2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Leader Selection in Multi-Agent Systems Subject to Partial Failure

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Abstract— **This paper studies the structural controllability of a leader-follower multi-agent system. A controllability condition is first provided based on the topology of the information flow graph. Conditions for controllability preservation in a multiple-leader system subject to failure in the agents and communication links are then investigated. The problem of optimal leader selection is introduced, which is concerned with finding the minimum number of agents whose selection as leaders increase the reliability of the network in terms of controllability. A polynomial-time algorithm is subsequently presented to solve the problem for undirected information flow graphs.**

I. Introduction

There has been a surge of interest in the use of multi-agent systems in a wide variety of engineering applications over the past several years. Important features of multi-agent systems and their superiority to traditional monolithic systems in terms of reliability, flexibility, and adaptability to unknown dynamic environments have been extensively investigated [1], [2], [3]. In particular, control and coordination of this type of system have received a great deal of interest in recent years [4], [5], [6], [7]. Cooperative control of multi-agent systems has a broad range of applications including formation flying of multiple unmanned aerial, ground, and underwater vehicles. Due to the importance of information sharing in the coordination of a group of agents, the information flow structure of the system must be taken into consideration in any control design algorithm [8], [9], [10].

Graph-theoretic techniques, on the other hand, are effective tools for the analysis of multi-agent systems. Such tools are often employed to analyze a number of related problems such as consensus, rendezvous, flocking, containment and leader-follower formation control, to name only a few [11], [12], [13], [14], [15]. The controllability problem in leader-follower multi-agent systems was first introduced in [16], where the classical notion of controllability was studied for a leader-based multi-agent system. Necessary and sufficient conditions were subsequently derived for controllability of the system in terms of the eigenvalues and eigenvectors of a sub-matrix of the graph's Laplacian. It was also substantiated in [16] that increasing the size of the information flow graph would not necessarily improve the controllability of the system. In [17], it was shown that a leader-symmetric interconnection network is uncontrollable.

Network equitable partitions were introduced in [18] to present a new necessary condition for the controllability of a multi-agent system. Using this notion, the controllability characterization methods were extended to the multiple-leader case [19]. More recently, the notion of *relaxed equitable partitions* was introduced in [20] to provide a graph-theoretic interpretation for the controllability subspace when the network is not completely controllable. The controllability of a single-leader multi-agent system under fixed and switching topologies for both continuous-time and discrete-time cases was studied in [21], [22]. It was shown in these papers that the controllability of the overall system does not require that the network be controllable for a fixed topology. A graphical characterization of the structural controllability for high-order multi-agent systems was given in [23].

In [24], the structural controllability, as opposed to the controllability of a fixed model, is studied for a single-leader multi-agent system. Then, the notions of p-link and q-agent controllability are introduced as quantitative measures for the structural controllability of a system subject to failure in the communication links and agents. Topology-based necessary and sufficient conditions are also given in [24] for controllability preservation under such failures. Polynomial-time algorithms are subsequently provided to find the maximum number of such failures for which the system remains structurally controllable. The results of [24] are very useful in the controllability analysis of a single-leader configuration; however, many real-world multi-agent system applications require more than one leader to ensure controllability (the reader is referred to [25], [26], [27] for some relevant problems).

In this paper, the structural controllability of a leader-follower multi-agent system with multiple leaders is investigated. A necessary and sufficient condition is provided, which extends the results of [24] to the multiple leader case. A necessary and sufficient condition is also provided in this case, to ensure the controllability of the system when the communication links between the agents are subject to failure. This gives a quantitative measure for the reliability of the multi-agent system with respect to its communication links, and can analogously be extended to the case of the agent loss. The problem of leader selection is subsequently introduced, where it is desired to determine the minimum number of leaders (to be selected from agents) for which the system maintains a certain degree of controllability. This helps the designer to improve the reliability and efficiency of a multi-agent system by properly assigning the leaders in the system.

This paper is organized as follows: Section II provides some preliminaries from graph theory, and introduces the notation used throughout the paper. The problem is defined

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in detail and is subsequently formulated in Section III. Sections IV and V contain the main results of the paper, and finally the concluding remarks are summarized in Section VI.

