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The Geometry of a Probabilistic Consensus of Opinion Algorithm

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Abstract— **We consider the problem of a group of agents whose objective is to asymptotically reach agreement of opinion. The agents exchange information subject to a communication topology modeled by a time varying graph. The agents use a probabilistic algorithm under which at each time instant an agent updates its state by probabilistically choosing from its current state/opinion and the ones of its neighbors. We show that under some minimal assumptions on the communication topology (infinitely often connectivity and bounded intercommunication time between agents), the agents reach agreement with probability one. We show that this algorithm has the same geometric properties as the linear consensus algorithm in $\mathbb{R}^n$. More specifically, we show that the probabilistic update scheme of an agent is equivalent to choosing a point from the (generalized) convex hull of its current state and the states of its neighbors; convex hull defined on a particular convex metric space where the states of the agents live and for which a detailed description is given.**

I. Introduction

A consensus problem consists of a group of dynamic agents who seek to agree upon certain quantities of interest by exchanging information among themselves according to a set of rules. This problem can model many phenomena involving information exchange between agents such as cooperative control of vehicles, formation control, flocking, synchronization, parallel computing, agreeing on a common decision, etc. Distributed computation over networks has a long history in control theory starting with the work of Borkar and Varaiya [1], Tsitsiklis, Bertsekas and Athans [25], [26] on asynchronous agreement problems and parallel computing. A theoretical framework for solving consensus problems was introduced by Olfati-Saber and Murray in [18], [19], while Jadbabaie et al. studied alignment problems [5] for reaching an agreement. Relevant extensions of the consensus problem were done by Ren and Beard [16], by Moreau in [11] or, more recently, by Nedic and Ozdaglar in [14], [13].

In the following, we first introduce a probabilistic algorithm which ensures convergence to the same opinion with probability one and which can be applied to the scenario discussed above. Under this algorithm the state of an agent is updated by choosing probabilistically from the set of current states/opinions of the agent and its neighbors. Initially, we prove the convergence of this algorithm by using purely probability theory arguments. Second, we address the geometric properties of the aforementioned algorithm and show that it has similar geometric properties as the linear consensus algorithm in $\mathbb{R}^n$ [2],[25],[26]. We introduce the notion of *convex metric space*, which is a metric space endowed with a convex structure. Using the convex structure, the notion of a convex hull is generalized. Next we refer to a result about the generalized asymptotic consensus problem [8], which states that if an agent updates its state by choosing as new state a point from (a subset of) the (generalized) convex hull of the current state of the agent and the states of its neighbors, then asymptotic agreement is guaranteed. We show that the probabilistic algorithm for reaching consensus of opinion fits the framework of the generalized asymptotic consensus problem for a particular convex metric space (with a particular convex structure). More specifically, choosing with some probability as new state an opinion from the current states of its own and its neighbors, an agent in fact chooses a point from the convex hull of its current state and the states of its neighbors, belonging to the particular metric space. We define the convex structure for this convex metric space and in particular we give a detailed description of the convex hull of a finite set of points. We end our analysis by taking a closer look on the probabilistic convergence properties of the agreement algorithm.

The paper is organized as follows. In Section II we formulate the problem studied in this paper and present the main assumptions. In Section III we introduce a probabilistic consensus of opinion algorithm together with a proof of convergence, using probability theory arguments. In Section IV we present a generalization of the consensus problem on convex metric spaces together with a general convergence result. In Section V we make use of the machinery introduced in the previous section to study the geometric properties of the probabilistic consensus of opinion algorithm.

Some basic notations: Given $W \in \mathbb{R}^{n\times n}$ by $[W]_{ij}$ we refer to the (i, j) element of the matrix. The *underlying graph* of W is a graph of order n for which every edge corresponds to a non-zero, non-diagonal entry of W. We will denote by $\mathbb{1}_{\{A\}}$ the indicator function of event A. Given some space $\mathcal{X}$ we denote by $\mathcal{P}(\mathcal{X})$ the set of all subsets of $\mathcal{X}$.

II. Problem formulation

We assume that the objective of a group of n agents indexed by i is to agree on an opinion. We model the set of opinions by a finite set of distinct integers, say $S = \{1, 2, \dots, s\}$ for some positive integer s, where each

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element of S refers to a particular opinion. The goal of the agents is to reach the same opinion by repeatedly exchanging messages among themselves. By denoting the time-index by k, the message exchanges are performed subject to a communication topology modeled by a time varying graph $G(k) = (V, E(k))$, where V represents the finite set of vertices (agents) and $E(k)$ represents the time varying set of edges. If an arc from node j to node i exits at time k, then this means that agent j sends a message to agent i.

