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Improved vertex control for time-varying and uncertain linear discrete-time systems with control and state constraints

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***Abstract*— This paper addresses the problem of regulating a discrete-time linear uncertain and time-varying system to the origin. It is shown that, based on an interpolation technique, by minimizing an appropriate objective function, how feasibility and a robustly and asymptotically stable closed-loop behavior can be achieved. It is shown that the control is a piecewise affine and continuous function of the state. Several simulations demonstrate the performance of our results.**

I. INTRODUCTION

The problem of controlling a linear discrete-time system with state and control constraints has seen many solutions. One is vertex control [1], others include implicit and explicit Model Predictive Control (MPC), see e.g. the text books [2], [3]. The vertex control solution was extended to the uncertain plant case by [4] and [5], while MPC is not readily extendable without great conservativeness or on-line computational burden.

A weakness of vertex control is that the full control range is exploited only on the border of the feasible positive invarianr set in the state space, and hence the time to regulate the plant to the origin is much longer than e.g. the time-optimal one. A way to overcome this is to switch to another, more aggressive, local controller, e.g. a state feedback controller $u_o = Kx$ when the state reaches the maximum feasible state set of the local controller. The disadvantage of this solution is that the control action becomes non-smooth [6].

Noting that the vertex control Lyapunov level curves are polyhedra parallel with the border of the vertex control feasible set polyhedron, we postulate a polyhedral feasible set for the local control. Then we suggest a smooth convex interpolation between the vertex control action u_v and the local control action u_o for the current state x, in the form $u = cu_v + (1-c)u_o$, $0 \leq c \leq 1$, whereby c is minimized in order to give maximal control action. We show that with this objective function there exist a Lyapunov function for the system controlled by the interpolated controller u, and hence stability is proved.

It is shown that from a computational point of view the minimization of c can be done by linear programming. In a companion paper it is further shown that that the minimization can be done off-line, yielding a polyhedral partition each with its affine control law. Thus, our controller can be compared with explicit MPC where the feasible set in the state space is also partitioned in polyhedra each of which with its own affine state feedback control law.

The difference between the new improved vertex control and explicit MPC is that while the explicit MPC is optimal with respect to the chosen criterion for one nominal plant case, the improved vertex control is proved to be stable for a given set of plants or for a time-varying plant, and is considerable simpler with much fewer polyhedral subsets and less on-line computational burden.

Our approach could also be compared to the barycentric interpolation in [7] which however yields a non-linear control law over the polytopic partition of the state space.

Several simulated examples illustrate our results.

II. PROBLEM STATEMENT

Consider the problem of regulating to the origin the following discrete-time linear time-varying system:

$$x(t+1) = A(t)x(t) + B(t)u(t) \quad (1)$$

where $x(t) \in R^n$ and $u(t) \in R^m$ are respectively the measurable state and the input, and with given matrices A_i and B_i, the matrices $A(t) \in R^{n\times n}$ and $B(t) \in R^{n\times m}$ satisfy

$$\begin{cases} A(t) = \sum_{i=1}^{s} \alpha_i A_i, B(t) = \sum_{i=1}^{s} \alpha_i B_i, \\ \alpha_i \geq 0, \forall i = 1, \ldots, s, \\ \sum_{i=1}^{s} \alpha_i = 1. \end{cases} \quad (2)$$

Remark 1: $A(t)$ and $B(t)$ given as

$$\begin{cases} A(t) = \sum_{i=1}^{v} \alpha_i A_i, B(t) = \sum_{i=1}^{r} \beta_i B_i, \\ \alpha_i \geq 0, \forall i = 1, \ldots, v, \\ \beta_i \geq 0, \forall i = 1, \ldots, r, \\ \sum_{i=1}^{v} \alpha_i = 1 \text{ and } \sum_{i=1}^{r} \beta_i = 1. \end{cases} \quad (3)$$