II. Preliminaries

This section provides a background to the problem under study in this paper. First, some useful concepts of graph theory are introduced, and then the notions of structured systems and structural controllability are presented.

A. Basic Concepts from Graph Theory

Throughout this paper, the set of integers $\{1, 2, \ldots, k\}$ is denoted by $\mathbb{N}_k$. The difference of the set X and Y which is the set containing those elements of X that do not belong to Y is denoted by $X\backslash Y$. The size of a set X is the number of its elements, and is represented by $|X|$. The ith member of an ordered set X is denoted by $X(i)$. Two sets X and Y are *intersecting* if the sets $X\backslash Y$, $Y\backslash X$, and $X \cap Y$ are all nonempty. A *directed graph* or *digraph* G is defined by a set of vertices $V = \{1, \ldots, n\}$ and a set of edges $E \subseteq V \times V$, and is represented by $G = (V, E)$. An edge of G is denoted by $e_{ij} := (i, j) \in E$, which is a directed arc from vertex i to vertex j. In such an ordered pair, the first vertex i is called a *tail* and the second vertex j is called a *head*. A *self-loop* $e_{ii} = (i, i)$ is an edge connecting vertex i to itself. Two edges are *anti-parallel* if the head/tail of one is the tail/head of the other. The set of all neighbors of vertex i is defined as $N_i := \{j \mid e_{ji} \in E\}$. The size of the vertex set and the edge set of a graph are called the *order* and *size* of the graph, respectively. A sequence of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$ is referred to as an $i_1 i_k$*-path* ($i_j \in V$, $j \in \mathbb{N}_k$), and the *parent function* of this path is defined as $\zeta(i_j) = i_{j-1}$, for any $j \in \mathbb{N}_k\backslash\{1\}$. The vertex i_1 in the above path is called the *origin* or *root*, and the vertex i_k is called the *end* of this path. Two paths are called *disjoint* if they consist of disjoint sets of vertices. An R*-rooted path* is a path whose origin is in the set $R \subset V$; the set R associated with such a path is called the *root set*. A vertex i is called *reachable* from the set R if there exists an R-rooted path whose end is the vertex i. A group of mutually disjoint R-rooted paths is called an R*-rooted path family*. A closed path consisting of distinct vertices is called a *cycle*. A set of disjoint cycles is called a *cycle family*. The *length* of a path or a cycle is the number of its edges (excluding self-loop edges). The set of all edges of G entering $X \subseteq V$ is denoted by $\partial_G^-(X)$, and is called the *incut* of X. The set of all edges of G leaving X, on the other hand, is denoted by $\partial_G^+(X)$, and is referred to as the *outcut* of X. The size of the incut and outcut associated with X are denoted by $d_G^-(X)$ and $d_G^+(X)$, and are called the *indegree* and *outdegree* of X, respectively. For two disjoint sets $X, Y \subset V$, let $\partial_{G_Y}^-(X) \subseteq \partial_G^-(X)$ be the set of all edges of G whose tails lie in Y and whose heads lie in X; denote the size of this set with $d_{G_Y}^-(X)$.

An *undirected graph* is a graph whose edges are all undirected (represented by plain lines). Throughout this paper, an undirected graph (and its edge set) will be distinguished by a bar over the symbol. Furthermore, given an undirected graph $\bar{G}$, its directed counterpart will be represented by $\vec{G}$, which is a digraph obtained by replacing every edges of $\bar{G} = (V, \bar{E})$ with a pair of anti-parallel directed edges.

B. Structured Systems and Structural Controllability

A matrix is called *structured* if its entries are either fixed zeros or independent free parameters [28]. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ be two structured matrices. A linear time-invariant (LTI) system whose state-space equation in the standard form is described by the structured pair (A, B) is called a *structured system*. The m-input, n-dimensional structured system $\mathbf{S}$ defined by the pair (A, B) can be represented by a digraph $G_\mathbf{S}$ with $n + m$ vertices, where the ijth entry of matrix $[A \mid B]$ corresponding to a nonzero parameter is associated with a directed edge from vertex j to vertex i. A structured system (A, B) is said to be *structurally controllable* if its free parameters can be set to some particular values such that the system is controllable.

