamiltonian system. This allows us to determine when the interconnection of an abstraction with a "virtual controller" yields the original system. The objective of the following section is to develop conditions under which a port-Hamiltonian system can be written as an interconnection of its abstraction with an additional "virtual controller" port-Hamiltonian system. It is conceptually preferable, for our purpose, to work with Dirac structures to represent linear port-Hamiltonian systems.

A. Dirac Structures

Dirac structures are generalizations of symplectic and Poisson structures, which are models for the interconnection structure of Hamiltonian systems [6]. They are algebraic structures that can be extended naturally to differentiable structures on manifolds (see [4] for further details). We are interested in Dirac structures that are induced by port-Hamiltonian systems. For this class of systems, the Dirac structure is interconnected via a set of internal ports to an energy storage element and a resistive element, that represent the storage or Hamiltonian function and the damping structure, respectively.

978-1-4577-0079-8/11/$26.00 ©2011 AACC

Definition 2.1: Let $\mathcal{F}$ be a finite dimensional real vector space and $\mathcal{F}^*$ its dual. A *Dirac structure* is a subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ such that $\dim \mathcal{D} = \dim \mathcal{F}$ and $\langle e, f \rangle = 0 \ \forall (f, e) \in \mathcal{D} \times \mathcal{D}^*$, where $\langle e, f \rangle$ denotes the duality product, that is, the linear functional $e \in \mathcal{F}^*$ acting on $f \in \mathcal{F}$.
The variables $f = [f_1, \ldots, f_n]^\top$ are called *flow variables* of $\mathcal{D}$ and their duals $e = [e^1, \ldots, e^n]^\top$ are called *effort variables*. For port-Hamiltonian systems, the state space $\mathcal{X}$ is the space of energy-variables denoted by x. The space of flow variables for the Dirac structure $\mathcal{D}$ is then the product $\mathcal{X} \times \mathcal{F}$ with $f_x \in \mathcal{X}$. By duality the space of effort variables is the product $\mathcal{X}^* \times \mathcal{F}^*$, with $e_x \in \mathcal{X}^*$. The flow variables of the energy storage element are given by $\dot{x}$ and the effort variables are given by $\frac{\partial H}{\partial x}$ such that the energy storage element satisfies the total energy balance $\dot{H} = \langle \frac{\partial H}{\partial x}, \dot{x} \rangle = 0$.
The Dirac structure is interconnected via the internal ports to the energy storage element through the interconnection:

$$f_x = -\dot{x} \quad \text{and} \quad e_x = \frac{\partial H}{\partial x}.$$

This yields the dynamical system

$$\left(-\dot{x}, \frac{\partial H}{\partial x}, f, e\right) \in \mathcal{D}.$$

Dirac structures admit different representations, see [1], [3], [10], [2] for further details.

We require the following representation of a Dirac structure. Let $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ be a Dirac structure, then

$$\mathcal{D} = \{(e, f) \in \mathcal{F} \times \mathcal{F}^* \mid Ff + Ee = 0\}$$

for $n \times n$ matrices F and E satisfying

$$\begin{aligned} EF^\top + FE^\top &= 0 \\ \operatorname{rank}\,[F|E] &= n. \end{aligned}$$

The pair (E, F) is called the *matrix kernel representation* of $\mathcal{D}$. If the image of F and E has dimension larger than the dimension of $\mathcal{F}$ then (E, F) is called *relaxed matrix kernel representation*.

Example 2.2: Port-Hamiltonian systems define Dirac structures. In matrix kernel representation, the Dirac structure associated to a linear port-Hamiltonian system is

$$\begin{bmatrix} I_n \\ 0 \end{bmatrix} f_x + \begin{bmatrix} J \\ B^\top \end{bmatrix} e_x + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -I_m \end{bmatrix} y = 0.$$

Hence, we have the matrix kernel representation (E, F) with

$$F = \begin{bmatrix} I_n & B \\ 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} J & 0 \\ B^\top & -I_m \end{bmatrix}.$$

We use the following notation for the matrices involved in the matrix kernel representation of port-Hamiltonian systems:

$$F_x = \begin{bmatrix} I_n \\ 0 \end{bmatrix}, F_p = \begin{bmatrix} B \\ 0 \end{bmatrix}, E_x = \begin{bmatrix} J \\ B^\top \end{bmatrix}, E_p = \begin{bmatrix} 0 \\ -I_m \end{bmatrix}.$$

