2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Input-state model matching for multirate systems

Mauro Cimino, *Student Member, IEEE,* and Prabhakar R. Pagilla, *Member, IEEE,*

***Abstract*— In this paper we address the model matching problem for multirate systems where the controller output is generated at a faster rate than the measurement update rate. The model matching problem is considered from the input-state viewpoint: given a desired LTI system, find conditions and provide a controller design procedure to achieve matching between the closed-loop system and the desired system state variables at the measurement update rate. We provide a solution to this problem using a particular time-varying controller structure. In addition we give conditions to avoid ripples in the steady-state output of the continuous-time plant.**

I. INTRODUCTION

Multirate systems are characterized by the presence of digital signals and systems updating at different rates. There are numerous applications where multirate control systems are employed. In some of these applications the update rate of the feedback measurement is slower than the controller update rate. Examples include the control of the Hard-Disk Drive (HDD) Read/Write (R/W) head [1], the octane rating control in the continuous catalytic reforming process [2], and the control of chemical concentrations in distillation columns [3]. The use of a control update rate faster than the measurement update rate was shown in [4] to allow input-output model matching with a desired LTI system. In this paper a state-space approach to design a controller for multirate systems is proposed. The multirate system comprises a continuous-time LTI plant whose state is available at the slow rate, $1/T_s$ (also referred to as measurement update rate), and is provided to a digital Linear Periodically Time-Varying (LPTV) controller operating at the faster rate, $1/T$. The first control design objectives is to ensure that the closed-loop state vector matches the state vector of a desired LTI system at the slow measurement rate $1/T_s$. This control problem, which can be referred to as the input-state matching problem, clearly differs from the classical input-output model matching problem by the fact that a desired dynamics can be assigned to each state of the closed-loop system.

The second control design objective is to ensure that the steady-state response of the closed-loop system to step reference signals is ripple-free. Ripples is a well known drawback of LPTV controllers in the multirate systems framework [4]. Sufficient conditions for the existence of a controller such that the closed-loop system exhibits ripple-free steady-state response to step reference signals are either the use of a discrete-time internal model of the reference [5], [6] or that the time-varying gains of the controller meet some particular conditions [7]. In this paper we solve the problem of ripples in the steady-state response from the model matching perspective. In other words, we find conditions under which a multirate control system designed to achieve input-state matching with another desired system also achieves a ripple-free steady-state response to step reference signals. In particular, we show that these conditions pose restrictions on the choice of the input matrix of the desired system.

II. PRELIMINARIES

The control system considered in this work involves digital signals and systems updating at the two different rates, $1/T$ and $1/T_s$, where $T_s = NT$, and N is a positive integer. To distinguish between the slow-rate and fast-rate signals (or system) the superscripts T and NT, respectively, will be utilized to refer to their update rate. No superscript will be utilized, instead, for continuous-time signals (or systems) and constant matrices. Notice that, due to the different update rates, the k-th sample of a fast-rate signal y_1^T (that is $y_1^T[k]$) is available at time $t = kT$, and that the k-th sample of a slow-rate signal y_2^{NT} (that is $y_2^{NT}[k]$) is available only at time $t = kNT$. For this reason, given a continuous-time signal y, and denoting with y^T and y^{NT} the corresponding fast-sampled and slow-sampled versions of y, respectively, the following relation holds: $y^{NT}[k] = y^T[kN]$ for every integer k. Since it is desired to keep track of both the sampling periods T_s and T, for fast-rate signals we will use the double index notation $kN+i$ to refer to their samples, where $i = 0,\dots,N-1$ serves to index the N samples available within the time windows $[kT_s,(k+1)T_s)$, and k is a non-negative integer needed to refer to a particular time window. By using this double index notation, the sample $y_1^T[kN+i]$ of the fast-rate signal y_1^T is available at the time instant $t = (kN+i)T = kT_s + iT$.

In the following we give the definition of the lifting operator and we show how the lifting operation can be applied to linear time-invariant (LTI) systems. Let V be the set of one sided, real-valued, fast-updating sequences $w^T[k]$ and let V^N be the set of sequences with elements as N dimensional vectors. The lifting operator $\mathscr{L}_N$ and the lifted sequence $w_{\mathscr{L}}^{NT}[k]$ are defined as $\mathscr{L}_N : V \to V^N : w^T[k] \to w_{\mathscr{L}}^{NT}[k]$, where $w_{\mathscr{L}}^{NT}[k] \triangleq \left[(w^T[kN])^\top, (w^T[kN+1])^\top, \cdots, (w^T[kN+N-1])^\top\right]^\top$ and $\top$ denotes the transpose operator.

Consider now the strictly proper LTI system

$$\begin{aligned} v^T[kN+i+1] &= A_v v^T[kN+i] + B_v w^T[kN+i] \\ y^{NT}[k] &= C_v v^T[kN] \end{aligned} \tag{1}$$

where the state variables v^T and input w^T update at the fast-rate $1/T$ and the output y^{NT} is available only at the slower-

M. Cimino and P.R. Pagilla are with the School of Mechanical and Aerospace Engineering, Oklahoma State University, Stillwater, OK, 74078 USA. E-mail: mauro.cimino@hotmail.com, pagilla@okstate.edu

rate $1/T_s$. We will refer to the lifted representation of the fast-updating system (1) as the following slow-updating single-rate system

$$\begin{aligned} v^T[(k+1)N] &= \tilde{A}_v v^T[kN] + \tilde{B}_v w^{NT}_{\mathscr{L}}[k] \\ y^{NT}[k] &= C_v v^T[kN] \end{aligned}$$

where the matrices $\tilde{A}_v$ and $\tilde{B}_v$, given by $\tilde{A}_v = A_v^N$ and $\tilde{B}_v = \left[A_v^{N-1}B_v, \cdots, A_vB_v, B_v\right]$, can be obtained by recursively applying (1) for $i = 0, \ldots, N-1$.