Similar to the communication models used in [26], [2], [12], we impose minimal assumptions on the connectivity of the communication graph $G(k)$. Basically, these assumptions consist of having the communication graph connected *infinitely often* and having *bounded intercommunication interval* between neighboring nodes.

Assumption 2.1 (Connectivity): The graph (V, E_∞) is connected, where E_∞ is the set of edges (i, j) representing agent pairs communicating directly infinitely many times, i.e.,

$$E_\infty = \{(i, j) \mid (i, j) \in E(k) \text{ for infinitely many indices } k\}$$

Assumption 2.2 (Bounded intercommunication interval): There exists an integer $B \geq 1$ such that for every $(i, j) \in E_\infty$ agent j sends his/her information to the neighboring agent i at least once every B consecutive time slots, i.e. at time k or at time $k+1$ or … or (at latest) at time $k+B-1$ for any $k \geq 0$.

Assumption 2.2 is equivalent to the existence of an integer $B \geq 1$ such that

$$(i, j) \in E(k) \cup E(k+1) \cup \ldots \cup E(k+B-1),\ \forall (i, j) \in E_\infty \text{ and } \forall k.$$

We denote by $\mathcal{X}$ the space of discrete random variables defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, taking values in the set S. We denote by $X_i(k)$ the state of agent i at time k, which are random processes defined on $(\Omega, \mathcal{F}, \mathcal{P})$.

Let $\mathcal{N}_i(k)$ denote the communication neighborhood of agent i, which contains all nodes sending information to i at time k, i.e. $\mathcal{N}_i(k) = \{j \mid e_{ji}(k) \in E(k)\} \cup \{i\}$, which by convention contains the node i itself. We denote by $A_i(k) \triangleq \{X_j(k), j \in \mathcal{N}_i(k)\}$ the set of the states of agent i's neighbors (its own included), and by $A(k) \triangleq \{X_i(k), i = 1 \ldots n\}$ the set of all states of the agents.

We introduce the following definitions of consensus/agreement.

Definition 2.1: We say that the agents reach agreement (a)

1) in *probability*, if

$$\lim_{k\to\infty} Pr\left(\max_{i\neq j} |X_i(k) - X_j(k)| = 0\right) = 1.$$

2) in *almost sure sense* (with probability one), if

$$Pr\left(\lim_{k\to\infty} \max_{i\neq j} |X_i(k) - X_j(k)| = 0\right) = 1.$$

In the following section we introduce a probabilistic algorithm which, under Assumptions 2.1 and 2.2 ensures convergence to agreement in the sense of Definition 2.1. In addition, the geometric properties of this algorithm will be investigated.

III. Probabilistic Consensus of Opinion Algorithm

In this section we introduce a simple probabilistic algorithm which ensures convergence to agreement in the sense of Definition 2.1. Using purely probability theory arguments, we prove the convergence of this algorithm for the special case where the communication topology is fixed (time invariant) and connected. However, we later generalized the result to hold under the more general Assumptions 2.1 and 2.2.

Let $\theta_i(k)$, $i = 1 \ldots n$ be a set of independent random processes defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, taking values in the finite set $\{1, 2, \ldots, n\}$. The statistics of $\theta_i(k)$ is known by agent i and is given by $Pr(\theta_i(k) = j) = w_{ij}(k)$, with $\sum_{j=1}^{n} w_{ij}(k) = 1$.

Assumption 3.1: The probability distributions of the random processes $\theta_i(k)$ have the properties:

1) $w_{ij}(k) > 0$ for $j \in \mathcal{N}_i(k)$ and $w_{ij}(k) = 0$ for $j \notin \mathcal{N}_i(k)$, for all i and k;
2) there exists a positive scalar $\underline{\lambda}$ small enough (i.e. much smaller than $1/n$) such that $w_{ij}(k) \geq \underline{\lambda}$ for $j \in \mathcal{N}_i(k)$, for all i and k.