III. Problem Statement

Consider a team of n agents, and let $x_i(t)$ and $u_i(t)$ represent the state and control input of agent i, respectively. Assume that the dynamics of each agent is given by a single integrator, i.e. $\dot{x}_i(t) = u_i(t)$, for each $i \in \mathbb{N}_n$. The interaction structure between the agents is specified by a given static information flow graph $G = (V, E)$ of order n, in which each vertex corresponds to an agent. There is a directed edge from vertex i to vertex j, if agent i transmits its state to agent j. Assume that some of the agents, say the last m agents, act as the leaders and are influenced by external control inputs denoted by $u_i(t) = u_{\text{ext}}^i(t)$, $i \in \mathbb{N}_n\backslash\mathbb{N}_{n-m}$, enabling them to move without any constraint. The set of vertices of G corresponding to the leaders is called the *root set*, and is denoted by R. The rest of the agents, called followers, are governed by the following control law

$$u_i(t) = \sum_{j \in N_i \cup \{i\}} \alpha_{ij} x_j(t), \qquad i \in \mathbb{N}_{n-m} \tag{1}$$

where the coefficients $\alpha_{ij} \in \mathbb{R}$ are fixed. The state of each agent is defined to be its absolute position. Throughout the paper, it is assumed that the agent dynamics is decoupled along each dimension, allowing for one-dimensional state representation for each agent.

Definition 1: [24] The information flow graph G is called *controllable* if one can choose α_{ij}'s in (1) in such a way that by moving the leaders properly, the followers can take any desired configuration.

Under the control law (1), the dynamics of the followers can be described as

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2}$$

where $x(t) = [x_1(t) \ \ldots \ x_{n-m}(t)]^T \in \mathbb{R}^{n-m}$, $u(t) = [x_{n-m+1}(t) \ \ldots \ x_n(t)]^T \in \mathbb{R}^m$ are the state and control input, respectively, and A, B are structured matrices of proper dimensions. Thus, the state equation (2) describes a structured system whose structural controllability is equivalent to the controllability of the underlying information flow graph G.

It is to be noted that although only the dynamics of the followers are considered in (2), the controllability of the corresponding information flow graph implies that all agents (including the leaders) can reach any desired position by a proper choice of the external input for the leaders.

In this paper, it is aimed to determine a minimum number of leaders required to achieve structural controllability. This problem, which is hereafter referred to as the *leader selection problem*, is addressed in the general case of a multi-agent system subject to failure in some communication links or loss of some agents.

IV. Structural Controllability of Multi-Agent Systems

In [24], the controllability of the information flow graph of a single-leader multi-agent system is studied. This section aims to extend the results of [24] to a multiple leader setting, that is when more than one agent can act as leaders. Consider a structured system with the state-space representation of the form (2). The following theorem is borrowed from [29], and provides necessary and sufficient conditions for the structural controllability of the system in terms of its digraph.

Theorem 1: [29] A structured system $\mathbf{S}$ of the form (2) with the digraph $G_{\mathbf{S}} = (V_{\mathbf{S}}, E_{\mathbf{S}})$ is structurally controllable if and only if both of the following conditions hold:

i) Every vertex in $G_{\mathbf{S}}$ is the end vertex of an R-rooted path.

ii) There exists a disjoint union of an R-rooted path family and a cycle family that covers all vertices.

where $R \subset V_{\mathbf{S}}$ is the set of vertices corresponding to the columns of the matrix B.

The following theorem provides a necessary and sufficient condition for the controllability of an information flow graph.

Theorem 2: The information flow graph G is controllable if and only if each vertex in $V \backslash R$ is reachable from the root set R.