may be translated into the form of (2) as follows,

$$\begin{aligned} x(t+1) &= \sum_{i=1}^{v} \alpha_i A_i x(t) + \sum_{j=1}^{r} \beta_j B_j u(t) \\ &= \sum_{i=1}^{v} \alpha_i A_i x(t) + \sum_{i=1}^{s} \alpha_i \sum_{j=1}^{r} \beta_j B_j u(t) \\ &= \sum_{i=1}^{v} \alpha_i (A_i x(t) + \sum_{j=1}^{r} \beta_j B_j u(t)) \\ &= \sum_{i=1}^{v} \alpha_i (\sum_{j=1}^{r} \beta_j A_i x(t) + \sum_{j=1}^{r} \beta_j B_j u(t)) \\ &= \sum_{i=1}^{v} \alpha_i (\sum_{j=1}^{r} \beta_j (A_i x(t) + B_j u(t))) \\ &= \sum_{i=1,j=1}^{v,r} \alpha_i \beta_j (A_i x(t) + B_j u(t)). \end{aligned}$$

Consider the polytope P_c, the vertices of which are given by taking all possible combinations of $\{A_i, B_j\}$ where $i = 1, \ldots, v$ and $j = 1, \ldots, r$. From $\sum_{i=1,j=1}^{v,r} \alpha_i \beta_j = \sum_{i=1}^{v} \alpha_i \sum_{j=1}^{r} \beta_j = 1$, it is clear that $\{A(t), B(t)\}$ can be expressed as a convex combination of the vertices of P_c.

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Both the state vector $x(t)$ and the control vector $u(t)$ are subject to polytopic constraints:

$$\begin{cases} x(t) \in X, X = \{x : F_x x \leq g_x\} \\ u(t) \in U, U = \{u : F_u u \leq g_u\} \end{cases} \forall t \geq 0 \qquad (4)$$

where the matrices F_x, F_u and the vectors g_x, g_u are assumed to be constant with $g_x > 0$, $g_u > 0$ such that the origin is contained in the interior of X and U.

III. Robust linear state feedback synthesis

The aim of this section is to find a linear feedback controller:

$$u(t) = Kx(t) \qquad (5)$$

that robustly asymptotically stabilizes (1) for some interior subset X_0, U of (4), $X_0 \subseteq X$. Then the closed-loop is

$$x(t+1) = (A(t) + B(t)K)x(t) \qquad (6)$$

It is well known that the existence of a common quadratic Lyapunov function, namely of a positive-definite Q^{-1} for the parameterized linear matrix inequalities (LMI), see [8],

$$(A(t) + B(t)K)^T Q^{-1}(A(t) + B(t)K) - Q^{-1} \prec 0 \qquad (7)$$

assures the satisfaction of the inequalities

$$x(0)^T Q^{-1} x(0) > x(1)^T Q^{-1} x(1) > \dots$$
$$> x(t)^T Q^{-1} x(t) > \dots > 0$$

and therefore stability is guaranteed. Here A^T denotes the transpose of matrix A and $P \prec 0 (\succ 0)$ denotes that matrix P is negative definite (positive definite).

The expression $x^T(A(t)+B(t)K)^T Q^{-1}(A(t)+B(t)K)x$ can be treated as a function of t and reaches the maximum on one of the vertices of $A(t), B(t)$ in (2), so the set of LMI conditions to be satisfied to check stability is the following:

$$(A_i + B_i K)^T Q^{-1}(A_i + B_i K) - Q^{-1} \prec 0, i = 1, \dots, s \qquad (8)$$

By pre- and post-multiplying both sides by Q and making the substitution $R = KQ$, one gets

$$(QA_i^T + R^T B_i^T)Q^{-1}(A_i Q + B_i R) - Q \prec 0, i = 1, \dots, s \qquad (9)$$

By applying Schur complements, one obtains

$$\begin{pmatrix} Q & QA_i^T + R^T B_i^T \\ A_i Q + B_i R & Q \end{pmatrix} \succ 0, i = 1, \dots, s \qquad (10)$$

This is an LMI condition. If a solution Q does exist, then

$$K = RQ^{-1} \qquad (11)$$

is a suitable robust control for the unconstrained case.