III. Problem formulation

Consider the continuous-time LTI system

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + B_c u(t) \\ y(t) &= C_c x(t) \end{aligned} \tag{2}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input vector, and $y(t) \in \mathbb{R}^{n_y}$ is the output vector to be regulated to the constant reference signal $r^{NT}[k] \in \mathbb{R}^{n_y}$. The control system structure shown in Fig. 1 is considered. The digital controller $\mathscr{C}$ operates at the sampling period T, and the measurements of the plant state $x(t)$ are available at the slow sampling period NT. We consider a causal, digital

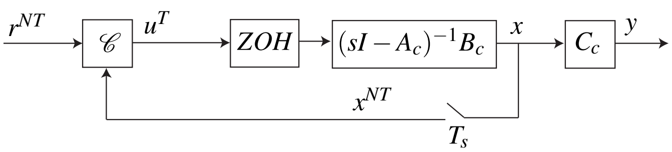

Fig. 1: Control system structure

controller $\mathscr{C}$ of the following LPTV form

$$\begin{aligned} \varphi^T[kN+i+1] &= K_{\varphi,i}\varphi^T[kN] + K_{x,i}x^{NT}[k] + L_i r^{NT}[k] \\ u^T[kN+i] &= C_\varphi \varphi^T[kN+i] \end{aligned} \tag{3}$$

where $\varphi^T \in \mathbb{R}^{n_\varphi}$ with $n_\varphi \in \mathbb{N}^+$. The control design objectives are stated in the following problem.

Problem 3.1: Find conditions and design the matrices C_φ, $K_{\varphi,i}$, $K_{x,i}$, L_i, $i = 0, \ldots, N-1$, of the controller (3) to achieve closed-loop state matching, at the slow rate $1/NT$, with the desired single-rate system

$$\begin{aligned} \zeta^{NT}[k+1] &= F\zeta^{NT}[k] + Gr^{NT}[k] \\ y^{NT}[k] &= H\zeta^{NT}[k] \end{aligned} \tag{4}$$

where $\zeta^{NT}[k] \in \mathbb{R}^{n_x+n_\varphi}$, $H \triangleq [C_c, 0_{n_y \times n_\varphi}]$, and the matrices F, G are chosen such that the desired system (4) is stable and exhibits zero steady-state regulation error to a unit step input. Moreover, the matrices C_φ, $K_{\varphi,i}$, $K_{x,i}$, and L_i have to be designed in order to achieve a ripple-free closed-loop response to step reference signals. By "ripple-free" it is meant that the continuous-time regulation error, $e(t) = y(t) - r_\infty$ (where $r_\infty \triangleq \lim_{k\to\infty} r^{NT}[k]$), has to be zero at steady-state within two consecutive measurement update instants, that is, $\lim_{k\to\infty} \int_{kT_s}^{(k+1)T_s} e^\top(t)e(t)dt = 0$.

To ensure the non-criticality of the sampling period T and the stabilizability of the periodic discrete-time system formed by the cascade connection of the zero-order hold, plant and slow-rate sampler, the following is assumed [8], [9], [10]:

Assumption 3.1: The sampling times involved in the multirate system are not critical, that is, $\lambda_a - \lambda_b \neq j2\pi k/(NT)$ and $\lambda_a \neq j2\pi k/(NT)$, $\forall k \in \mathbb{N}$, $k \neq 0$, where λ_a and λ_b are any two distinct eigenvalues of A_c.

This assumption can be readily satisfied with proper selection of T and N.

IV. Multirate control design

In this section we provide a solution to problem 3.1. In particular, conditions to achieve input-state model matching with the desired system (4) are given in section IV-A; in section IV-B we provide conditions such that the closed-loop system designed in section IV-A also exhibits ripple-free steady-state response to step reference signals. Before addressing the above mentioned problems it is convenient to rewrite the closed-loop system into a different form. Consider the zero-order hold equivalent of the plant (2) at the sampling period T

$$\begin{aligned} x^T[kN+i+1] &= \Phi x^T[kN+i] + \Gamma u^T[kN+i] \\ y^T[kN+i] &= C_c x^T[kN+i] \end{aligned} \tag{5}$$

where $\Phi = e^{A_cT}$ and $\Gamma = \int_0^T e^{A_c\lambda}B_c d\lambda$. The discrete-time system (5) and the proposed time-varying controller (3) can be rewritten in the following compact form:

$$\begin{aligned} \xi^T[kN+i+1] &= \bar{\Phi}\xi^T[kN+i] + \bar{\Gamma}w^T[kN+i] \\ y^{NT}[k] &= H\xi^T[kN] \end{aligned} \tag{6}$$

where $\xi^T[kN+i] \triangleq [(x^T[kN+i])^\top, (\varphi^T[kN+i])^\top]^\top$ is the extended state vector, the matrices $\bar{\Phi}$, $\bar{\Gamma}$ are given by

$$\bar{\Phi} \triangleq \begin{bmatrix} \Phi & \Gamma C_\varphi \\ 0_{n_\varphi \times n_x} & 0_{n_\varphi \times n_\varphi} \end{bmatrix}, \quad \bar{\Gamma} \triangleq \begin{bmatrix} 0_{n_x \times n_\varphi} \\ I_{n_\varphi \times n_\varphi} \end{bmatrix},$$

and $w^T[kN+i]$ can be thought of as a pseudo-control action given by

$$w^T[kN+i] \triangleq K_i\xi^T[kN] + L_i r^{NT}[k] \tag{7}$$

with $K_i \triangleq \left[K_{x,i}, K_{\varphi,i}\right]$. Let $w^{NT}_{\mathscr{L}}$ be the lifted pseudo-control signal obtained from w^T as shown in section II. The induced lifted version of the system (6) is given by:

$$\begin{aligned} \xi^T[(k+1)N] &= \bar{\Phi}_{\mathscr{L}}\xi^T[kN] + \bar{\Gamma}_{\mathscr{L}}w^{NT}_{\mathscr{L}}[k] \\ y^{NT}[k] &= H\xi^T[kN] \end{aligned} \tag{8}$$

where $\bar{\Phi}_{\mathscr{L}} \triangleq \bar{\Phi}^N$, $\bar{\Gamma}_{\mathscr{L}} \triangleq \left[\bar{\Phi}^{N-1}\bar{\Gamma}, \cdots, \bar{\Phi}\bar{\Gamma}, \bar{\Gamma}\right]$, $w^{NT}_{\mathscr{L}}[k] \triangleq \left[(w^T[kN])^\top, \cdots, (w^T[kN+N-1])^\top\right]^\top$. Considering that ξ^T in (7) is available only at the measurement update rate, the lifted version of the pseudo-control (7) takes the form

$$w^{NT}_{\mathscr{L}}[k] = K_{\mathscr{L}}\xi^T[kN] + L_{\mathscr{L}}r^{NT}[k] \tag{9}$$

where $K_{\mathscr{L}}$ and $L_{\mathscr{L}}$ are constant matrices given by

$$K_{\mathscr{L}} \triangleq \begin{bmatrix} K_0 \\ K_1 \\ \vdots \\ K_{N-1} \end{bmatrix} = \begin{bmatrix} K_{x,0} & K_{\varphi,0} \\ K_{x,1} & K_{\varphi,1} \\ \vdots & \vdots \\ K_{x,N-1} & K_{\varphi,N-1} \end{bmatrix}, L_{\mathscr{L}} \triangleq \begin{bmatrix} L_0 \\ L_1 \\ \vdots \\ L_{N-1} \end{bmatrix}. \tag{10}$$

Notice that the lifted *single-rate* LTI system formed by (8) and (9) describes the dynamics of the original LPTV closed-loop system, formed by (3) and (5), at the sampling period T_s. In particular, the closed-loop single-rate LTI system obtained by combining (8) and (9) can be rewritten as

$$\xi^T[(k+1)N] = (\bar{\Phi}_{\mathscr{L}} + \bar{\Gamma}_{\mathscr{L}} K_{\mathscr{L}})\xi^T[kN] + \bar{\Gamma}_{\mathscr{L}} L_{\mathscr{L}} r^{NT}[k]$$
$$y^{NT}[k] = H\xi^T[kN]. \quad (11)$$

A. Model matching problem: design of $K_{\mathscr{L}}$ and $L_{\mathscr{L}}$

From the closed-loop system (11) it is clear that, for a given matrix C_φ, input-state matching is achieved at the slow-rate $1/T_s$ with the desired system (4) if and only if the periodically time-varying matrices $K_{x,i}$, $K_{\varphi,i}$, and L_i, $i = 0,\dots,N-1$, are selected such that $K_{\mathscr{L}}$ and $L_{\mathscr{L}}$ satisfy

$$\bar{\Phi}_{\mathscr{L}} + \bar{\Gamma}_{\mathscr{L}} K_{\mathscr{L}} = F \quad (12)$$
$$\bar{\Gamma}_{\mathscr{L}} L_{\mathscr{L}} = G. \quad (13)$$

In this section, necessary and sufficient conditions are given for the existence of a solution $(K_{\mathscr{L}}, L_{\mathscr{L}})$ to (12) and (13) for any pair (F,G) characterizing the desired system (4). The following lemma gives a preliminary result needed to obtain those conditions.

Lemma 4.1 (On the existence of a right inverse of $\bar{\Gamma}_{\mathscr{L}}$): The matrix $\bar{\Gamma}_{\mathscr{L}}$, defined for the system (8), has a right inverse $\bar{\Gamma}^+_{\mathscr{L}}$ for any integer n_φ if and only if

1. $N \geq n_x + 1$
2. the pair $(\Phi, \Gamma C_\varphi)$ is controllable

Proof: Considering that $\bar{\Phi}^p$, where $p \in \mathbb{N}^+$, is given by

$$\bar{\Phi}^p = \begin{bmatrix} \Phi^p & \Phi^{p-1}\Gamma C_\varphi \\ 0_{n_\varphi \times n_x} & 0_{n_\varphi \times n_\varphi} \end{bmatrix},$$

$\bar{\Gamma}_{\mathscr{L}}$ can be rewritten as

$$\bar{\Gamma}_{\mathscr{L}} = \begin{bmatrix} \tilde{R} & 0_{n_x \times n_\varphi} \\ 0_{n_\varphi \times n_\varphi(N-1)} & I_{n_\varphi} \end{bmatrix} \quad (14)$$

where

$$\tilde{R} \triangleq \begin{bmatrix} \Phi^{N-2}\Gamma C_\varphi & \cdots & \Phi\Gamma C_\varphi & \Gamma C_\varphi \end{bmatrix}. \quad (15)$$

For $\bar{\Gamma}^+_{\mathscr{L}}$ to exist, $\bar{\Gamma}_{\mathscr{L}}$ must have full row rank. By inspection this occurs if and only if $\tilde{R}$ has full row rank, that is, n_x.

Sufficiency ($\Leftarrow$). Since $N \geq n_x + 1$ and the pair $(\Phi, \Gamma C_\varphi)$ is controllable, $\tilde{R}$ is a full row rank matrix.

