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Distributed computation of the average of multiple time-varying reference signals

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Abstract— **We present a distributed discontinuous control algorithm for a team of agents to track the average of multiple time-varying reference signals with bounded derivatives. We use tools from nonsmooth analysis to analyze the stability of the system. For time-invariant undirected connected network topologies, we prove that the states of all agents will converge to the average of the time-varying reference signals with bounded derivatives in finite time provided that the control gain is properly chosen. The validity of this result is also established for scenarios with switching undirected connected network topologies. For time-invariant directed network topologies with a directed spanning tree, we show that all agents will still reach a consensus in finite time, but the convergent value is generally not the average of the time-varying reference signals with bounded derivatives.**

Notation

$\mathbb{R}$	Real numbers.
$\mathbb{R}^+$	Nonnegative real numbers.
$\mathbf{1}$	Vectors with all ones.
$\text{sgn}(\cdot)$	Signum function.
$\|\cdot\|_1$	1-norm of a vector.
$\|\cdot\|_2$	2-norm of a vector.
$\|\cdot\|_\infty$	∞-norm of a vector.
$\|\cdot\|$	Any norm of a vector.
$\lvert\cdot\rvert$	Absolute value of a scalar.
$\#S$	Cardinality of the set S.
co	Convex hull.
$\overline{\text{co}}$	Convex closure.
A^T	Transpose of A.
$u(\Omega)$	Lebesgue measure of Ω.
$B(x, r)$	Open ball of radius r centered at x.
$\nabla \mathfrak{f}$	Gradient of $\mathfrak{f}$.
a.e.	Almost everywhere.
$\mathcal{B}(\mathbb{R}^d)$	Collection of subsets of $\mathbb{R}^d$.

I. Introduction

In multi-agent systems, consensus means to reach an agreement on a quantity of interest. Here, the states of all agents usually converge to the average or the weighted average of the initial conditions of these agents, which is a constant value (see, e.g., [1], [2]). In a consensus problem, when there exists a dynamic leader (e.g., an agent that moves by itself regardless of the other agents) or a time-varying reference signal, the consensus problem becomes a coordinated tracking problem. Here, the objective is that the states of all agents track the state of the dynamic leader or the time-varying reference signal [3], [4], [5]. When there exist multiple reference signals, dynamic average consensus problems were studied. Here, the objective is that the states of all agents track the average of the reference signals. In [6], a distributed algorithm was proposed to guarantee that a consensus is reached on the average of multiple reference signals with steady-state values. The result was proved by frequency-domain techniques and was applied in [7] to obtain least-squares fused estimates based on spatially distributed measurements, which is robust to changes in the underlying network topology. In [8], two dynamic average consensus algorithms were proposed, namely, a proportional (or high-pass) algorithm and a proportional-integral algorithm. With properly chosen parameters, the proportional algorithm can guarantee the tracking of the average of multiple constant reference signals provided that the estimators are correctly initialized. With an integral term introduced in the estimator, the proportional-integral algorithm can guarantee the tracking of the average of multiple constant reference signals without the need for the correct estimator initialization. However, for time-varying reference signals, a tracking error is expected. These two algorithms were used in [9] to build a framework for decentralized estimation and control, which is a good complement to purely reactive memoryless controller design. A consensus filter was used in [10] to study swarm dynamics where inter-agent forces are governed by repulsive-attractive forces. The consensus filter generates a collective estimate of the swarm center that is used by the agents to guide their movements. It was shown that if the network topology is regular, i.e., the maximum out-degree equals to the minimum out-degree, then the consensus filter can reach a consensus regardless of the swarm size. But for a general network topology, a consensus error is expected.

The contributions of this paper lie in the following facts. We propose a simple but compelling control algorithm to solve the distributed average tracking problem by using the signum function. To the best of our knowledge, the proposed algorithm is the first distributed algorithm that guarantees accurate tracking of the average of multiple time-varying reference signals with bounded derivatives. Because of the discontinuity in the control algorithm, we exploit tools from nonsmooth analysis to investigate the stability of the closed-loop system. Under a time-invariant undirected network topology, we show that all agents will track the average of multiple time-varying reference signals with bounded derivatives in finite time as long as the control gain is

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properly chosen and the network topology is connected. We also establish the validity of the result for scenarios with switching but connected network topologies. For a time-invariant directed network topology with a directed spanning tree, we show that all agents can still reach a consensus in finite time, but the convergent value is generally not the average of the time-varying reference signals with bounded derivatives.