Remark 2: Condition (10) is necessary and sufficient for a polytopic system to be quadratically stabilizable with linear state feedback control [8].

Remark 3: The results reported by Kothare *et al* [9] can, in principle, be employed to take into account the constraints on control and state (4). However we do not pursue this idea further in this paper and concentrate on a synthesis procedure which allows the constraints to be active.

IV. Invariant set construction

A. Maximal robustly admissible set

For some given K from (11) denote $H(t) = A(t) + B(t)K$, and $H_i = A_i + B_i K, \forall i = 1, \dots, s$.

Definition 1: (Robustly positively invariant set) The set Ω is a positively invariant set with respect to $x(t+1) = H(t)x(t)$ if and only if

$$\forall x \in \Omega \Rightarrow H_i x \in \Omega, \forall i = 1, \dots, s \qquad (12)$$

Definition 2: (Robustly admissible set) The set Ω is a robustly positively admissible set for the system (1) with a feedback controller $u = Kx$ and with respect to constraints (4) if and only if the trajectories $x(t)$ of the system (1), starting from any point $x_0 \in \Omega$ satisfy

$$x(t) \in X, \; Kx(t) \in U \qquad (13)$$

The largest positively invariant admissible set is generally called the maximal admissible set (MAS)[10].

It is well known that the existence of a stabilizing feedback controller $u = Kx$ for (1) implies the existence of a contractive ellipsoid which in turn implies the existence of a polyhedral invariant set [11], and a finitely determined maximal polyhedral invariant set for which a constructive procedure is described in [11]. The MAS is denoted:

$$\Omega = \{x : F_w x \leq g_w\} \qquad (14)$$

B. Robustly positively invariant set for any $u \in U$

Recall the following definitions [12]:

Definition 3: (Robustly positively controlled invariant set) Given the polytopic system (1), the set Φ is invariant if for any $x(t) \in \Phi$ there exists a control $u(t)$ such that $x(t+1) \in \Phi$.

Definition 4: (Pre-image set) Given the polytopic system (1), the one-step pre-image set of the set $P_0 = \{x : F_0 x \leq g_0\}$ is given by all the states that can by brought in one step into P_0 by a suitable control. The pre-image set, called $P_1 = \text{Pre}(P_0)$, can be shown to be

$$P_1 = \{x \in R^n : \exists u \in U : F_0(A_i x + B_i u) \leq g_0\} \qquad (15)$$

Remark 4: It is clear that if set Φ is contained in its pre-image set then Φ is invariant.

Define P_N as the set of states, that can be steered to the MAS Ω 14 in no more than N steps along an admissible trajectory, i.e. one that satisfies (4). P_N can be generated recursively by the following procedure.

Procedure 1: Invariant set computation.

1) Set $k = 0$ and $P_0 = \Omega$
2) Define
$$P_{k+1} = \text{Pre}(P_k) \bigcap X$$
3) If $P_{k+1} = P_k$, then stop and set $P_N = P_k$. Else continue.
4) If $k = N$, then stop else continue.
5) Set $k = k + 1$ and go to the step 2.

It is clear that if Ω is an invariant set, then for each k, it holds that $P_{k-1} \subset P_k$, and therefore P_k is an invariant set and a sequence of nested polytopes. Note that the complexity of P_N does not have an analytic dependence on N and may increase without bound, thus placing a practical limitation on the choice of N. Denote

$$P_N = \{x : F_N x \le g_N\} \tag{16}$$

V. Interpolation based controller with linear programming

The purpose of this section is to show how an interpolation technique can be used together with linear programming.