Necessity ($\Rightarrow$) The necessity of conditions 1 and 2 is shown by contradiction. Assume that the pair $(\Phi, \Gamma C_\varphi)$ is not controllable but that condition 1 holds. As a consequence of the Cayley-Hamilton theorem, the matrices Φ^p, where $p \geq n_x$, can be expressed as a linear combination of the set of matrices Φ^q, $q = 1,\dots,n_x - 1$. Therefore $\tilde{R}$ does not have full row rank, that is, $\text{Rank}(\tilde{R}) < n_x$. Conversely, assume that the pair $(\Phi, \Gamma C_\varphi)$ is controllable but that $n_x > (N-1)$. Then, for any integer $n_\varphi < n_x/(N-1)$, the number of columns of $\tilde{R}$ is $n_\varphi(N-1) < n_x$. This contradicts the hypothesis that $\bar{\Gamma}^+_{\mathscr{L}}$ has a right inverse. ■

The following theorem provides the solution to the input-state matching problem.

Theorem 4.1: Under the assumption 3.1, there exists a controller of the form (3) such that the state of the closed-loop multirate system matches the state of the desired single-rate system (4) at the rate $1/T_s$, for any pair (F,G) and any integer n_φ, if and only if

a. $N \geq n_x + 1$
b. the pair (A_c, B_c) is controllable
c. $\text{Rank}(C_\varphi) = n_u$

Proof: Sufficiency ($\Leftarrow$). Let conditions *a*, *b* and *c* hold. We have to show the existence of matrices $K_{\mathscr{L}}$ and $L_{\mathscr{L}}$ that satisfy (12) and (13). Let $\lambda_\star$ be an eigenvalue of Φ and let $v^\top \neq 0$ be the corresponding left eigenvector, that is, $v^\top\Phi = \lambda_\star v^\top$. Since (A_c, B_c) is controllable, by assumption 3.1 (Φ, Γ) is also controllable. Therefore, it has to be that $v^\top[\Phi - \lambda_\star I, \Gamma] \neq 0$, that is, $v^\top\Gamma \neq 0$. Since C_φ is a full row rank matrix, we also have $v^\top\Gamma C_\varphi \neq 0$, that is, $v^\top\left[\Phi - \lambda_\star I, \Gamma C_\varphi\right] \neq 0$. Therefore, the pair $(\Phi, \Gamma C_\varphi)$ is controllable, and by lemma 4.1, the right inverse $\bar{\Gamma}^+_{\mathscr{L}}$ of $\bar{\Gamma}_{\mathscr{L}}$ exists. The solution $(K_{\mathscr{L}}, L_{\mathscr{L}})$ to (12) and (13) is then given by

$$K_{\mathscr{L}} = \bar{\Gamma}^+_{\mathscr{L}}(F - \bar{\Phi}_{\mathscr{L}}) \quad (16)$$
$$L_{\mathscr{L}} = \bar{\Gamma}^+_{\mathscr{L}} G \quad (17)$$

Necessity ($\Rightarrow$). Since there exists a solution $(K_{\mathscr{L}}, L_{\mathscr{L}})$ to (12) and (13), $[F - \bar{\Phi}_{\mathscr{L}}, G] \in \text{Span}(\bar{\Gamma}_{\mathscr{L}})$. Since this has to hold for any pair (F,G) and any integer n_φ, it must be that $\text{Rank}(\bar{\Gamma}_{\mathscr{L}}) = n_x + n_\phi$. In other words, $\bar{\Gamma}_{\mathscr{L}}$ must have a right inverse. By lemma 4.1, this implies that $n_x \leq N-1$ and that the pair $(\Phi, \Gamma C_\varphi)$ is controllable. Let $\lambda_\star$ be an eigenvalue of Φ and let $v^\top \neq 0$ be the corresponding left eigenvector, that is, $v^\top\Phi = \lambda_\star v^\top$. The proof will proceed by contradiction. Assume first that $\text{Rank}(C_\varphi) = n_u$ but that the pair (A_c, B_c) is not controllable. Then (Φ, Γ) is not controllable, that is, there exists an eigenvector of Φ such that $v^\top\Gamma = 0$. In turn, this implies $v^\top\Gamma C_\varphi = 0$, which is a contradiction since the pair $(\Phi, \Gamma C_\varphi)$ is controllable. Conversely, assume that the pair (A_c, B_c) is controllable but that $\text{Rank}(C_\varphi) < n_u$. Since the pair (A_c, B_c) is controllable, also the pair (Φ, Γ) is controllable. Therefore, $v^\top\Phi = \lambda_\star v^\top$ and $v^\top\Gamma \neq 0$ for every eigenvector-eigenvalue pair $(v^\top, \lambda_\star)$ of Φ. However, since $\text{Rank}(C_\varphi) < n_u$ there may exist an eigenvector $v^\top$ of Φ such that $v^\top\Gamma C_\varphi = 0$. However, this contradicts the fact that the pair $(\Phi, \Gamma C_\varphi)$ is controllable. ■

Remark 4.1 (On the values of N and n_φ): Notice that conditions *a* and *c* of Theorem 4.1 require the selection of N and n_φ, respectively, such that $N \geq n_x + 1$ and $n_\varphi \geq n_u$.

B. Conditions for ripple-free steady-state response

In this section we investigate under which conditions the closed-loop multirate system, obtained by designing $K_{\mathscr{L}}$ and $L_{\mathscr{L}}$ according to (16) and (17), also exhibits ripple-free steady-state response to step reference signals. We start by denoting with $M \triangleq -(F-I)^{-1}G$ the $(n_x + n_\varphi) \times n_y$ matrix characterizing the steady-state value of the desired system states, that is, $\zeta^{NT}[k] \to Mr_\infty$ as $k \to \infty$, and by referring to M_a and M_b respectively as the $n_x \times n_y$ and the $n_\varphi \times n_y$ partitions of M such that $M \triangleq [M_a^\top, M_b^\top]^\top$. Moreover, denote with $\mathscr{N}_s \triangleq [S_a^\top, S_b^\top]^\top$ the matrix whose columns form a basis for the null space of $[\Phi - I, \Gamma]$, and hence, $[\Phi - I, \Gamma][S_a^\top, S_b^\top]^\top = 0$, where the row-dimensions of S_a and S_b are n_x and n_u, respectively.