We assume that at each time instant, the agents update their state according to the scheme

$$X_i(k+1) = \sum_{j\in\mathcal{N}_i(k)} \mathbb{1}_{\{\theta_i(k)=j\}} X_j(k),\ \forall i, \tag{1}$$

where the initial states $X_i(0)$'s take values in S with some probability distribution. We note that (1) can be equivalently written as

$$X_i(k+1) = X_j(k) \quad \text{with probability } w_{ij}(k), \tag{2}$$

for all i and $j \in \mathcal{N}_i(k)$. However, (1) exhibits a linearity property in terms of the neighboring states (familiar to the consensus problem in $\mathbb{R}^n$) which is going to be useful in proving some later results.

In what follows, using purely probability theory arguments, we prove that under the update scheme (1), indeed the agents converge to the same opinion in the sense of Definition 2.1. For simplicity, we assume that the communication graph is fixed for all time and connected. We will however prove the convergence results under more general assumptions, but using a different *machinery*.

Proposition 3.1: Let Assumption 3.1 hold and assume that the communication graph $G(k)$ is time invariant and connected. In addition, assume that the coefficients $w_{ij}(k)$ defining the probability distributions of the random process $\theta_i(k)$ are constant for all time instances. If the agents update their states according to the scheme (1), i.e.

$$X_i(k+1) = \sum_{j\in\mathcal{N}_i} \mathbb{1}_{\{\theta_i(k)=j\}} X_j(k), \tag{3}$$

then the agents reach agreement with probability one in the sense of Definition 2.1.

Proof: We define the random process $Z(k) = (X_1(k), X_2(k), \ldots, X_n(k))$ which has a maximum of s^s states and we introduce the *agreement space*

$$\mathcal{A} \triangleq \{(o, o, \ldots, o) \mid o \in S\}.$$

From (3), the probability of $X_i(k+1)$ conditioned on $X_j(k), j \in \mathcal{N}_i$ is given by

$$Pr(X_i(k+1) = o_i | X_j(k) = o_j, j \in \mathcal{N}_i) = \sum_{j \in \mathcal{N}_i} w_{ij} \mathbb{1}_{\{o_i = o_j\}}. \quad (4)$$

It is not difficult to note that $Z(k)$ is a finite state, homogeneous Markov chain. We will show that $Z(k)$ has s absorbing states and all other $s^s - s$ states are transient, where the absorbing states correspond to the states in the agreement space $\mathcal{A}$. Using the independence of the random processes $\theta_i(k)$, the entries of the probability transition matrix of $Z(k)$ can be derived from (4) and are given by

$$\begin{aligned} &Pr(X_1(k+1) = o_{l_1}, \ldots, X_n(k+1) = o_{l_n} | X_1(k) = o_{p_1}, \ldots \\ &\ldots, X_n(k) = o_{p_n}) = \textstyle\prod_{i=1}^n \sum_{j \in \mathcal{N}_i} w_{ij} \mathbb{1}_{\{l_i = p_j\}}. \end{aligned} \quad (5)$$

We note from (5) that once the process reaches an agreement state it will stay there indefinitely, i.e.

$$\begin{aligned} &Pr(X_1(k+1) = o, \ldots, X_n(k+1) = o | X_1(k) = o, \ldots \\ &\ldots, X_n(k) = o) = 1, \ \forall o \in S, \end{aligned}$$

and hence the agreement states are absorbing states. We will show next that, under the connectivity assumption, the agreement space $\mathcal{A}$ is reachable from any other state, and therefore all other states are transient. We are not saying that all agreement states are reachable from any other state, but that from any state at least one agreement state is reachable. Let $(o_1, o_2, \ldots, o_n) \notin \mathcal{A}$, with $o_j \in S$, $j = 1 \ldots n$ be an arbitrary state. We first note that from this state only agreement states of the form $(o_j, o_j, \ldots, o_j)$ can be reached. Given that $X_j(0) = o_j$, we show that with positive probability the agreement vector $(o_j, o_j, \ldots, o_j)$ can be reached. At time slot one, with probability w_{jj} agent j keeps its initial choice, while its neighbors to which it sends information can choose o_j with some positive probability, i.e. $X_i(1) = o_j$ with probability w_{ij}, for all i such that $j \in \mathcal{N}_i$. Due to the connectivity assumption there exists at least one i such that $j \in \mathcal{N}_i$. At the next time-index all the agents which have already chosen o_j keep their opinion with positive probability, while their neighbors will choose o_j with positive probability. Since the communication network is assumed connected, every agent will be able to choose o_j with positive probability in at most $n-1$ steps, therefore an agreement state can be reached with positive probability. Hence, from any initial state $(o_1, o_2, \ldots, o_n) \notin \mathcal{A}$, all agreement states of the form $(o_j, o_j, \ldots, o_j)$ with $j = 1 \ldots n$ are reachable with positive probability. Since the agreement states are absorbing states, it follows that $(o_1, o_2, \ldots, o_n) \notin \mathcal{A}$ is a transient state. Therefore, the probability for the Markov chain $Z(k)$ to be in a transient state converges asymptotically to zero, while the probability to be in one of the agreement states converges asymptotically to one, i.e.