1) Interconnection of port-Hamiltonian systems and Dirac structures: The following discussion can be found in [2]. We consider two types of composition of Dirac structures, composition and gyrative composition. We study the composition of two Dirac structures with partially shared variables. Consider the Dirac structure $\mathcal{D}_A$ on a product space $\mathcal{F}_1 \times \mathcal{F}_2$ of two linear spaces $\mathcal{F}_1$ and $\mathcal{F}_2$, and another Dirac structure $\mathcal{D}_B$ on $\mathcal{F}_2 \times \mathcal{F}_3$, with $\mathcal{F}_3$ being an additional linear space. The space $\mathcal{F}_2$ is the space of *shared flow variables*, and $\mathcal{F}_2^*$ the space of *shared effort variables*. We make the following definitions:

Definition 2.3: Let $\mathcal{D}_A$ and $\mathcal{D}_B$ be two Dirac structures on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$ and $\mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$, respectively. The *composition* (or canonical interconnection) of $\mathcal{D}_A$ and $\mathcal{D}_B$ is defined as

$$\begin{aligned} \mathcal{D}_A || \mathcal{D}_B = \{ &(f_1, e_1, f_3, e_3) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^* \mid \\ &\exists (f_2, e_2) \in \mathcal{F}_2 \times \mathcal{F}_2^* \text{ s.t.} \\ &(f_1, e_1, f_2, e_2) \in \mathcal{D}_A \text{ and } (-f_2, e_2, f_3, e_3) \in \mathcal{D}_B \}. \end{aligned}$$

Definition 2.4: Let $\mathcal{D}_A$ and $\mathcal{D}_B$ be two Dirac structures on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$ and $\mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$, respectively. The *gyrative composition* of $\mathcal{D}_A$ and $\mathcal{D}_B$ is defined as

$$\begin{aligned} \mathcal{D}_A \wedge \mathcal{D}_B = \{ &(f_1, e_1, f_3, e_3) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^* \mid \\ &\exists (f_2, e_2) \in \mathcal{F}_2 \times \mathcal{F}_2^* \text{ s.t.} \\ &(f_1, e_1, f_2, e_2) \in \mathcal{D}_A \text{ and } (e_2, -f_2, f_3, e_3) \in \mathcal{D}_B \}. \end{aligned}$$

Remark 2.5: Note that $\mathcal{D}_A \wedge \mathcal{D}_B$ can also be constructed via the composition with the symplectic Dirac structure

$$\mathcal{D}_I = \{(f_{IA}, e_{IA}, f_{IB}, e_{IB}) \mid f_{IA} = -e_{IB}, f_{IB} = e_{IA}\},$$

such that $\mathcal{D}_A || \mathcal{D}_I || \mathcal{D}_B = \mathcal{D}_A \wedge \mathcal{D}_B$.
It can be shown that, if $\mathcal{D}_A$ and $\mathcal{D}_B$ are two Dirac structures with ports as defined above, then $\mathcal{D}_A || \mathcal{D}_B$ and $\mathcal{D}_A \wedge \mathcal{D}_B$ are Dirac structures with respect to the bilinear on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$. Furthermore, if Σ_1 and Σ_2 are two port-Hamiltonian systems and $\mathcal{D}_1$ and $\mathcal{D}_2$ are their Dirac structures, constructed as in Example 2.2, then $\mathcal{D}_1 \wedge \mathcal{D}_2$ is again a port-Hamiltonian system and is the feedback interconnection of Σ_1 and Σ_2.

We have the following proposition, due to [2], that proposes a matrix kernel representation of the composition of two Dirac structures in terms of the matrix kernel representation of the individual Dirac structures.

Proposition 2.6: Let $\mathcal{F}_i, i = 1, 2, 3$ be a finite-dimensional linear space with $\dim \mathcal{F}_i = n_i$. Consider the Dirac structures

$$\begin{aligned} \mathcal{D}_A &\subset \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^*, \quad n_A = \dim \mathcal{F}_1 \times \mathcal{F}_2 = n_1 + n_2 \\ \mathcal{D}_B &\subset \mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^*, \quad n_B = \dim \mathcal{F}_2 \times \mathcal{F}_3 = n_2 + n_3 \end{aligned}$$

given by the relaxed matrix kernel representation

$$\begin{aligned} (F_A, E_A) &= ([F_1 | F_{2A}], [E_1 | E_{2A}]), \\ (F_B, E_B) &= ([F_{2B} | F_3], [E_{2B} | E_3]) \end{aligned}$$

$n'_A \times n_A$ matrices and $n'_B \times n_B$ matrices, respectively with $n'_A \geq n_A$ and $n'_B \geq n_B$. Define the $(n'_A + n'_B) \times 2n_2$ matrix

$$M = \begin{bmatrix} F_{2A} & E_{2A} \\ -F_{2B} & E_{2B} \end{bmatrix}$$

and let L_A, L_B be $m \times n'_A$, respectively $m \times n'_B$, matrices with $L = [L_A | L_B]$ and $\ker L = \operatorname{im} M$. Then

$$F = [L_A F_1 | L_B F_3], \quad E = [L_A E_1 | L_B E_3],$$

is a relaxed matrix kernel representation of $\mathcal{D}_A || \mathcal{D}_B$.
Similar, we have for the gyrative composition $\mathcal{D}_A \wedge \mathcal{D}_B$.