The following lemma provides preliminary conditions on the matrix M that are necessary and sufficient in order to guarantee ripple-free steady-state response to step reference signals.

Lemma 4.2: Let the conditions of Theorem 4.1 be satisfied, and let $K_\mathscr{L}$ and $L_\mathscr{L}$ be given by (16) and (17), respectively. Under assumption 3.1, the closed-loop system in Fig. 1 exhibits ripple-free steady-state response to step reference signals if and only if

$$L_i = M_b - K_i M, \quad i = 0, \dots, N-1 \tag{18}$$

Proof: Sufficiency ($\Leftarrow$). As $\zeta^{NT}[k] \to M r_\infty$, also $\xi^T[kN] \to M r_\infty$ by the state matching condition. Therefore, $x^{NT}[k] \to M_a r_\infty$ and $\varphi^T[kN] \to M_b r_\infty$ as $k \to \infty$. If (18) holds, the result follows by considering that the controller output $u^T[kN+i]$ becomes constant as $k \to \infty$.

Necessity ($\Rightarrow$). The closed-loop system exhibits a ripple-free response to step reference signals and state matching at the slow-rate with $\zeta^{NT}[k]$. Therefore, since $\zeta^{NT}[k]$ tends to a constant as k goes to infinity, also $\varphi^T[kN+i]$ tends to a constant. Because of the state matching condition, $\varphi^T[kN+i] \to M_b r_\infty$ as $k \to \infty$. As a result, $K_i M r_\infty + L_i r_\infty = M_b r_\infty$. ■

Based on the result given in Lemma 4.2, the following theorem provides insights on the structure of the matrix M such that ripple-free closed-loop response is achieved.

Theorem 4.2: Let the conditions of Theorem 4.1 be satisfied, and let $K_\mathscr{L}$ and $L_\mathscr{L}$ be given by (16) and (17), respectively. Moreover, let $\tilde{M}_b$ be the $(N-1)$-blocks matrix where each block is equal to M_b, that is, $\tilde{M}_b \triangleq [M_b^\top, \cdots, M_b^\top]^\top$, and let $\tilde{R}$ be defined as in (15), that is, $\tilde{R} \triangleq [\Phi^{N-2}\Gamma C_\varphi, \cdots, \Phi\Gamma C_\varphi, \Gamma C_\varphi]$. Under assumption 3.1, the closed-loop system in Fig. 1 exhibits a ripple-free steady-state response to step reference signals if and only if

(a) $\tilde{M}_b \in \mathrm{Span}\{\tilde{R}^\top\}$ and
(b) the columns of the matrix

$$\begin{bmatrix} M_a \\ C_\varphi M_b \end{bmatrix} \tag{19}$$

are contained in the range space of $\mathscr{N}_s$

Proof: By lemma 4.2 it is necessary and sufficient to show that $L_\mathscr{L} + K_\mathscr{L} M = M_{b,\mathscr{L}}$ where $M_{b,\mathscr{L}} \triangleq [\tilde{M}_b^\top, M_b^\top]^\top$. Using (16) and (17), and considering that $(F-I)M + G = 0$, the following holds:

$$\begin{aligned} L_\mathscr{L} + K_\mathscr{L} M &= \bar{\Gamma}_\mathscr{L}^+ G + \bar{\Gamma}_\mathscr{L}^+ (F - \bar{\Phi}_\mathscr{L}) M \\ &= \bar{\Gamma}_\mathscr{L}^+ \big(G + (F-I)M - (\bar{\Phi}_\mathscr{L} - I)M \big) \\ &= -\bar{\Gamma}_\mathscr{L}^+ (\bar{\Phi}_\mathscr{L} - I) M \\ &= -\bar{\Gamma}_\mathscr{L}^+ \begin{bmatrix} (\Phi^N - I)M_a + \Phi^{N-1}\Gamma C_\varphi M_b \\ -M_b \end{bmatrix}. \end{aligned} \tag{20}$$

Sufficiency ($\Leftarrow$). Since the columns of the matrix in (19) are in the range space of $\mathscr{N}_s$, $\left[\Phi - I, \Gamma C_\varphi\right] M = 0$. Therefore, $(\Phi - I)M_a = -\Gamma C_\varphi M_b$. Considering also that $(\Phi^N - I) = (\Phi^{N-1} + \cdots + \Phi + I)(\Phi - I)$, (20) can be further expanded as

$$L_\mathscr{L} + K_\mathscr{L} M = \begin{bmatrix} \tilde{R}^\top (\tilde{R}\tilde{R}^\top)^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{R} & 0 \\ 0 & I \end{bmatrix} M_{b,\mathscr{L}}$$

where the first matrix on the right-hand side is the right inverse, $\bar{\Gamma}_\mathscr{L}^+$, of the matrix $\bar{\Gamma}_\mathscr{L}$ given in (14). Therefore,

$$L_\mathscr{L} + K_\mathscr{L} M = \begin{bmatrix} \tilde{R}^\top (\tilde{R}\tilde{R}^\top)^{-1}\tilde{R} & 0 \\ 0 & I \end{bmatrix} M_{b,\mathscr{L}}$$

Let (U, V, Σ) be the singular value decomposition of $\tilde{R}$, that is, $\tilde{R} = U[\Sigma, 0]V^\top$, where U and V are $n_x \times n_x$ and $n_\varphi(N-1) \times n_\varphi(N-1)$ unitary matrices, respectively, and Σ is a square, diagonal, nonsingular $n_x \times n_x$ matrix (because $\mathrm{Rank}\{\tilde{R}\} = n_x$). It is possible to show that

$$\tilde{R}^\top (\tilde{R}\tilde{R}^\top)^{-1}\tilde{R} = V \begin{bmatrix} \Sigma^\top (\Sigma\Sigma^\top)^{-1}\Sigma & 0 \\ 0 & 0 \end{bmatrix} V^\top = V \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} V^\top \tag{21}$$