$$\lim_{k \to \infty} Pr(Z(k) \notin \mathcal{A}) = \lim_{k \to \infty} Pr\left(\bigcup_{i \neq j} \left\{ X_i(k) \neq X_j(k) \right\} \right) = 0. \quad (6)$$

Given an arbitrary $\epsilon > 0$, we define the event

$$B_k(\epsilon) \triangleq \left\{ \omega : \max_{i \neq j} |X_i(k) - X_j(k)| > \epsilon \right\}.$$

But since

$$B_k(\epsilon) = \bigcup_{i \neq j} \left\{ |X_i(k) - X_j(k)| > \epsilon \right\} \subseteq \bigcup_{i \neq j} \left\{ X_i(k) \neq X_j(k) \right\},$$

from (6) it follows that

$$\lim_{k \to \infty} Pr(B_k(\epsilon)) \leq \lim_{k \to \infty} Pr\left(\bigcup_{i \neq j} \left\{ X_i(k) \neq X_j(k) \right\} \right) = 0,$$

and hence the agents asymptotically agree in probability sense. In addition, due to the geometric decay toward zero of the probability $Pr(Z(k) \notin \mathcal{A})$, by the Borel-Cantelli Lemma the result follows. ■

In the next sections we study the geometric properties of the consensus of opinion algorithm introduced above, based on the theory of *convex metric spaces*.

IV. The Asymptotic Consensus Problem on Convex Metric Spaces

In this section we present a generalization of the consensus problem on convex metric spaces. We first give a brief introduction on convex metric spaces followed by the results concerning the consensus problem. This theory will be used as framework for studying the geometric properties of the probabilistic consensus of opinion algorithm presented in the previous section. A detailed version of this section with the detailed proofs of the results stated here can be found in [7], [8].

A. Definitions and Results on Convex Metric Spaces

For more details about the following definitions and results the reader is invited to consult [27],[28].

Definition 4.1: Let $(\mathcal{X}, d)$ be a metric space. A mapping $\psi : \mathcal{X} \times \mathcal{X} \times [0,1] \to \mathcal{X}$ is said to be a *convex structure* on $\mathcal{X}$ if

$$d(u, \psi(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y), \quad (7)$$

for all $x, y, u \in \mathcal{X}$ and for all $\lambda \in [0,1]$.

Definition 4.2: The metric space $(\mathcal{X}, d)$ together with the convex structure ψ is called *convex metric space*.

Definition 4.3: Let $\mathcal{X}$ be a convex metric space. A nonempty subset $K \subset \mathcal{X}$ is said to be *convex* if $\psi(x, y, \lambda) \in K$, $\forall x, y \in K$ and $\forall \lambda \in [0,1]$.

We define the set valued mapping $\tilde{\psi} : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ as

$$\tilde{\psi}(A) \triangleq \{\psi(x, y, \lambda) \mid \forall x, y \in A, \forall \lambda \in [0,1]\}, \quad (8)$$

where A is an arbitrary set in $\mathcal{X}$.

In [28] it is shown that, in a convex metric space, an arbitrary intersection of convex sets is also convex and therefore the next definition makes sense.

Definition 4.4: The *convex hull* of the set $A \subset \mathcal{X}$ is the intersection of all convex sets in $\mathcal{X}$ containing A and is denoted by *conv*(A).

Another characterization of the convex hull of a set in $\mathcal{X}$ is given in what follows. By defining $A_m \triangleq \tilde{\psi}(A_{m-1})$ with $A_0 = A$ for some $A \subset \mathcal{X}$, it is shown in [27] that the set sequence $\{A_m\}_{m \geq 0}$ is increasing and $\limsup A_m$ exits, and $\limsup A_m = \liminf A_m = \lim A_m = \bigcup_{m=0}^{\infty} A_m$.