Proposition 2.7: Let $\mathcal{F}_i, i = 1, 2, 3$ and $\mathcal{D}_A, \mathcal{D}_B$ be defined as in Proposition 2.6. Define the $(n'_A + n'_B) \times 2n_2$ matrix

$$M = \begin{bmatrix} E_{2A} & F_{2A} \\ -F_{2B} & E_{2B} \end{bmatrix}$$

and let L_A, L_B be $m \times n'_A$, respectively $m \times n'_B$, matrices with $L = [L_A | L_B]$ and $\ker L = \operatorname{im} M$. Then

$$F = [L_A F_1 | L_B F_3], \quad E = [L_A E_1 | L_B E_3],$$

is a relaxed matrix kernel representation of $\mathcal{D}_A \wedge \mathcal{D}_B$.

Proof: Let $\mathcal{D}_A \wedge \mathcal{D}_B = \mathcal{D}_A || \mathcal{D}_I || \mathcal{D}_B$ the gyrative composition of $\mathcal{D}_A$ and $\mathcal{D}_B$. Then the proof follows the proof of Theorem 4 in [2] with matrix representation of the shared flow and effort variables

$$M = \begin{bmatrix} E_{2A} & F_{2A} \\ -F_{2B} & E_{2B} \end{bmatrix}.$$

■

Next, we define regularity of a composition.

Definition 2.8: Given two Dirac structures $\mathcal{D}_A$ and $\mathcal{D}_B$ defined as above. Their composition is said to be *regular* if the values of the power variables in $\mathcal{F}_2 \times \mathcal{F}_2^*$ are uniquely determined by the values in the power variables in $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$; that is, the following implication holds:

$$\begin{aligned} &(f_1, e_1, f_2, e_2) \in \mathcal{D}_A \text{ and } (-f_2, e_2, f_3, e_3) \in \mathcal{D}_B \\ &(f_1, e_1, \widetilde{f}_2, \widetilde{e}_2) \in \mathcal{D}_A \text{ and } (-\widetilde{f}_2, e_2, \widetilde{f}_3, e_3) \in \mathcal{D}_B \\ \Rightarrow &f_2 = \widetilde{f}_2, \ e_2 = \widetilde{e}_2. \end{aligned}$$

B. Linear Abstraction

Next, we introduce the concepts of linear abstractions as presented in [8]. Linear abstractions have been introduced for hierarchical control where the high level control system is modelled by aggregating the details of the lower, more complex, control systems. Furthermore, under some assumptions, controllability of the abstraction is equivalent to the controllability of the full system.

Definition 2.9: Let $C : \mathcal{X} \to \mathcal{Y}$ be a surjective linear map. Consider the linear control systems

$$\begin{aligned} \Sigma_1 : &\quad \dot{x} = A_1 x + B_1 u \\ \Sigma_2 : &\quad \dot{y} = A_2 y + B_2 v \end{aligned}$$

on $\mathcal{X}$ and $\mathcal{Y}$, respectively. They are said to be *C-related* if for all $x \in \mathcal{X}$ and $u \in \mathcal{U}$ there exists $v \in \mathbb{R}^l$ such that

$$C(A_1 x + B_1 u) = A_2 C x + B_2 v. \tag{1}$$

This definition is equivalent to the ability to find a control v such that the trajectory generated by it is identical to the projection of any trajectory of Σ_1 under C. For a given control system Σ and surjective map C, we can always construct a control system which is C-related to Σ_1 via the following proposition.

Proposition 2.10 ([8]): Consider the linear system

$$\Sigma_1 : \quad \dot{x} = A_1 x + B_2 u$$

and a surjective map $y = Cx$. Let

$$\Sigma_2 : \quad \dot{y} = A_2 y + B_2 v$$

be a linear control system on $\mathcal{Y}$ where

$$\begin{aligned} A_2 &= C A_1 C^+ \\ B_2 &= \begin{bmatrix} C B_1 & C A_1 v_1 & \cdots & C A_1 v_r \end{bmatrix} \end{aligned}$$

with C^+ the pseudoinverse of C (*i.e.*, $C^+ = (C^\top C)^{-1} C^\top$) and $v_1, \ldots, v_r$ such that $\operatorname{span}\{v_1, \ldots, v_r\} = \ker C$. Then Σ_1 and Σ_2 are C-related.