A. Vertex control law [1]

Given a positve invariant polytope $P_N \in R^n$. This polytope can be decomposed as a sequence of simplices P_N^k, each formed by n vertices $x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}$ and the origin. These simplices have following properties:

- P_N^k has nonempty interior,
- $Int(P_N^k \cap P_N^l) = \emptyset$ if $k \ne l$,
- $\bigcup_k P_N^k = P_N$,

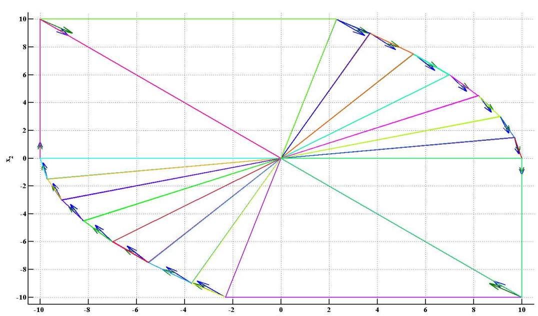

Fig. 1. Vertex control law and vector field

Denote by $X^{(k)} = (x_1^{(k)}\ x_2^{(k)}\ \ldots\ x_n^{(k)})$ the square matrix defined by the vertices generating P_N^k. Since P_N^k has nonempty interior, $X^{(k)}$ is invertible. Let $U^{(k)} = (u_1^{(k)}\ u_2^{(k)}\ \ldots\ u_n^{(k)})$ be the matrix defined by the admissible control values at these vertices. For $x \in P_N^k$ consider the following linear gain K^k:

$$K^k = U^{(k)}(X^{(k)})^{-1} \tag{17}$$

Remark 5: By the admissible control value we understand any control action, that keeps the state inside the invariant set. Generally one would like to maximize this control action which may be done by the following program.

$$J = \max \|u\|_p \text{ s.t. } \begin{cases} F_N(A_i x + B_i u) \le g_N, \ \forall i = 1, \ldots, s, \\ F_u u \le g_u. \end{cases} \tag{18}$$

where $\|u\|_p$ is the p -norm of the vector u. Due to the properties of the positive invariant set, (18) is always feasible.

Theorem 1: The piecewise linear control $u = K^k x$ is feasible for all $x \in P_N$.

Proof: A proof is given in [1]. Here a simpler proof is proposed.

For all $x(t) \in P_N$ there exists an index k such that $x(t) \in P_N^k$, and $x(t)$ can be expressed by convex combination of vertices of P_N^k: $x(t) = \sum_{i=1}^n \alpha_i x_i^k$, which is equivalent with

$$x(t) = X^{(k)}\alpha$$

and by consequence $\alpha = (X^{(k)})^{-1}x(t)$, $\alpha \ge 0$ and $\sum_{i=1}^n \alpha_i \le 1$. For feasibility one has to ensure $\forall x(t) \in P_N$: $F_u u(t) \le g_u$ and $x(t+1) = A(t)x(t) + B(t)u(t) \in P_N$. With simple manipulations

$$\begin{aligned} F_u u(t) &= F_u U^{(k)}(X^{(k)})^{-1}x(t) = F_u U^{(k)}\alpha \\ &= \sum_{i=1}^n \alpha_i F_u u_i^k \le \sum_{i=1}^n \alpha_i g_u \le g_u \end{aligned}$$

and

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) \\ &= A(t)(X^{(k)})\alpha + B(t)U^{(k)}\alpha \\ &= \sum_{i=1}^n \alpha(A(t)x_i^k + B(t)u_i^k) \end{aligned}$$

$\forall i = \overline{1,n}$ we have $A(t)x_i^k + B(t)u_i^k \in P_N$, it follows that $x(t+1) \in P_N$ □

Theorem 1 states that for any $x \in P_N$, by using the vertex control law, one has recursive feasibility. Moreover, in [1] and [4] it was proved that the resulting closed loop system with this controller is also asymptotically stable.