Therefore, (V, V, I) is the singular value decomposition of the symmetric and square matrix $\tilde{R}^\top (\tilde{R}\tilde{R}^\top)^{-1}\tilde{R}$. Let V be partitioned as $V = [V_1, V_2]$, where V_1 is an $n_\varphi(N-1) \times n_x$ matrix. Then, the matrix $\tilde{R}^\top (\tilde{R}\tilde{R}^\top)^{-1}\tilde{R}$ acts as an identity operator for all the vectors in the range space of V_1, that is, $\tilde{R}^\top (\tilde{R}\tilde{R}^\top)^{-1}\tilde{R} V_1 = V_1$. Since also $\tilde{R}^\top (\tilde{R}\tilde{R}^\top)^{-1}\tilde{R}\tilde{R}^\top = \tilde{R}^\top$, the columns of $\tilde{R}^\top$ must be in the range space of V_1. However, since $\mathrm{Rank}\{V_1\} = \mathrm{Rank}\{\tilde{R}^\top\} = n_x$, $\mathrm{Span}\{V_1\} \equiv \mathrm{Span}\{\tilde{R}^\top\}$. Since by hypothesis $\tilde{M}_b \in \mathrm{Span}\{\tilde{R}^\top\}$, (20) can be rewritten as $L_\mathscr{L} + K_\mathscr{L} M = M_{b,\mathscr{L}}$.

Necessity ($\Rightarrow$). Because of the state matching at the slow-rate $1/T_s$, and since the closed-loop response is ripple-free at steady-state, $x^T[kN+i] \to M_a r_\infty$ and $\varphi^T[kN+i] \to M_b r_\infty$ for every $i = 0, \dots, N-1$ as $k \to \infty$. Therefore, from (5), $(\Phi - I)M_a r_\infty + \Gamma C_\varphi M_b r_\infty = 0$, which proves the second condition of the theorem. Also, since $L_\mathscr{L} + K_\mathscr{L} M = M_{b,\mathscr{L}}$, it must be that $\tilde{M}_b \in \mathrm{Span}\{\tilde{R}^\top\}$. ■

Since $M \triangleq -(F-I)^{-1}G$, the conditions given in theorem 4.2 place constraints on pair (F, G) of the desired system. Based on the result given in Theorem 4.2, the following lemma provides sufficient conditions on the choice of the pair (F, G) such that a ripple-free steady-state response is guaranteed.

Lemma 4.3: Let the conditions of Theorem 4.1 be satisfied, and let $K_\mathscr{L}$ and $L_\mathscr{L}$ be given by (16) and (17), respectively. Under assumption 3.1, the closed-loop system in Fig. 1 exhibits ripple-free steady-state response to step reference signals if

(a) the matrix G of the desired system is given by

$$G = -(F-I) \begin{bmatrix} S_a P \\ 0 \end{bmatrix} \tag{22}$$

where S_a is the first row-block of $\mathscr{N}_s$ and P is any $(n_u \times n_u)$ matrix,
(b) the number of Jordan blocks of Φ associated with the eigenvalue $\lambda = 1$ is greater than or equal to n_u,
(c) the pair (A_c, B_c) is controllable.

Proof: Let G be as in (22). Then, $M \triangleq -(F-I)^{-1}G = [(S_aP)^\top, 0]^\top$ implies $M_a = S_aP$ and $M_b = 0$. Notice that, with this choice of G, the columns of M_a are in the range space of S_a. Notice that this is a necessary condition for the columns of the matrix in (19) to be in the range space of $\mathscr{N}_s$. By Theorem 4.2, and considering that $M_a = S_aP$, for the closed-loop system to achieve a ripple-free steady-state response there have to exist matrices Q and C_φ such that the following equations are satisfied:

$$M_b = (\Phi^p \Gamma C_\varphi)^\top Q, \quad p = 0, \dots, N-2, \tag{23}$$

$$C_\varphi M_b = S_b P. \tag{24}$$

The above equations correspond to the conditions (a) and (b) of Theorem 4.2 rewritten for the case in which $M_a = S_aP$. Since $M_b = 0$ (because of the structure of G), equations (23) and (24) are satisfied for any matrix P if $S_b = 0$. Therefore, it remains to show that $S_b = 0$.

By definition, the matrices S_a and S_b are such that $[\Phi - I, \Gamma]\left[S_a^\top, S_b^\top\right]^\top = 0$. Since the pair (A_c, B_c) is controllable, the dimension of the null space of $[\Phi - I, \Gamma]$ is always n_u. Let $n_{J_1} \geq n_u$ be the number of Jordan blocks of Φ associated with the eigenvalue $\lambda = 1$. Then, every basis for the null space of $(\Phi - I)$ has cardinality equal to $n_{J_1} \geq n_u$. Let $\mathscr{N}_{\Phi-I}$ be a $(n_u \times n_u)$-matrix whose columns generate a linearly independent set in the null space of $(\Phi - I)$. Then, the columns of the matrix $[\mathscr{N}_{\Phi-I}^\top, 0_{n_u \times n_u}]^\top$ form a basis for the null space of $[\Phi - I, \Gamma]$. In other words, $S_b = 0$. ■

Remark 4.2 (On the steady-state regulation error): Since the pair (F, G) of the desired system is supposed to be selected in order to achieve zero steady-state regulation error to a step reference signal, it has to be $H(I-F)^{-1}G = I$. If G is designed through (22), it is straightforward to show that zero steady-state regulation error can be achieved only if $P = (C_cS_a)^{-1}$. In turn, this is always possible considering that the square matrix (C_cS_a) is nonsingular (since S_a and C_c are full column-rank and full row-rank matrices, respectively).