We refer to the system Σ_2 as a *linear abstraction* of Σ_1. For the linear port-Hamiltonian systems we make the following definition: Given a linear port-Hamiltonian system and a surjective map $C : \mathcal{X} \to \mathcal{X}_A$, C induces an interconnection and damping structure on $\mathcal{X}_A$ given by

$$J_A = C^\top J (C^+)^\top.$$

Furthermore, C is structure preserving with respect to this structure. Let Σ_A be an abstraction of the linear port-Hamiltonian system Σ. If Σ_A is a port-Hamiltonian system such that C is structure preserving then Σ_A is called a *linear port-Hamiltonian abstraction* of Σ.

Proposition 2.11: Let Σ be a linear port-Hamiltonian system and Σ_A a linear abstraction of Σ with respect to a surjective map C, then $H_A = H \circ C$ if Σ_A is a linear port-Hamiltonian abstraction.

Proof: If Σ_A is a linear port-Hamiltonian abstraction then C is structure preserving and $A_A = CAC^+$ which implies that we can write $Q_A = (C^+)^\top Q C^+$ such that $A_A = J_A Q_A$ with $J_A = CJ(C^+)^\top$. Hence, $H_A = H \circ C$ up to addition of a constant. ■

An existence condition for a C-relation between Σ and Σ_A is given by the following lemma.

Lemma 2.12: Let Σ and Σ_A be linear port-Hamiltonian systems with $\dim \mathcal{U} < \dim \mathcal{U}_A$. There exists a surjective map $C : \mathcal{X} \to \mathcal{X}_A$ such that Σ and Σ_A are C-related if there exists a linear port-Hamiltonian system Σ_B such that $\Sigma = \Sigma_A \wedge \Sigma_B$.

Proof: Let us write

$$\begin{aligned} \Sigma_A : &\ \dot{x}_A = J_A Q_A x_A + B_A u_A \\ \Sigma_B : &\ \dot{x}_B = J_B Q_B x_B + B_B u_B \end{aligned}$$

then their feedback interconnection $\Sigma = \Sigma_A \wedge \Sigma_B$ takes the form

$$\begin{bmatrix} \dot{x}_A \\ \dot{x}_B \end{bmatrix} = \begin{bmatrix} J_A & B_A B_B^\top \\ -B_B B_A^\top & J_B \end{bmatrix} \begin{bmatrix} Q_A & 0 \\ 0 & Q_B \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} + \begin{bmatrix} B_A & 0 \\ 0 & B_B \end{bmatrix} \begin{bmatrix} u_A \\ u_B \end{bmatrix}.$$

Hence, the control systems Σ and Σ_A are C-related with $C = [I_{n_a}, 0]$ since

$$C(JQx + Bu) = J_A Q_A Cx + B_A(-B_B^\top Q_B x_B + u_A).$$

■

III. Main Result

In this section, we combine abstractions of linear port-Hamiltonian systems and composition of Dirac structures to give conditions under which a given linear port-Hamiltonian system can be written as an interconnection of two lower dimensional systems. We motivate this with a simple example of an LC circuit (see [9]).

Example 3.1: Consider a controlled LC-circuit (see Fig. 1) consisting of two inductors with magnetic energies $H_1(\phi_1), H_2(\phi_2)$, where ϕ_1 and ϕ_2 are the magnetic flux linkages, and a capacitor with electrical energy $H_3(q)$, where q is the charge. Assuming that the elements are linear, then their total energy is

$$H_1(\phi_1) = \frac{1}{2L_1}\phi_1^2,\ H_2(\phi_2) = \frac{1}{2L_2}\phi_2^2 \text{ and } H_3(q) = \frac{1}{2C}q^2,$$

respectively. Furthermore, let $V = u$ denote the voltage source, then Kirchhoff's laws yield the linear port-Hamiltonian system

$$\begin{bmatrix} \dot\phi_1 \\ \dot\phi_2 \\ \dot q \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{L_1} & 0 & \\ 0 & \frac{1}{L_2} & 0 \\ 0 & 0 & \frac{1}{C} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ q \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, \tag{2}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{L_1} & 0 & \\ 0 & \frac{1}{L_2} & 0 \\ 0 & 0 & \frac{1}{C} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ q \end{bmatrix} \tag{3}$$

with Hamiltonian function $H = H_1 + H_2 + H_3$. An alternative

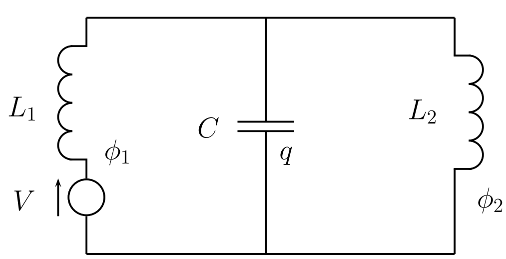

Fig. 1. LC circuit

way to establish this representation is to consider each element as a linear port-Hamiltonian system. Let us define