B. Interpolation via linear programming

Any state $x(t)$ in P_N can be decomposed as follows:

$$x(t) = c x_v(t) + (1-c)x_o(t) \tag{19}$$

where $x_v(t) \in P_N$, $x_o(t) \in \Omega$ and $0 \le c \le 1$.

Consider the following control law:

$$u(t) = c u_v(t) + (1-c)u_o(t) \tag{20}$$

where $u_v(t)$ is obtained by applying the vertex control law and $u_o(t) = Kx_o(t)$ is the control law, that is feasible in Ω.

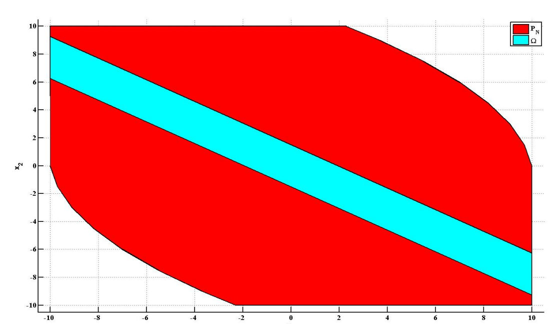

Fig. 2. Feasible regions for example 1. The blue one is the MAS Ω, when applying the control law $u = Kx$. The red one is the positively invariant set P_N.

Theorem 2: The above linear control is feasible for all $x \in P_N$.

Proof: Corresponding to the decomposition, the control law is given by (20).

One has to prove that $F_u u(t) \leq g_u$ and $x(t+1) = A(t)x(t) + B(t)u(t) \in P_N$ for all $x(t) \in P_N$. One has

$$\begin{aligned} F_u u(t) &= F_u(cu_v(t) + (1-c)u_o(t)) \\ &= cF_u u_v(t) + (1-c)F_u u_o(t) \\ &\leq cg_u + (1-c)g_u = g_u \end{aligned}$$

and

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) \\ &= A(t)(cx_v(t) + (1-c)x_o(t)) + \\ &\quad + B(t)(cu_v(t) + (1-c)u_o(t)) \\ &= c(A(t)x_v(t) + B(t)u_v(t)) + \\ &\quad + (1-c)(A(t)x_o(t) + B(t)u_o(t)) \end{aligned}$$

We have $A(t)x_v(t) + B(t)u_v(t) \in P_N$ and $A(t)x_o(t) + B(t)u_o(t) \in \Omega \subset P_N$. It follows that $x(t+1) \in P_N$. □

Referring to the discussion in the Introduction about maximal control action, one would like to minimize c, so the following program is given,

$$c^*(x) = \min_{c, x_v, x_o} c, \text{ s.t.} \begin{cases} F_N x_v \leq g_N, \\ F_w x_o \leq g_w, \\ cx_v + (1-c)x_o = x, \\ 0 \leq c \leq 1 \end{cases} \quad (21)$$

Denote $r_v = cx_v$, $r_o = (1-c)x_o$. It is clear that $r_v \in cP_N$ and $r_o \in (1-c)\Omega$ or equivalently $F_N r_v \leq cg_N$ and $F_w r_o \leq (1-c)g_w$. The above non-linear program is translated into a linear program as follows.

Interpolation based on linear programming

$$c^*(x) = \min_{c, r_v} c, \text{ s.t.} \begin{cases} F_N r_v \leq cg_N \\ F_w(x - r_v) \leq (1-c)g_w \\ 0 \leq c \leq 1 \end{cases} \quad (22)$$

Remark 6: If one would like to maximize c, it is obvious that $c = 1$ for all $x \in P_N$. In this case the controller turns out to be the vertex controller.

Theorem 3: The control law using interpolation based on linear programming (19), (20), (22) guarantees robustly asymptotic stability for all initial states $x(0) \in P_N$.

Proof: First of all we will prove that all solutions starting in P_N will reach the set Ω in finite time. For this purpose, consider the positive function $V(x) = c^*$ for all $x(t) \in P_N \setminus \Omega$. $V(x)$ is the Lyapunov function candidate.