Proof: The reachability of each vertex of the set $V \backslash R$ from the root set R is equivalent to each member of the above set being the end vertex of an R-rooted path in $G_{\mathbf{S}}$. The vertices in R can be considered as R-rooted paths of length zero, and a self-loop on each vertex $x \in V \backslash R$ constructs a cycle family whose union with these zero-length R-rooted paths span the vertex set $V = V_{\mathbf{S}}$. The proof follows now from Theorem 1. □

Definition 2: [24] The information flow graph G is said to be *p-link controllable* if p is the largest number for which the controllability of G is preserved after removing any group of at most $p-1$ edges.

In a p-link controllable digraph, p is the minimum number of edges whose removal makes the digraph uncontrollable. For a digraph G with the root set R, this number will hereafter be denoted by $lc(G;R)$, and will be referred to as the *link-controllability degree* of G. Define $lc(G;\emptyset) = 0$ and $lc(G;V) = \infty$, and let $lc(G,x;R)$ be the minimum number of edges whose removal makes the vertex $x \in V \backslash R$ unreachable from the root set R. Clearly, $lc(G;R) = \min_{x \in V \backslash R} lc(G,x;R)$. It is important to note that $lc(G;R)$ is, in fact, a quantitative measure for the reliability of the multi-agent system w.r.t. communication failure. The following theorem gives a necessary and sufficient condition for p-link controllability based on the information flow graph G.

Theorem 3: The information flow graph $G = (V, E)$ with the root set R is p-link controllable if and only if

$$\mathop{min}_{R \subseteq X \subset V} d_G^+(X) = p.$$

Proof: It is clear from the definition of outcut that removing the set $\partial_G^+(X)$ from the edge set E for every $X \subset V$ with $R \subseteq X$ makes the set $V \backslash X$ unreachable from R. On the other hand, suppose that F is the minimal set of edges whose removal makes at least one of the vertices unreachable from R. Let X_F be the set of reachable vertices from R after removing those edges which belong to F. The proof follows now on noting that F includes all members of the outcut of X_F, i.e., $\partial_G^+(X_F) \subseteq F$. □

One can use Theorem 3 to find the value of $lc(G;R)$ in a given digraph G. However, calculating the outdegree of all possible subsets of the vertex set V takes exponential time, and hence is intractable for high-order digraphs. Therefore, it is desired to develop a polynomial-time algorithm to find this value for any digraph.

To find the value of $lc(G,x;R)$ for an arbitrary vertex $x \in V$, let a new digraph $G' = (V', E')$ be constructed from G by extending the sets V and E as follows: Consider a new vertex r, and define $V' = V \cup \{r\}$ and $E' = E \cup \{(r,i),\ \forall i \in R\}$. The digraph G' will be referred to as the expanded digraph of G w.r.t. R. As an illustrative example, Fig. 1(a) shows a digraph G with the root set $R = \{4,5\}$, and Fig. 1(b) demonstrates how the digraph G' is constructed from G.

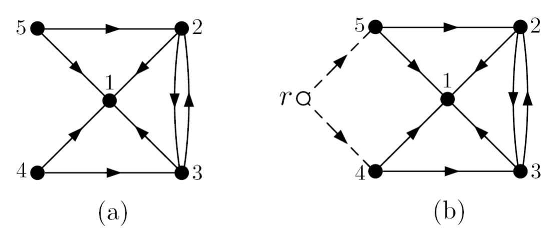

Fig. 1. (a) An information flow graph G, and (b) the corresponding expanded digraph G' w.r.t. $R = \{4,5\}$.

Consider the expanded digraph $G' = (V', E')$ corresponding to a given digraph G and the root set R, and let $x \in V' \backslash \{R \cup \{r\}\}$ be a specified vertex of G'. Construct a new digraph G'_{new} by reversing the direction of all edges of any rx-path, except for those edges which belong to $\{r\} \times R$, if any. Repeat the same procedure for G'_{new} and continue until a digraph G'_{final} is obtained in which x is unreachable from the root r. Denote with $X_{r,G'}$ the set of all reachable vertices from r in G'_{final} (note that $X_{r,G'} \subset V'$).