$$\Sigma_1 : \begin{array}{ll} \dot\phi_1 & = u_1 \\ y_1 & = \frac{\partial H_1}{\partial \phi_1} = \frac{\phi_1}{L_1} \end{array} \qquad \Sigma_2 : \begin{array}{ll} \dot\phi_2 & = u_2 \\ y_2 & = \frac{\partial H_2}{\partial \phi_2} = \frac{\phi_2}{L_2} \end{array}$$

$$\Sigma_3 : \begin{array}{ll} \dot q & = u_3 \\ y_3 & = \frac{\partial H_3}{\partial q} = \frac{q}{C}, \end{array}$$

where u_i and y_i, $i = 1, \ldots, 3$ are the voltages and currents of each element, respectively. The systems Σ_2 and Σ_3 are then interconnected via the following rule

$$\begin{array}{ll} u_2 & = y_3 + v_2 \\ u_3 & = -y_2 + v_3, \end{array}$$

which can also be expressed by a symplectic Dirac structure. This yields the intermediate linear port-Hamiltonian system

$$\Sigma' : \quad \begin{bmatrix} \dot\phi_2 \\ \dot q \end{bmatrix} = \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} v_2 \\ v_3 \end{bmatrix}$$

$$\begin{bmatrix} y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial H_2}{\partial \phi_2} \\ \frac{\partial H_3}{\partial q} \end{bmatrix} = \begin{bmatrix} \frac{\phi_2}{L_2} \\ \frac{q}{C} \end{bmatrix},$$

where we define a new input $v' = [v_2, v_3]^\top$ and new output $y' = [y_2, y_3]^\top$. Furthermore, the Hamiltonian function of Σ' is the sum of H_1 and H_2. Next, we interconnect Σ' to Σ_1 via the feedback interconnection

$$u_1 = \begin{bmatrix} 0 & 1 \end{bmatrix} y' + u$$
$$v' = -\begin{bmatrix} 0 \\ 1 \end{bmatrix} y_1,$$

that can also be expressed by a symplectic Dirac structure. The resulting interconnection constitutes the final system (2).

Next, we want to check whether one can deduce the intrinsic interconnection structure given only the overall system, assuming only that the final system was constructed using a lossless feedback (symplectic) interconnection. We begin with the state that is directly influenced by the control input, we assume that this subsystem consisting of the ϕ_1 dynamics is lossless with quadratic storage function $H_1(\phi_1)$, which is given by $H_1 = \frac{1}{2}B^\top QB\phi_1^2$. If the system is not lossless, it is clear that we can find a preliminary feedback that cancels damping in this state. Next, we need to construct the subsystem consisting of the remaining states and the interconnection with the ϕ_1-dynamics.

First, let us define the subsystem consisting of the remaining states. Motivated by the idea of linear abstractions, we quotient the states space by $\mathrm{Im}\,(B)$, using the identification of the state space with the tangent space at each point, to get the reduced state space. We denote the projection by C, then the reduced dynamics take the form

$$\begin{bmatrix} \dot\phi_2 \\ \dot q \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L_2} & 0 \end{bmatrix} \begin{bmatrix} \phi_2 \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L_1}\phi_1 \end{bmatrix},$$

where the last term is the "virtual port" for the reduced dynamics. Note that this only specifies a virtual port, not a virtual control input which is used, for example, in backstepping. The remaining states must have a scalar input $\bar u$ (to be determined) and a scalar output $\bar y$ dual to $\bar u$. Moreover, the input $\bar u$ must be a function of ϕ_1.

We are therefore left with the determination of $\bar u$ and $\bar y$. By the assumption of symplectic interconnection we have that $\bar u = y_1$. If one further assumes that $B' = [0, 1]^\top$ and $F' = CFC^\top$ then there exists a Q' such that Σ' is a linear port-Hamiltonian abstraction. Consequently, the reduced dynamics can be written in a linear port-Hamiltonian form.

We are interested in formalizing this approach using the idea of achievable Dirac structures and linear abstraction. We need the following proposition, due to [2], in the proof of the main proposition of this section.