For any $x(t) \in P_N$, one has $x(t) = c^*(t)x_v(t) + (1 - c^*(t))x_o(t)$ and $u(t) = c^*(t)u_v(t) + (1 - c^*(t))u_o(t)$. It follows that

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) \\ &= c^*(t)x_v(t+1) + (1 - c^*(t))x_o(t+1) \end{aligned}$$

where $x_v(t+1) = A(t)x_v(t) + B(t)u_v(t) \in P_N$ and $x_o(t+1) = A(t)x_o(t) + B(t)u_o(t) \in \Omega$.

By using the interpolation based on linear programming, one gets that $x(t+1) = c^*(t+1)x_v^o(t+1) + (1 - c^*(t+1))x_o^o(t+1)$ where $x_v^o(t+1) \in P_N$ and $x_o^o(t+1) \in \Omega$. It follows that $c^*(t+1) \leq c^*(t)$, and $V(x)$ is non-increasing.

The asymptotically stability property of the vertex control law and the feasibility of the controller entering in the interpolation over Ω assures that there is no initial condition $x(0) \in P_N \setminus \Omega$ such that $c^*(t) = c^*(0), \forall t \geq 0$. It follows that $V(x) = c^*$ is a Lyapunov function for $x(t) \in P_N \setminus \Omega$.

Using the vertex controller, an interpolation between the vertices of the feasible invariant set and the origin is obtained. Conversely using the controller (19), (20), (22), an interpolation is constructed between the vertices of the feasible invariant set and those of the MAS which contains the origin as an interior point. This last property proves that the vertex controller is a feasible choice for the interpolation based technique. From these facts we conclude that the closed sets defined by the Lyapunov function level curves for the closed loop system with the with the controller (19), (20), (22) are subsets of the closed sets defined by the corresponding Lyapunov function level curves for the closed loop with vertex control. The latter ones are, in fact, homothetical polyhedra with respect to the border of the vertex control feasible invariant set.

The proof is complete by noting that inside Ω the feasible stabilizing control $u = Kx$ is contractive, and thus the interpolation-based controller assures asymptotic stability for all $x \in P_N$. □

Theorem 4: The control law from the interpolation based on linear programming (19), (20), (22) can be represented as a continuous and piecewise affine function of the state.

Proof: Equation (22) can be interpreted as a multi-parametric linear optimization problem [13], with the state being the vector of parameters. An explicit solution can be constructed for the optimal arguments in terms of a continuous piecewise affine function defined over a polyhedral partition of the parameters space [14]. Thus the ultimate control action represents a function of the state (22). The stated properties are concluded. □

VI. Examples

To show the effectiveness of the proposed approach, two examples are presented in this section. To solve problem (10) in both examples we used CVX, a package for specifying and solving convex programs [15], [16]. To solve linear programs we used the Multi-parametric toolbox [17].

A. Example 1

Consider the uncertain discrete-time system:

$$x(t+1) = A(t)x(t) + B(t)u(t) \quad (23)$$

where

$$\begin{aligned} A(t) &= \alpha(t)A_1 + (1-\alpha(t))A_2, \\ B(t) &= \alpha(t)B_1 + (1-\alpha(t))B_2 \end{aligned}$$

and

$$A_1 = \begin{pmatrix} 1 & 0.1 \\ 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} 1 & 0.2 \\ 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 1.5 \end{pmatrix}$$

At each sampling time $\alpha(t) \in [0,1]$ is an uniformly distributed pseudorandom number. The constraints are $-10 \leq x_1 \leq 10$, $-10 \leq x_2 \leq 10$ and $-1 \leq u \leq 1$.

Solving (10) gives the feedback gain $K = (-0.5160 \ \ -0.6644)$.

Using procedures in [11] and our procedure 1 above one can obtain the sets Ω and P_N as showed in Figure 2. Note that $P_{17} = P_{18}$, so P_{17} is the MAS for (23).