Theorem 4: The outcut of $X_{r,G'}$ in G' is a minimal set whose removal makes the vertex $x \in V' \backslash \{R \cup \{r\}\}$ (defined above) unreachable from R; in particular, $d^+_{G'}(X_{r,G'}) = lc(G,x;R)$.

Proof: The proof is similar to that of Theorem 4 in [24], and is omitted here. □

One can use Theorem 4 to develop a polynomial-time procedure for finding the value of $lc(G,x;R)$. The following algorithm is presented for this purpose.

Algorithm 1:

$H = G'$.
Main: $X = \{r\}$ and $\zeta(j) = \emptyset$ $(\forall j \in V')$.
    while $\exists\, e_{zy} \in \partial^+_H(X)$,
        $X = X \cup \{y\}$ and $\zeta(y) = z$.
    end while
    if $x \in X$,
        In H, reverse the direction of all edges in the rx-path obtained by using the parent function ζ, except the paths of the form (r,i), $i \in R$, and then go to *Main*.
    end if
$lc(G,x;R) = d^+_{G'}(X)$.
return $lc(G,x;R)$.

As an example, applying the above algorithm to the digraph shown in Fig. 1 yields $lc(G,1;R) = 4$, $lc(G,2;R) = 2$, and $lc(G,3;R) = 2$; hence, for that digraph $lc(G;R) = 2$.

Remark 1: Analogously to the problem of p-link controllability, one can define the problem of q-agent controllability, which is concerned with the controllability preservation after the failure of at most $q-1$ agents [24]. This problem can be converted to the problem of q-link controllability by using the node-duplication technique as discussed in [24].

V. Leader Selection

In general, any problem concerned with finding an optimal set of sources in a network to achieve certain requirements is called the *source location problem* [25]. Various types of this problem with different objectives are investigated in the literature [30], [31], [26], [27]. In [32], a source location problem with edge-connectivity requirement is studied. This section investigates a special case of the source location problem for multi-agent systems, which will be referred to as the *leader selection* problem. To this end, some important ideas are borrowed from [32].

Leader selection in a multi-agent system deals with the problem of finding a minimum number of agents, which if selected as leaders, the overall system becomes structurally controllable. It is desired in the sequel to find a vertex set $R \subseteq V$ of the smallest size such that the information flow graph G is at least p-link controllable, for a given p. This can be formulated in the framework of the q-agent controllability problem discussed in Remark 1.

Definition 3: Given a digraph $G = (V,E)$, a set $X \subset V$ is called *p-deficient* if $d^-_G(X) < p$. A p-deficient set is minimal if none of its proper subsets is p-deficient. As an example, Fig. 2 shows a digraph with two minimal 2-deficient sets $X_1 = \{1,2,3\}$ and $X_2 = \{5\}$.

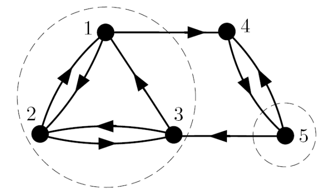

Fig. 2. A digraph with two minimal 2-deficient sets.

Definition 4: The set $R \subseteq V$ is called a *p-link root set* if $lc(G;R) \geq p$.

Theorem 5: Given a digraph $G = (V,E)$, a set $R \subseteq V$ is a p-link root set if and only if any p-deficient set X in G intersects R.

Proof: Assume X is a p-deficient set disjoint from R. Since $d^-_G(X) < p$, hence for every vertex $x \in X$, $lc(G,x;R) < p$. This contradicts the initial assumption that R is a p-link root set. Consider now a set R $(R \subseteq V)$ for which $lc(G;R) < p$, and assume R intersects any p-deficient set of G. According to Theorem 3, there exists a set $X \subset V$ with $R \subseteq X$, such that $d^+_G(X) < p$, or equivalently $d^-_G(V\backslash X) < p$. This means that $V\backslash X$ is a p-deficient set disjoint from R, which contradicts the assumption that R intersects any p-deficient set of G. This contradiction completes the proof. □

One can deduce from Theorem 5 that a set R is a p-link root set if and only if any minimal p-deficient set intersects R. Using this result, it is desired to present a polynomial-time procedure to solve the leader selection problem in a network with bidirectional communication links, represented by the information flow graph $\bar{G} = (V,\bar{E})$. To this end, certain properties of undirected graphs (which do not apply to directed graphs, in general) will be discussed in the sequel.