Proposition 3.2: Consider a (given) plant Dirac structure $\mathcal{D}_P$ with port variables (f_1, e_1, f_2, e_2), and a desired Dirac structure $\mathcal{D}$ with port variables (f_1, e_1, f_3, e_3). Then, there exists a controller Dirac structure $\mathcal{D}_C$ such that $\mathcal{D} = \mathcal{D}_P || \mathcal{D}_C$ if and only if the following two equivalent conditions are satisfied

$$\mathcal{D}_P^0 \subset \mathcal{D}^0$$
$$\mathcal{D}^\pi \subset \mathcal{D}_P^\pi$$

where

$$\mathcal{D}_P^0 = \{(f_1, e_1) \mid (f_1, e_1, 0, 0) \in \mathcal{D}_P\}$$
$$\mathcal{D}_P^\pi = \{(f_1, e_1) \mid \exists (f, e) \text{ s.t. } (f_1, e_1, f, e) \in \mathcal{D}_P\}$$
$$\mathcal{D}^0 = \{(f_1, e_1) \mid (f_1, e_1, 0, 0) \in \mathcal{D}\}$$
$$\mathcal{D}^\pi = \{(f_1, e_1) \mid \exists (f, e) \text{ s.t. } (f_1, e_1, f, e) \in \mathcal{D}\}.$$

The proof of this proposition is based on the definition of a "copy" $\mathcal{D}_P^*$ of the plant Dirac structure $\mathcal{D}_P$. We define a copy of a Dirac structure $\mathcal{D}$ by

$$\mathcal{D}^* = \{(f_1, e_1, f, e) \mid (-f_1, e_1, -f, e) \in \mathcal{D}\}.$$

One possible controller Dirac structure is then constructed as $\mathcal{D}_C = \mathcal{D}_P^* || \mathcal{D}$.

Proposition 3.3: Let Σ linear port-Hamiltonian system and Σ_A a linear port-Hamiltonian abstraction. Then there exists a Dirac structure $\mathcal{D}_B$ such that $\mathcal{D} = \mathcal{D}_A || \mathcal{D}_B$, with $\mathcal{D}$ and $\mathcal{D}_A$ two Dirac structures canonically associated to each control system.

Proof: Let $x \in \mathcal{X}$, $x_A \in \mathcal{X}_A$ and $u \in \mathcal{U}$, $u_A \in \mathcal{U}_A$, respectively. We define the following vector spaces accordingly

$$\mathcal{F}_1 = \mathcal{X}_A \times \mathcal{U}, \quad \mathcal{F}_2 = \mathbb{R}^k, \quad \mathcal{F}_3 = \mathcal{X}/\mathcal{X}_A,$$

where $\mathcal{U}_A = \mathcal{U} \times \mathbb{R}^k$. To show that there exists a Dirac structure $\mathcal{D}_B$ such that $\mathcal{D} = \mathcal{D}_A || \mathcal{D}_B$, we verify that $\mathcal{D}^\pi \subset \mathcal{D}_A^\pi$, which is necessary and sufficient for the existence of $\mathcal{D}_B$ by Proposition 3.2. Assume $(f_1, e_1) \in \mathcal{D}^\pi$, then there exists (f_3, e_3) such that $(f_1, e_1, f_3, e_3) = (f_x, e_x, u, y)$ satisfies

$$F_x f_x + F_p u + E_x e_x + E_p y = 0. \tag{4}$$

Here f_3 is defined by $f = [f_3, f_2]^\top = \Lambda [f_x, u]^\top$ with the non-singular matrix

$$\Lambda = \begin{bmatrix} C_\perp & 0 \\ C & 0 \\ 0 & I_m \end{bmatrix}.$$

Premultiplying equation (4) by Λ implies that $C(f_x + Bu + Je_x) = 0$. Since Σ and Σ_A are C-related, there exists a $u_A \in \mathcal{U}_A$ such that

$$f_x^A + B_A u_A + J_A e_x^A = 0.$$

Let $y_A = [y, \bar{y}_A]^\top$ and $B_A = [CB, \bar{B}_A]$ then $e_x = C^\top e_x^A$ by Proposition 2.11. Choose $\bar{y}_A = \bar{B}_A^\top e_x^A$ then

$$F_x^A f_x^A + F_p^A u_A + E_x^A e_x^A + E_p^A y_A = 0,$$

and therefore there exists a pair (f_2, e_2) such that $(f_1, e_1, f_2, e_2) = (f_x^A, u_A, e_x^A, y_A) \in \mathcal{D}_A$. ■

Remark 3.4: One possible implementation for the Dirac structure $\mathcal{D}_B$ is $\mathcal{D}_A^* || \mathcal{D}$.

We establish a similar result for the gyrative composition under additional assumptions.

Proposition 3.5: Let Σ linear port-Hamiltonian system and Σ_A its linear port-Hamiltonian abstraction. There exists a Dirac structure $\mathcal{D}_B$ such that $\mathcal{D} = \mathcal{D}_A \wedge \mathcal{D}_B$, where $\mathcal{D}$ and $\mathcal{D}_A$ are Dirac structures associated to Σ and Σ_A, respectively if and only if

$$(C_\perp F + \bar{B}_A^\top C) e_x \in \operatorname{Im} C_\perp B$$
$$\text{and} \quad CF - F_A C - \bar{B}_A C_\perp = 0$$

where $B_A = [CB, \bar{B}_A]$.