The set of vertices of P_N is given by the matrix $V(P_N)$ below, together with the control matrix U_v

$$V(P_N) = (V_1 \ ; \ -V_1)$$

where

$$V_1 = \begin{pmatrix} 10 & 9.7 & 9.1 & 8.2 & 7 & 5.5 & 3.7 & 2.3 & -10 \\ 0 & 1.5 & 3 & 4.5 & 6 & 7.5 & 9 & 10 & 10 \end{pmatrix}$$

and

$$U_v = (U_1 \ ; \ -U_1)$$

where

$$U_1 = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}$$

Using the algorithm (22), Figure 3 shows the state space partition and three different trajectories of the closed loop system, depending on the realization of $\alpha(t)$.

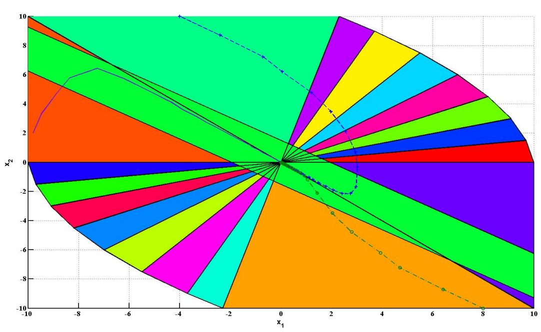

Fig. 3. State space partition and trajectories of the closed loop system for example 1.

For the initial condition $x(0) = (-4 \ \ 10)^T$, Figure 4 shows the realization of $\alpha(t)$, the state and input trajectories and the interpolating coefficient c as a function of t. As expected, $c(t)$ is positive and non-increasing.

B. Example 2

This example is taken from [18]. Consider the uncertain discrete time system:

$$x(t+1) = A(t)x(t) + Bu(t) \tag{24}$$

where

$$A(t) = \alpha(t)A_1 + (1-\alpha(t))A_2$$

$$A_1 = \begin{pmatrix} 1 & 0.1 \\ 0 & 0.99 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0.1 \\ 0 & 0 \end{pmatrix},$$

and $B(t) = (0 \ \ 0.0787)^T$. At each sampling time $\alpha(t) \in [0,1]$ is an uniformly distributed pseudorandom number. The

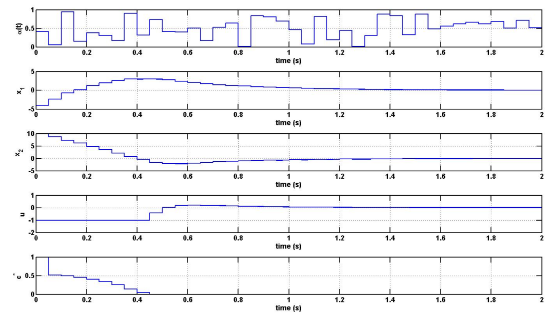

Fig. 4. The realization of $\alpha(t)$, state and input trajectories and interpolating coefficient for example 1.

constraints are $-1 \leq x_1 \leq 1$, $-1 \leq x_2 \leq 1$ and $-2 \leq u \leq 2$. Solving (10) gives the feedback gain $K = (-92.8160 \ \ -14.4876)$.

The set of vertices of P_N is given by the matrix $V(P_N)$ below, together with the control matrix U_v

$$V(P_N) = (10^{-1}V_1 \ ; \ -10^{-1}V_1)$$

where

$$V_1 = \begin{pmatrix} 3.4 & 2.5 & 2.2 & 1.7 & 1.1 & 0.3 & -0.7 & -1.8 & -2 & -3.3 \\ -8.5 & 0 & 1.6 & 3.2 & 4.8 & 6.5 & 8.1 & 9.8 & 10 & 10 \end{pmatrix}$$

and

$$U_v = (U_1 \ ; \ -U_1)$$

where

$$U_1 = \begin{pmatrix} -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \end{pmatrix}$$

Using the algorithm (22) gives Figure 5 showing the state space partition and four different trajectories of the closed loop system, depending on the realization of $\alpha(t)$.