Proposition 4.1 ([27]): Let $\mathcal{X}$ be a convex metric space. The convex hull of a set $A \subset \mathcal{X}$ is given by

$$conv(A) = \lim A_m = \bigcup_{m=0}^{\infty} A_m. \quad (9)$$

It follows immediately from above that if $A_{m+1} = A_m$ for some m, then $conv(A) = A_m$.

For a positive integer n, let $A = \{x_1, \ldots, x_n\}$ be a finite set in $\mathcal{X}$ with convex hull $conv(A)$ and let z belong to $conv(A)$. By Proposition 4.1 it follows that there exists a positive integer m such that $z \in A_m$. But since $A_m = \tilde{\psi}(A_{m-1})$ it follows that there exists $z_1, z_2 \in A_{m-1}$ and $\lambda_{(1,2)} \in [0,1]$ such that $z = \psi(z_1, z_2, \lambda_{(1,2)})$. Similarly, there exists $z_3, z_4, z_5, z_6 \in A_{m-2}$ and $\lambda_{(3,4)}, \lambda_{5,6} \in [0,1]$ such that $z_1 = \psi(z_3, z_4, \lambda_{(3,4)})$ and $z_2 = \psi(z_5, z_6, \lambda_{(5,6)})$. By further decomposing z_3,z_4,z_5 and z_6 and their followers until they are expressed as functions of elements of A and using a graph theory terminology, we note that z can be viewed as the root of a weighted binary tree with leaves belonging to the set A. Each node α (except the leaves) has two children α_1 and α_2, and are related through the operator ψ in the sense $\alpha = \psi(\alpha_1, \alpha_2, \lambda)$ for some $\lambda \in [0,1]$. The weights of the edges connecting α with α_1 and α_2 are given by λ and $1 - \lambda$ respectively.

From the above discussion we note that for any point $z \in conv(A)$ there exists a non-negative integer m such that z is the root of a binary tree of height m, and has as leaves elements of A. The binary tree rooted at z may or may not be a *perfect binary tree*, i.e. a full binary tree in which all leaves are at the same depth. That is because on some branches of the tree the points in A are reached faster than on others. Let n_i denote the number of times x_i appears as a leaf node, with $\sum_{i=1}^{n} n_i \leq 2^m$ and let m_{i_l} be the length of the i_l^{th} path from the root z to the node x_i, for $l = 1 \ldots n_i$. We formally describe the paths from the root z to x_i as the set

$$P_{z,x_i} \triangleq \left\{ \left(\{y_{i_l,j}\}_{j=0}^{m_{i_l}}, \{\lambda_{i_l,j}\}_{j=1}^{m_{i_l}} \right) \mid l = 1 \ldots n_i \right\}, \quad (10)$$

where $\{y_{i_l j}\}_{j=0}^{m_{i_l}}$ is the set of points forming the i_l^{th} path, with $y_{i_l,0} = z$ and $y_{i_l,m_{i_l}} = x_i$ and where $\{\lambda_{i_l,j}\}_{j=1}^{m_{i_l}}$ is the set of weights corresponding to the edges along the paths, in particular $\lambda_{i_l,j}$ being the weight of the edge $(y_{i_l,j-1}, y_{i_l,j})$. We define the aggregate weight of the paths from root z to node x_i as

$$\mathcal{W}(P_{z,x_i}) \triangleq \sum_{l=1}^{n_i} \prod_{j=1}^{m_{i_l}} \lambda_{i_l,j}. \quad (11)$$

It is not difficult to note that all the aggregate weights of the paths from the root z to the leaves $\{x_1, \ldots, x_n\}$ sum up to one, i.e.

$$\sum_{i=1}^{n} \mathcal{W}(P_{z,x_i}) = 1.$$

Next, we define an approximation of the convex hull of a finite set of points.

Definition 4.5: Given a small enough positive scalar $\underline{\lambda} < 1$, we denote by $C_{\underline{\lambda}}(A)$ the sub-set of $conv(A)$ consisting in all points in $conv(A)$ for which there exists at least one tree representation whose aggregate weights are lower bounded by $\underline{\lambda}$, i.e.

$$C_{\underline{\lambda}}(A) \triangleq \{z \mid z \in conv(A), \mathcal{W}(P_{z,x_i}) \geq \underline{\lambda}, \ \forall x_i \in A\}. \quad (12)$$

Remark 4.1: By a *small enough* value of $\underline{\lambda}$ we understand a value such that the inequality $\mathcal{W}(P_{z,x_i}) \geq \underline{\lambda}$ is satisfied for all i. Obviously, for n agents $\underline{\lambda}$ needs to satisfy $\underline{\lambda} \leq \frac{1}{n}$, but usually we would want to choose a value much smaller then $1/n$ since this implies a richer set $C_{\underline{\lambda}}(A)$.