Proof: Necessity can be shown by straight forward computation. Sufficiency is shown as follows. Assume that Σ and Σ_A are C-related linear port-Hamiltonian systems. Let $x \in \mathcal{X}$, $x_A \in \mathcal{X}_A$ and $u \in \mathcal{U}$, $u_A \in \mathcal{U}_A$, respectively. Let us also define two Dirac structures canonically associated to each control system, denoted by $\mathcal{D}$ and $\mathcal{D}_A$. We define the vector spaces $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ as in Proposition 3.3. Define furthermore $\bar{\mathcal{D}}_A = \mathcal{D}_A || \mathcal{D}_I$, where $\mathcal{D}_I$ is a full rank symplectic Dirac structure. Then we have to check that $\mathcal{D}^\pi \subset \bar{\mathcal{D}}_A^\pi$ to establish the existence of $\mathcal{D}_B$. Assume $(f_1, e_1) \in \mathcal{D}^\pi$, then there exists (f_3, e_3) such that $(f_1, e_1, f_3, e_3) = (f_x, e_x, u, y)$ satisfies

$$F_x f_x + F_p u + E_x e_x + E_p y = 0.$$

By definition $(f_1, e_1, e_2, e_2) \in \bar{\mathcal{D}}_A$ if there exists a pair $(\bar{f}_2, \bar{e}_2)$ such that

$$(f_1, e_1, \bar{f}_2, \bar{e}_2) \in \mathcal{D}_A \tag{5}$$
$$\text{and} \quad (-\bar{f}_2, \bar{e}_2, f_2, e_2) \in \mathcal{D}_I, \tag{6}$$

where (6) is equivalent to $\bar{f}_2 = e_3$ and $\bar{e}_2 = f_3$. Hence, using the same notation to define f_3 as in the proof of Proposition 3.3 we have

$$\bar{f}_2 = C_\perp e_x$$
$$\bar{e}_3 = -C_\perp \dot{x},$$

and it remains to show that (5) holds. We have

$$\dot{x}_A = J_A e_x^A + B_A u_A$$
$$y_A = B_A e_x^A.$$

Let us define $\bar{u}_A$ by $u_A = [u, \bar{u}_A]^\top$ and similar $y_A = [y, \bar{y}_A]^\top$, then

$$\dot{x}_A = J_A e_x^A + CBu + \bar{B}_A C_\perp e_x$$
$$-C_\perp (J e_x + Bu) = \bar{B}_A^\top e_x^A.$$

By definition of C-relation we have that

$$CF e_x - J_A e_x^A = \bar{B}_A C_\perp e_x,$$

furthermore $e_x^A = Ce_x$ (if $x \in (\ker C)_\perp$) then,

$$(CJ - J_A C - \bar{B}_A C_\perp)e_x = 0$$

which is true by assumption. Hence, equation (5) can be satisfied since we have that if $(C^\perp J + \bar{B}_A^\top C)e_x \in \operatorname{Im} C^\perp B$ then we can find a preliminary linear feedback $\alpha(x)$ such that

$$\bar{e}_2 = \bar{B}_A^\top e_x^A = -C_\perp(Je_x + B\alpha(x))$$

which implies that $(f_1, e_1, f_2, e_2) \in \bar{\mathcal{D}}_A$. ■

Next, a possible candidate for the virtual controller Dirac structure $\mathcal{D}_B$ is presented. Note that a Dirac structure embodies *generalized* port-Hamiltonian systems [4] that may include algebraic constraints on the dynamical system. Before proceeding, the following technical lemma is needed.

Lemma 3.6: Let $\mathcal{D}_A$ and $\mathcal{D}_B$ be two Dirac structures, then $(\mathcal{D}_A||\mathcal{D}_B)^* = \mathcal{D}_B^*||\mathcal{D}_A^*$.

Proof: We show $\mathcal{D}_B^*||\mathcal{D}_A^* \subset (\mathcal{D}_A||\mathcal{D}_B)^*$ and $(\mathcal{D}_A||\mathcal{D}_B)^* \subset \mathcal{D}_B^*||\mathcal{D}_A^*$. If $(f_1, e_1, f_3, e_3) \in (\mathcal{D}_A||\mathcal{D}_B)^*$ then $(-f_1, e_2, -f_3, e_3) \in \mathcal{D}_A||\mathcal{D}_B$ which implies that there exists $(-f_2, e_2)$ such that $(-f_1, e_1, -f_2, e_2) \in \mathcal{D}_A$ and $(f_2, e_2, -f_3, e_3) \in \mathcal{D}_B$. Hence, $(f_1, e_1, f_3, e_3) \in \mathcal{D}_B^*||\mathcal{D}_A^*$. Now, assume $(f_1, e_1, f_3, e_3) \in \mathcal{D}_B^*||\mathcal{D}_A^*$ then there exists (f_2, e_2) such that $(-f_1, e_1, -f_2, e_2) \in \mathcal{D}_A$ and $(f_2, e_2, -f_3, e_3) \in \mathcal{D}_B$. It follows that $(-f_1, e_1, -f_3, e_3) \in \mathcal{D}_A||\mathcal{D}_B$ and hence $(f_1, e_1, f_3, e_3) \in (\mathcal{D}_A||\mathcal{D}_B)^*$. ■