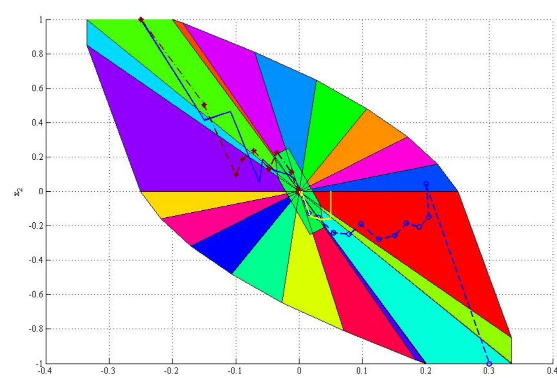

Fig. 5. State space partition and trajectories of the closed loop system for example 2.

For the initial condition $x(0) = (0.05 \ \ 0)^T$, Figure 6 shows the realization of $\alpha(t)$, the state and input trajectories, and the interpolating coefficient c as a function of t. As a comparison, Figure 7 shows the state and input trajectories for the same initial condition, using algorithms described in [18].

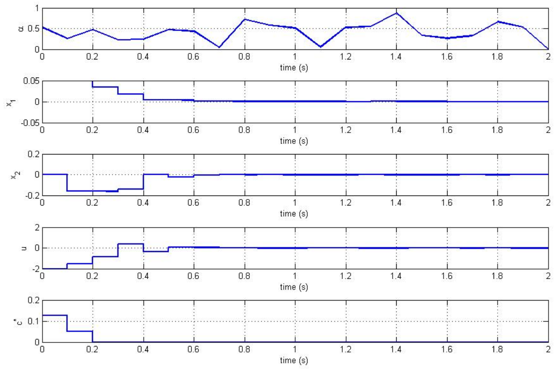

Fig. 6. The realization of $\alpha(t)$, state and input trajectories and the interpolating coefficient c for example 2

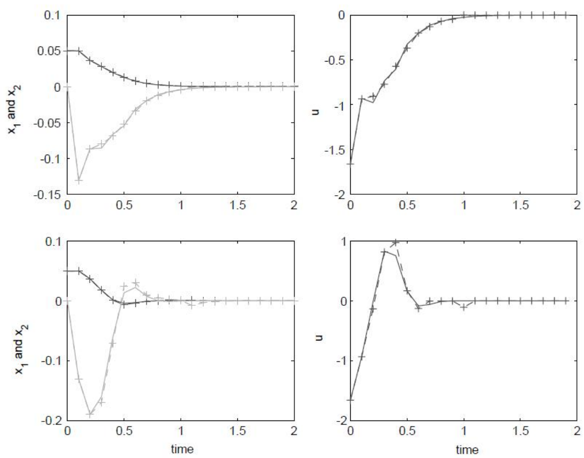

Fig. 7. State and input trajectories for example 2, using algorithms in [18].

VII. Conclusion

In this paper a novel interpolation scheme by linear programming is introduced for time-varying and uncertain linear discrete-time plants with polyhedral state and control constraints. The interpolation is done between global vertex control and local unconstrained robust optimal control. A proof of asymptotic stability is given. Several simulation examples are presented including a comparison with an earlier solution from the literature.

The resulting control law is affine over a polyhedral partition of the state space and is thus similar to Explicit Model Predictive Control. In a companion paper we show how to partition the state space and merge polyhedral cells by off-line computation of the proposed control law. However, the implicit version with the on-line LP-solution as presented here is computationally attractive, since the size of the LP-problem solved in each sampling instant is independent of the any prediction horizon.

In contrast to most MPC schemes, the present controller is suitable for uncertain and time-varying plants, and gives local optimal control We believe that it will prove attractive for some fast MPC-like applications in industry.

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