Remark 4.3 (On the condition (b) of Lemma 4.3): The condition (b) of Lemma 4.3 is a consequence of the internal model principle for constant exogenous signals. However, it differs from the analogous sufficient condition provided in [5] for multirate systems, where the number of Jordan blocks associated with the eigenvalue $\lambda = 1$ is required to equal the number n_y of controlled outputs.

It is important to note that the condition (b) of Lemma 4.3 does not require the presence of any continuous-time integrators in the forward path of the control system, but only of digital integrators updating at the fast rate $1/T$. Therefore, as shown in Fig. 2, if the continuous-time plant does not embed those integrators a fast-updating digital linear time-invariant precompensator $\mathscr{D}$ in cascade with the plant can be designed to provide them. The theory shown in this work can then be applied by considering the precompensator $\mathscr{D}$ as being part of the plant. However, in order to do so, such a precompensator has to be designed so that the cascade system made of the discretized plant (5) and the precompensator $\mathscr{D}$ remains controllable. If we denote with n_{J_1} the number of Jordan blocks of Φ associated with the eigenvalue $\lambda = 1$, it is possible to show that a suitable precompensator has the following state-space representation $(A_\mathscr{D}, B_\mathscr{D}, C_\mathscr{D}, D_\mathscr{D})$:

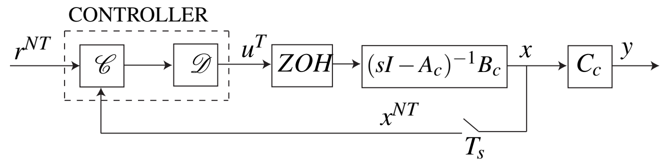

Fig. 2: Control system structure including the precompensator $\mathscr{D}$

$$\begin{aligned} A_\mathscr{D} &\triangleq I_{n_u - n_{J_1}}, \quad B_\mathscr{D} \triangleq \begin{bmatrix} I_{n_u - n_{J_1}} & 0_{(n_u - n_{J_1}) \times n_{J_1}} \end{bmatrix} \\ C_\mathscr{D} &\triangleq \begin{bmatrix} I_{n_u - n_{J_1}} \\ 0_{n_{J_1} \times (n_u - n_{J_1})} \end{bmatrix}, \quad D_\mathscr{D} \triangleq I_{n_u} \end{aligned} \tag{25}$$

For convenience, in the following we will refer to the system $\mathscr{F} \triangleq (A_\mathscr{F}, B_\mathscr{F}, C_\mathscr{F}, D_\mathscr{F})$ as the cascade connection of $\mathscr{D}$ with the plant (5), and hence such that

$$A_\mathscr{F} = \begin{bmatrix} \Phi & \Gamma C_\mathscr{D} \\ 0 & A_\mathscr{D} \end{bmatrix}, B_\mathscr{F} = \begin{bmatrix} \Gamma D_\mathscr{D} \\ B_\mathscr{D} \end{bmatrix}, C_\mathscr{F} = \begin{bmatrix} C_c & 0 \end{bmatrix}, D_\mathscr{F} = 0.$$

V. Controller Design Procedure and Example

In this section we show a step-by-step procedure to apply the obtained results, and an example based on a double integrator system.

- *Step 1*. Obtain the discrete equivalent representation, (Φ, Γ), of the continuous-time controllable plant (2), where Γ is a full column rank matrix.
- *Step 2*. If the number of Jordan blocks of Φ associated with the eigenvalue $\lambda = 1$ is $n_{J_1} < n_u$, design a digital precompensator $\mathscr{D}$ as given in (25). Also, take $(\Phi, \Gamma, C_c) = (A_\mathscr{F}, B_\mathscr{F}, C_\mathscr{F})$ as the new plant.
- *Step 3*. Let n_x be the dimension of the square matrix Φ (which may include the precompensator dynamics as explained in Step 2), and select N to satisfy assumption 3.1 and $N \geq n_x + 1$.
- *Step 4*. Find a basis, $\mathscr{N}_s \triangleq [S_a^\top, S_b^\top]^\top$, for the null space of the matrix $[\Phi - I, \Gamma]$.
- *Step 5*. Select the dimension n_φ for the controller (3) such that $n_\varphi \geq n_u$.
- *Step 6*. Choose any matrix F for the desired stable system (4) and select the matrix G as in (22) with $P = (C_cS_a)^{-1}$.
- *Step 7*. Select C_φ to be any full row rank matrix.
- *Step 8*. Construct the matrices $\bar{\Phi}_\mathscr{L}$, $\bar{\Gamma}_\mathscr{L}$ and compute the right inverse $\bar{\Gamma}_\mathscr{L}^+ = \bar{\Gamma}_\mathscr{L}^\top(\bar{\Gamma}_\mathscr{L}\bar{\Gamma}_\mathscr{L}^\top)^{-1}$.
- *Step 9*. After computing the matrices $K_\mathscr{L}$ and $L_\mathscr{L}$ according to (16) and (17), obtain the controller matrices $K_{x,i}$, $K_{\varphi,i}$ and L_i as given in (10).
- *Step 10*. For implementation purposes, the actual controller is given by the cascade of $\mathscr{C}$ with $\mathscr{D}$, as shown in Fig. 2.