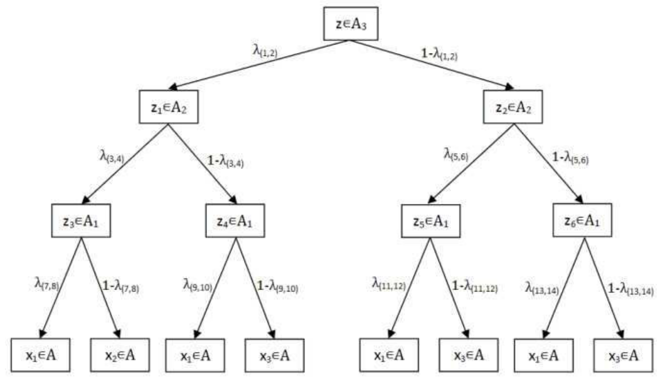

Fig. 1. The decomposition of a point $z \in A_3$ with $A = \{x_1, x_2, x_3\}$

Remark 4.2: We can iteratively generate points for which we can make sure that they belong to the convex hull of a finite set $A = \{x_1, \ldots, x_n\}$. Given a set of positive scalars $\{\lambda_1, \ldots, \lambda_{n-1}\} \in (0,1)$, consider the iteration

$$y_{i+1} = \psi(y_i, x_{i+1}, \lambda_i) \text{ for } i = 1 \ldots n-1 \text{ with } y_1 = x_1. \quad (13)$$

It is not difficult to note that y_n is guaranteed to belong to $conv(A)$. In addition, if we impose the condition

$$\underline{\lambda}^{\frac{1}{n-1}} \leq \lambda_i \leq \frac{1-(n-1)\underline{\lambda}}{1-(n-2)\underline{\lambda}}, \ i = 1 \ldots n-1, \quad (14)$$

and $\underline{\lambda}$ respects the inequality

$$\underline{\lambda}^{\frac{1}{n-1}} \leq \frac{1-(n-1)\underline{\lambda}}{1-(n-2)\underline{\lambda}}, \quad (15)$$

then $y_n \in C_{\underline{\lambda}}(A)$. We should note that for any $n \geq 2$ we can find a small enough value of $\underline{\lambda}$ such that the inequality (15) is satisfied.

B. Consensus algorithm on convex metric spaces

We consider a convex metric space $(\mathcal{X}, d, \psi)$ and a set of n agents indexed by i which take values on $\mathcal{X}$. *The communication model is identical to the one described in Section II.* The Assumptions 2.1 and 2.2 on the communication model and the notations for the communication neighborhoods are kept valid throughout this section, as well. In what follows we denote by $x_i(k)$ the value or *state* of agent i at time k.

Definition 4.6: We say that the agents asymptotically reach *consensus* (or agreement) if

$$\lim_{k\to\infty} d(x_i(k), x_j(k)) = 0, \ \forall i, j, \ i \neq j. \quad (16)$$

The following theorem introduces the generalized asymptotic consensus algorithm on convex metric spaces.

Theorem 4.1: Let Assumptions 2.1 and 2.2 hold for $G(k)$ and let $\underline{\lambda} < 1$ be a positive scalar sufficiently small. If agents update their state according to the scheme

$$x_i(k+1) \in C_{\underline{\lambda}}(A_i(k)), \ \forall i, \quad (17)$$

then they asymptotically reach consensus, i.e.

$$\lim_{k\to\infty} d(x_i(k), x_j(k)) = 0, \ \forall i, j, \ i \neq j. \quad (18)$$

In addition, the decrease of the distances between the state of the agents is (at least) geometric, i.e. there exist two positive constants c and γ, with $\gamma < 1$, such that

$$d(x_i(k), x_j(k)) \leq c\gamma^k,\ \forall i \neq j. \tag{19}$$

The proof of this theorem can be found in Section IV of [8] (proof of Theorem 3.1).

In fact, we can show that not only the distances between the agents' states converge to zero, but the agents' states converge to some fixed value, common for all agents.