We define $\mathcal{D}_B = \mathcal{D}_I^*||\mathcal{D}_A^*||\mathcal{D}$ as the virtual controller Dirac structure, whose matrix kernel representation can be computed using the results in Section II.

Let us revisit Example 3.1 to illustrate our findings. The abstraction of the LC circuit is generate by the projection C such that $C_\perp = B^\top$, hence $C_\perp B = a \neq 0$. Hence, the first condition in 3.3 is satisfied. The second condition yields the matrix equation

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{v_1}{L_1} & 0 & 0 \end{bmatrix}$$

where $v_1 = \mathbb{R}/\{0\}$ parameterizes the kernel of C. Hence, for $v_1 = L_1$ this condition is also satisfied and the abstraction is a linear port-Hamiltonian system.

IV. Conclusions

We established a methodology to decompose linear port-Hamiltonian systems based on the concept of linear abstractions and achievable Dirac structures. We showed how the decomposition can be constructed based on the composition of Dirac structures. A second type of decomposition, motivated by power conserving feedback interconnections of port-Hamiltonian systems, has also been considered. The conditions for the existence in this case are stronger than in the first case. We have not considered any damping structure in this paper. The extension to linear port-Hamiltonian with damping structures requires one to replace the conditions in Proposition 3.2 with similar conditions for Dirac structures with resistive elements given in [2].

An intended application of this result is the introduction of an inductive approach to solving the linear matching problem arising in IDA-PBC for linear systems. This approach is computationally less efficient than other solutions proposed in the literature, but it has the advantage that the procedure gives clear insight to the interplay between the energy variables of the system (in some sense extracts the energy representation inherent in the system, rather than imposing it). A future direction of research, following ideas in [2], is to analyze the (achievable) Casimir function of the interconnection. The development presented applies exclusively to linear control systems. The largest obstacle to extend decomposition of port-Hamiltonian systems to nonlinear control-affine systems is that the extension of abstractions cannot guarantee that the abstractions are in control-affine form even though the concept of port-Hamiltonian systems relies on this control-affine form.

V. Acknowledgments

The first author would like to thank Prof. van der Schaft for his hospitality and support during his visit at the university of Groningen.

References

[1] A.M. Bloch and P.E. Crouch. Representations of Dirac structures on vector spaces and nonlinear LC circuits. In *Symposia in Pure Mathematics, Differential Geometry and Control Theory*, pages 103–117. American Mathematical Society, 1999.

[2] J. Cervera, A. J. van der Schaft, and A. Banos. Interconnection of port-Hamiltonian systems and composition of Dirac structures. *Automatica*, 43(2):212 – 225, 2007.

[3] T.J. Courant. Dirac manifolds. *Trans. Amer. Math. Soc*, 319(2):631–661, 1990.

[4] M. Dalsmo and A. J. van der Schaft. On Representations and Integrability of Mathematical Structures in Energy-Conserving Physical Systems. *SIAM Journal on Control and Optimization*, 37(1):54–91, 1998.

[5] V. Duindam, A. Macchelli, and S. Stramigioli. *Modeling and Control of Complex Physical Systems: The Port-Hamiltonian Approach*. Springer Verlag, 2009.

[6] P. Libermann and C. M. Marle. *Symplectic Geometry and Analytical Mechanics*. Kluwer Academic Publishers, 1987.

[7] R. Ortega, A. Loria, P. J. Nicklasson, and H. Sira-Ramirez. *Passivity-based control of Euler-Lagrange systems*. Springer Verlag, 1998.

[8] G.J. Pappas, G. Lafferriere, and S. Sastry. Hierarchically consistent control systems. *IEEE Transactions on Automatic Control*, 45(6):1144–1160, 2000.

[9] R. Polyuga and A.J. van der Schaft. Structure preserving model reduction of port-hamiltonian systems. In *Proc. 18th Int. Symposium on Mathematical Theory of Networks and Systems*, 2008.

[10] A. J. van der Schaft. *L_2-Gain and Passivity Techniques in Nonlinear Control*, volume 218 of *Lecture Notes in Control and Information Science*. Springer-Verlag, Berlin, 1999.