A. Example

Consider a double-integrator system with state-space representation

$$A_c = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_c = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Since $n_x = 2$ it is required to choose $N \geq 3$. For simplicity, let us consider the case $N = 3$ and $T = 1$. The zero-order hold equivalent of the plant operating at the fast rate $1/T$ is given by

$$\Phi = \begin{bmatrix} 1 & 0 \\ T & 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} T \\ 5T^2 \end{bmatrix}, \quad C_c = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Notice that Φ has enough Jordan blocks associated with the eigenvalue $\lambda = 1$, and hence, there is no need for the precompensator $\mathscr{D}$. A basis for the null space of $[\Phi - I, \Gamma]$ is $\mathscr{N}_s = [0,1,0]^\top$. Therefore, $S_a = [0,1]^\top$ and $S_b = 0$. Let us consider a controller of order 1 (that is $n_\varphi = n_u = 1$), let the matrix F of the desired system (4) be zero-valued, and let G be selected according to (22). Hence, the state-space representation of the desired system is given by:

$$F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Let $C_\varphi = 1$. Then, the matrices $K_{x,i}$, $K_{\varphi,i}$ and L_i of the controller (3), designed according to (16) and (17), are given by $K_{x,0} = [-5/(2T), -1/(10T^2)]$, $K_{\varphi,0} = -2$, $L_0 = 1/(10T^2)$, $K_{x,1} = [3/(2T), 1/(10T^2)]$, $K_{\varphi,1} = 1$, $L_1 = -1/(10T^2)$, $K_{x,2} = [0,0]$, $K_{\varphi,2} = 0$, $L_2 = 0$. The step response of the closed-loop multirate system comprising the continuous-time plant and the digital controller is compared with the response of the desired system as shown in Fig. 3. From Fig. 3 it is clear that the state of the closed-loop system matches at every measurement sampling instant the state of the desired system (4), and that the response of the closed-loop system is ripple-free at steady-state.

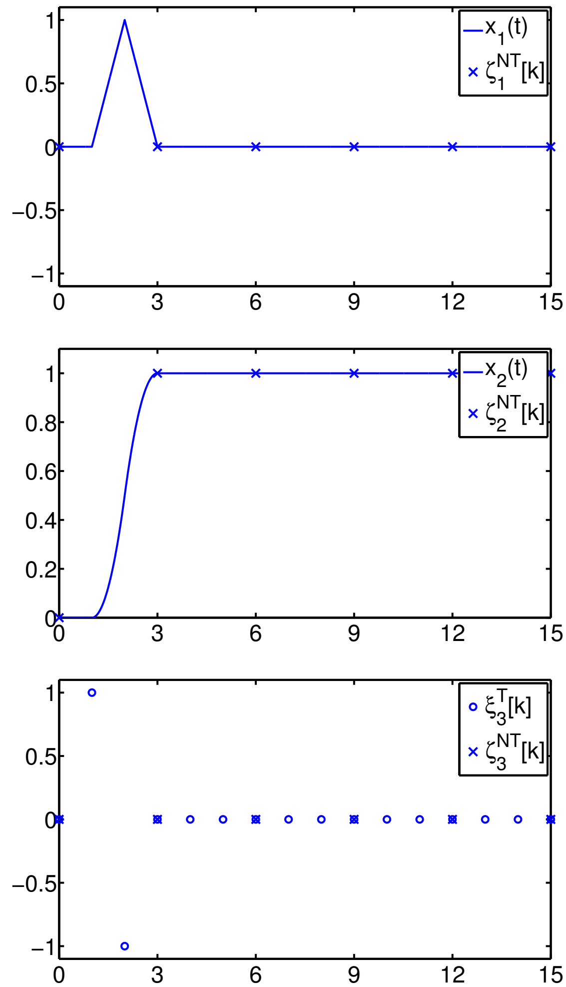

Fig. 3: Closed-loop system response to a step reference signal. The signals x_1, x_2 are the velocity and position, respectively, of the double integrator continuous-time plant. The closed-loop system output is $y(t) = x_2(t)$.

VI. Conclusions

In this work we addressed the input-state matching problem for multirate systems. In particular, given any desired single-rate LTI system operating at the measurement update rate, we provided conditions and a controller design procedure for which the closed-loop system state matches the state of the desired system at that measurement update rate. Moreover, we showed that, if the input matrix of the desired system is properly selected, a ripple-free steady-state response of the closed-loop system can be obtained. Despite the constraints on the input matrix of the desired system, the developed design procedure gives full freedom on the choice of the closed-loop eigenstructure.

References

[1] S.-H. Lee, "Multirate digital control system design and its application to computer disk drives," *IEEE Transactions on Control Systems Technology*, vol. 14, no. 1, pp. 124–133, January 2006.

[2] D. Li, S. Shah, T. Chen, and K. Qi, "Application of dual-rate modeling to CCR octane quality inferential control," *IEEE Transactions on Control Systems Technology*, vol. 11, no. 1, pp. 43 – 51, jan 2003.

[3] R. Scattolini, "Self-tuning multirate control of binary distillation columns," in *International Conference on Control*, 13-15 1988, pp. 35 –40.

[4] K. L. Moore, S. P. Bhattacharyya, and M. Dahleh, "Capabilities and limitations of multirate control schemes," *Automatica*, vol. 29, no. 4, pp. 941–951, 1993.

[5] O. M. Grasselli, L. Jetto, and S. Longhi, "Ripple-free dead-beat tracking for multirate sampled-data systems," *International Journal of Control*, vol. 61, no. 6, pp. 1437 – 1455, 1995.

[6] G. F. Franklin and A. Emami-Naeini, "Design of ripple-free multivariable robust servomechanisms," *IEEE Transactions On Automatic Control*, vol. AC-31, no. 7, July 1986.

[7] A. Tangirala, D. Li, R. Patwardhan, S. Shah, and T. Chen, "Ripple-free conditions for lifted multirate control systems," *Automatica*, vol. 37, no. 10, pp. 1637 – 1645, 2001.

[8] B. Francis and T. Georgiou, "Stability theory for linear time-invariant plants with periodic digital controllers," *IEEE Transactions on Automatic Control*, vol. 33, no. 9, pp. 820–832, 1988.

[9] R. E. Kalman, Y. C. Ho, and K. S. Narendra, "Controllability of linear dynamic systems," *Contributions to Differential Equations*, vol. 1, pp. 189–213, 1963.

[10] T. Chen and B. Francis, "Linear time-varying H2-optimal control of sampled-data systems," *Automatica*, vol. 27, no. 6, pp. 963–974, NOV 1991.