Lemma 1: Let X and Y be two intersecting subsets of V and define $Z = V\backslash\{X \cup Y\}$; then
$d^-_{\vec{G}}(X) + d^-_{\vec{G}}(Y) = d^-_{\vec{G}}(X\backslash Y) + d^-_{\vec{G}}(Y\backslash X) + 2d^-_{\vec{G}_Z}(X\cap Y)$.

Proof: It is straightforward to show that

$$\begin{aligned}
d^-_{\vec{G}}(X) &= d^-_{\vec{G}_Z}(X\backslash Y) + d^-_{\vec{G}_{Y\backslash X}}(X\backslash Y) \\
&\quad + d^-_{\vec{G}_Z}(X\cap Y) + d^-_{\vec{G}_{Y\backslash X}}(X\cap Y), \\
d^-_{\vec{G}}(Y) &= d^-_{\vec{G}_Z}(Y\backslash X) + d^-_{\vec{G}_{X\backslash Y}}(Y\backslash X) \\
&\quad + d^-_{\vec{G}_Z}(X\cap Y) + d^-_{\vec{G}_{X\backslash Y}}(X\cap Y), \\
d^-_{\vec{G}}(X\backslash Y) &= d^-_{\vec{G}_Z}(X\backslash Y) + d^-_{\vec{G}_{Y\backslash X}}(X\backslash Y) + d^-_{\vec{G}_{X\cap Y}}(X\backslash Y), \\
d^-_{\vec{G}}(Y\backslash X) &= d^-_{\vec{G}_Z}(Y\backslash X) + d^-_{\vec{G}_{X\backslash Y}}(Y\backslash X) + d^-_{\vec{G}_{X\cap Y}}(Y\backslash X).
\end{aligned} \tag{3}$$

It can also be easily shown that $d^-_{\vec{G}_{Y\backslash X}}(X\cap Y) = d^-_{\vec{G}_{X\cap Y}}(Y\backslash X)$, and that $d^-_{\vec{G}_{X\backslash Y}}(X\cap Y) = d^-_{\vec{G}_{X\cap Y}}(X\backslash Y)$. Now, the proof follows directly from (3). □

Theorem 6: Given a graph $\bar{G} = (V, \bar{E})$, all minimal p-deficient sets of its directed counterpart $\vec{G}$ are pairwise disjoint.

Proof: Let $X_1, X_2 \subset V$, $X_1 \neq X_2$ be two minimal p-deficient sets, and assume $X_1 \cap X_2 \neq \emptyset$. It follows from the definition of a minimal p-deficient set that X_1 and X_2 are intersecting. On the other hand, since $X_1 \backslash X_2 \subset X_1$ and $X_2 \backslash X_1 \subset X_2$, hence the sets $X_1 \backslash X_2$ and $X_2 \backslash X_1$ are not p-deficient. From Lemma 1

$$d_{\vec{G}}^{-}(X_1) + d_{\vec{G}}^{-}(X_2) = d_{\vec{G}}^{-}(X_1 \backslash X_2) + d_{\vec{G}}^{-}(X_2 \backslash X_1) + 2d_{\vec{G}_Z}^{-}(X_1 \cap X_2)$$

where $Z := V \backslash \{X_1 \cup X_2\}$. The facts that $d_{\vec{G}}^{-}(X_1) < p$, $d_{\vec{G}}^{-}(X_2) < p$, $d_{\vec{G}}^{-}(X_1 \backslash X_2) \geq p$, and $d_{\vec{G}}^{-}(X_2 \backslash X_1) \geq p$ imply $d_{\vec{G}_Z}^{-}(X_1 \cap X_2) < 0$. This result is not true because the indegree of a set cannot be negative. This contradiction completes the proof. $\square$

The result of the above theorem is not valid for a general digraph. In other words, the minimal p-deficient sets of a digraph are not mutually disjoint in general. For example, the digraph shown in Fig. 3 has two intersecting minimal 2-deficient sets $X_1 = \{1, 2, 3\}$ and $X_2 = \{1, 4, 5\}$.