Corollary 4.1: Let Assumptions 2.1 and 2.2 hold for $G(k)$ and let $\underline{\lambda} < 1$ be a positive scalar sufficiently small. If agents update their state according to the scheme

$$x_i(k+1) \in C_{\underline{\lambda}}(A_i(k)),\ \forall i, \tag{20}$$

then there exists $x^* \in \mathcal{X}$ such that

$$\lim_{k\to\infty} d(x_i(k), x^*) = 0,\ \forall i. \tag{21}$$

V. The Geometry of the Consensus of Opinion Algorithm

In this section we show that (2) respects the update scheme introduced in Theorem 5.1 for the generalized consensus problem, for a particular metric space and convex structure.

A. Geometric framework

As introduced in Section II, in what follows, by $\mathcal{X}$ we understand the space of discrete random variables taking values in S. We introduce the operator $d: \mathcal{X}\times\mathcal{X} \to \mathbb{R}$, defined as

$$d(X,Y) = E[\rho(X,Y)],$$

where $\rho: \mathbb{R}\times\mathbb{R} \to \{0,1\}$ is the discrete metric, i.e.

$$\rho(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

It is not difficult to note that the operator d can also be written as $d(X,Y) = E[\mathbb{1}_{\{X\neq Y\}}] = Pr(X \neq Y)$, where $\mathbb{1}_{\{X\neq Y\}}$ is the indicator function of the event $\{X \neq Y\}$.

We note that for all $X,Y,Z \in \mathcal{X}$, the operator d satisfies the following properties (a)

1) $d(X,Y) = 0$ if and only if $X = Y$ with probability one,
2) $d(X,Z) + d(Y,Z) \geq d(X,Y)$ with probability one,
3) $d(X,Y) = d(Y,X)$,
4) $d(X,Y) \geq 0$,

and therefore is a metric on $\mathcal{X}$. The space $\mathcal{X}$ together with the operator d define the *metric space* $(\mathcal{X}, d)$.

Let $\theta \in \{1,2\}$ be an independent random variable, with probability mass function $Pr(\theta = 1) = \lambda$ and $Pr(\theta = 2) = 1-\lambda$, where $\lambda \in [0,1]$. We define the mapping $\psi: \mathcal{X}\times\mathcal{X}\times[0,1] \to \mathcal{X}$ given by

$$\psi(X_1, X_2, \lambda) = \mathbb{1}_{\{\theta=1\}}X_1 + \mathbb{1}_{\{\theta=2\}}X_2,\ \forall X_1, X_2 \in \mathcal{X}, \lambda \in [0,1]. \tag{22}$$

Proposition 5.1: The mapping ψ is a convex structure on $\mathcal{X}$.

Proof: For any $U, X_1, X_2 \in \mathcal{X}$ and $\lambda \in [0,1]$ we have

$$\begin{aligned} d(U, \psi(X_1, X_2, \lambda)) &= E[\rho(U, \psi(X_1, X_2, \lambda))] = \\ &= E[E[\rho(U, \psi(X_1, X_2, \lambda))|U, X_1, X_2]] = \\ &= E[E[\rho(U, \mathbb{1}_{\{\theta=1\}}X_1 + \mathbb{1}_{\{\theta=2\}}X_2)]|U, X_1, X_2] = \\ &= E[\lambda\rho(U, X_1) + (1-\lambda)\rho(U, X_2)] = \\ &= \lambda d(U, X_1) + (1-\lambda) d(U, X_2). \end{aligned}$$

■

From the above proposition it follows that $(\mathcal{X}, d, \psi)$ is a *convex metric space*.

The next theorem characterizes the convex hull of a finite set in $\mathcal{X}$.

Theorem 5.1: Let n be a positive integer and let $A = \{X_1, \ldots, X_n\}$ be a set of points in $\mathcal{X}$. Consider an independent random variable θ taking values in the set $\{1,2,\ldots,n\}$ with probability mass function given by $Pr(\theta = i) = w_i$, for some non-negative scalars w_i, with $\sum_{i=1}^n w_i = 1$. Then

$$conv(A) = \left\{ Z \in \mathcal{X} \mid Z = \sum_{i=1}^n \mathbb{1}_{\{\theta=i\}}X_i,\ \forall w_i \geq 0, \sum_{i=1}^n w_i = 1 \right\}. \tag{23}$$

The proof of this result is given in [8] as the proof of Theorem 6.1.