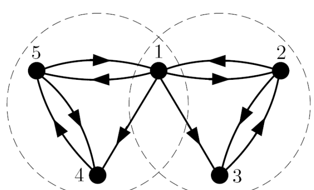

Fig. 3. An information flow graph with two intersecting minimal 2-deficient sets.

Theorems 5 and 6 imply that in an undirected information flow graph $\bar{G}$, a minimal p-link root set contains one vertex from each minimal p-deficient set of $\vec{G}$. One can use the result of Theorem 6 to find a minimal p-link root set in any undirected information flow graph without explicitly identifying the minimal p-deficient sets. This is spelled out in the next theorem. Note first that it is straightforward to show that $lc(\bar{G}, x; R) = lc(\vec{G}, x; R)$, for any $R \subseteq V$ and $x \in V \backslash R$.

Theorem 7: Given an undirected graph $\bar{G}$, let R be a p-link root set of its directed counterpart $\vec{G}$. For a vertex $x \in R$, if $lc(\vec{G}, x; R\backslash\{x\}) \geq p$, then $R\backslash\{x\}$ is a p-link root set as well. Moreover, if $lc(\vec{G}; R\backslash\{x\}) < p$, then there exists a minimal p-deficient set X whose only common element with R is x.

Proof:
Case *i*) Assume that $lc(\vec{G}, x; R\backslash\{x\}) \geq p$. Since R is a p-link root set, it intersects any p-deficient set. It can be shown that either $R\backslash\{x\}$ intersects any p-deficient set too, or there exists a p-deficient set X with $x \in X$, such that X is disjoint from $R\backslash\{x\}$. This implies that $lc(\vec{G}, x; R\backslash\{x\}) \leq d_{\vec{G}}^{-}(X) < p$ which is a contradiction. The proof in this case follows from Theorem 5.

Case *ii*) Assume now that $lc(\vec{G}; R\backslash\{x\}) < p$. This implies that $R\backslash\{x\}$ is not a p-link root set. Since R is a p-link root set, x should belong to a minimal p-deficient set in order for $R\backslash\{x\}$ not to intersect any minimal p-deficient set. Clearly, $R \cap X \neq \emptyset$ and $(R\backslash\{x\}) \cap X = \emptyset$, which completes the proof. $\square$

Theorem 7 is used next to develop a polynomial-time procedure for finding a minimal p-link root set. It is to be noted that the minimal p-link root set is not necessarily unique.

Algorithm 2:

$R = V$.
for $i = 1$ to n,
    if $lc(\vec{G}, i; R\backslash\{i\}) \geq p$,
        $R = R\backslash\{i\}$.
    end if
end for
return R.

Consider the graph depicted in Fig. 4. Applying the above algorithm to this example, one arrives at the set $R = \{1, 4\}$ as the minimal 3-link root set.

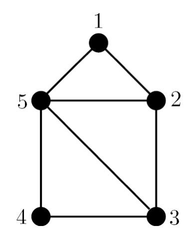

Fig. 4. An undirected information flow graph.

VI. Conclusions and Future Work

The structural controllability of a team of single integrator agents is investigated in this work. A leader-follower formation configuration is considered, where multiple agents can simultaneously act as leaders. Necessary and sufficient conditions for the system to remain structurally controllable in the case of failure of some of the communication links are derived in terms of the topology of the information flow graph. Then the problem of leader selection in undirected information flow graphs is investigated, where it is aimed to find a minimal set of agents that if selected as leaders, the resultant information flow graph is p-link controllable. Analogous results can also be developed for the case of agents loss. Future research is planned to extend the results obtained here to directed information flow graphs.

Acknowledgments

This work has been supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC) under Grant STPGP-364892-08, and in part by Motion Metrics International Corp.

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