Corollary 5.1: Let n be a positive integer, let $A = \{X_1, \ldots, X_n\}$ be a set of points in $\mathcal{X}$ and let $\underline{\lambda} < 1$ be a positive scalar sufficiently small. Consider an independent random variable θ taking values in the set $\{1,2,\ldots,n\}$ with probability mass function given by $Pr(\theta = i) = w_i$, for some positive scalars w_i, with $\sum_{i=1}^n w_i = 1$. Then

$$C_{\underline{\lambda}}(A) = \left\{ Z \in \mathcal{X} \mid Z = \sum_{i=1}^n \mathbb{1}_{\{\theta=i\}}X_i,\ \forall w_i \geq \underline{\lambda}, \sum_{i=1}^n w_i = 1 \right\}. \tag{24}$$

Proof: Follows immediately from Definition 4.5 and Theorem 5.1. ■

B. Convergence of the Consensus of Opinion Algorithm

In this section we give a general convergence result for the state update scheme (1), by using the machinery introduced in Section IV.

Corollary 5.2: Let Assumptions 2.1 and 2.2 hold for $G(k)$ and let Assumption 3.1 hold for the random processes $\theta_i(k)$. If the agents update their state according to the scheme (1), then the agents converge to consensus with probability one (in the sense of Definition 2.1).

Proof: By Corollary 5.1, from (1) we have that

$$X_i(k+1) \in C_{\underline{\lambda}}(A_i(k)),\ \forall i,k$$

However, by Theorem 4.1, we get that

$$\lim_{k\to\infty} d(X_i(k), X_j(k)) = 0,\ \forall i \neq j.$$

Recall that we defined the distance between two points $X_1, X_2 \in \mathcal{X}$ as

$$d(X_1, X_2) = E[\rho(X_1, X_2)] = Pr(X_1 \neq X_2).$$

Therefore, we have that

$$\lim_{k\to\infty} Pr(X_i(k) \neq X_j(k)) = 0. \tag{25}$$

This says that the measure of the set on which $X_i(k)$ and $X_j(k)$ are different converges to zero as k goes to infinity, i.e. *the agents asymptotically agree in probability sense*. In what follows we show that in fact the agents asymptotically agree with probability one (or in almost sure sense).

Given an arbitrary $\epsilon > 0$, we define the event

$$B_k(\epsilon) \triangleq \{\omega : \max_{i \neq j} |X_i(k) - X_j(k)| > \epsilon\}.$$

An upper bound on the probability of the event $B_k(\epsilon)$ is given by

$$\begin{aligned} Pr(B_k(\epsilon)) = Pr\Big(\bigcup_{i\neq j}\Big\{\omega : |X_i(k) - X_j(k)| > \epsilon\Big\}\Big) \leq \\ \leq \sum_{i\neq j} Pr\Big(|X_i(k) - X_j(k)| > \epsilon\Big) \leq \sum_{i\neq j} Pr\Big(X_i(k) \neq X_j(k)\Big). \end{aligned} \quad (26)$$

From (25) and (26) we obtain

$$\lim_{k\to\infty} Pr(B_k(\epsilon)) = 0.$$

By Theorem 4.1, we have that $d(X_i(k), X_j(k)) = Pr(X_i(k) \neq X_j(k))$, converge at least geometrically to zero. Therefore

$$\sum_{k\geq 0} Pr(B_k(\epsilon)) < \infty,$$

and by the Borel-Cantelli lemma it follows that

$$Pr\left(\lim_{k\to\infty} \max_{i\neq j} |X_i(k) - X_j(k)| = 0\right) = 1.$$

■

Remarkably, it can be shown that $X_i(k)$ converge in distribution to a random variable X^* given by

$$X^* = \sum_{j=1}^{n} \mathbb{1}_{\{\theta^* = j\}} X_j(0),$$

where $Pr(\theta^* = j) = \frac{1}{n}$. Note that X^* is a point in the convex hull of $\{X_1(0), \ldots, X_n(0)\}$ generated by associating equal weights to the initial values $X_j(0)$. Hence, X^* can be interpreted as the *(empirical) average* of the initial values.

VI. Conclusions

In this paper we investigated the geometric properties of a probabilistic consensus of opinion algorithm. We showed that this algorithm is an example of a generalized consensus algorithm defined on convex metric spaces; an algorithm which consists of updating the state of an agent by choosing as new state, a point from the (generalized) convex hull of the agent's current state and the current states of its neighbors. More specifically, we defined the convex metric space and the convex structure underlying the probabilistic of opinion algorithm.

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