II. Problem Descriptions And Mathematical Preliminaries

A. Problem Descriptions

Suppose that there are n time-varying reference signals, $r_i(t) \in \mathbb{R}^m$, $i = 1, \ldots, n$, satisfying the following dynamics:

$$\dot{r}_i(t) = f_i(t).$$

Here $f_i(t) \in \mathbb{R}^m$ is assumed to be measurable and bounded, i.e., $\sup_t \|f_i(t)\|_\infty \leq \bar{f}$ for all $i = 1, \ldots, n$, where $\bar{f}$ is a positive constant. Suppose that there are n agents with single-integrator dynamics given by:

$$\dot{x}_i(t) = u_i(t), \qquad i = 1, \ldots, n, \tag{1}$$

where $x_i(t) \in \mathbb{R}^m$ is the state of agent i and $u_i(t) \in \mathbb{R}^m$ is the control input that needs to be designed.

We assume that agent i has access to f_i and $r_i(0)$ [and hence $r_i(t)$], but does not have access to the other reference signals $r_j(t)$, $j \neq i$. We also assume that agent i can obtain information from a subset of the other agents, called its neighbors and denoted by $\mathcal{N}_i$. Here we assume that $i \notin \mathcal{N}_i$.

We use a graph $\mathcal{G} \triangleq \{\mathcal{V}, \mathcal{E}\}$ to describe the network topology between the agents, where $\mathcal{V} \triangleq \{1, \ldots, n\}$ is the node set and $\mathcal{E} \triangleq \{(i,j) | i \in \mathcal{N}_j\}$ is the edge set. A graph is undirected if $j \in \mathcal{N}_i$ implies $i \in \mathcal{N}_j$. If $i \in \mathcal{N}_j$, node i is the parent node while node j is the child node. A directed path from node i to node j is a sequence of edges of the form (i_1, i_2), (i_2, i_3), $\ldots$ in a directed graph. A directed tree is a directed graph, where every node has exactly one parent except for one node, called the root, and the root has directed paths to every other node. A directed spanning tree of a directed graph is a direct tree that contains all nodes of the directed graph. A directed graph has a spanning tree if there exists a directed spanning tree as a subset of the directed graph. An undirected path in an undirected graph is defined analogously. An undirected graph is connected if there is an undirected path between every pair of distinct nodes.

Our main objective is to design $u_i(t)$ based on $f_i(t)$, $r_i(0)$, and $x_j(t)$, $j \in \mathcal{N}_i$, such that all the agents will finally track the average of the n time-varying reference signals, i.e.,

$$\|x_i(t) - \frac{1}{n}\sum_{j=1}^{n} r_j(t)\| \to 0 \qquad \text{as } t \to \infty.$$

We initialize the states of all agents as

$$x_i(0) = r_i(0), \tag{2}$$

and design the following control law

$$u_i(t) = f_i(t) + \alpha \sum_{j \in \mathcal{N}_i(t)} \text{sgn}[x_j(t) - x_i(t)], \tag{3}$$

where $\alpha > 0$ is a constant, and $\text{sgn}(\cdot)$ is the signum function defined component-wise. Using (3) for (1), we obtain the following closed-loop system

$$\dot{x}_i(t) = f_i(t) + \alpha \sum_{j \in \mathcal{N}_i(t)} \text{sgn}[x_j(t) - x_i(t)] \tag{4}$$

with the initial conditions (2). We note that each component of $x_i(t)$ is decoupled in (4). Therefore, in the following, we will only tackle the one-dimensional case, i.e., $m = 1$. The same conclusions hold for any $m \geq 2$.

B. Mathematical Preliminaries

In the following, we present some preliminaries in nonsmooth analysis that will be frequently referred to. Because the right-hand side of (4) is discontinuous, we first need to define "What is a solution of the differential equation (4)?". In the existing literature on nonsmooth systems, we have seen several definitions of solutions such as Filippov solutions and Caratheodoory solutions. Here, we choose Filippov solutions because some elegant tools have been developed to analyze the stability of Filippov solutions, and these tools have been applied very well in the context of multi-agent systems. Filippov solutions are defined below.

Definition 1: [11] For a vector field $\mathfrak{f}(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, define the Filippov set-valued map $K[\mathfrak{f}](t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathcal{B}(\mathbb{R}^d)$ by

$$K[\mathfrak{f}](t,x) \triangleq \cap_{\delta > 0} \cap_{u(N)=0} \overline{\text{co}}\mathfrak{f}(t, B(x,\delta) - N),$$

where $\cap_{u(N)=0}$ denotes the intersection over all sets of Lebesgue measure zeroes.

Some useful rules are developed to simply the calculation of Filippov set-valued maps, which are summarized in the following lemma.

Lemma 1: [11]

1) Assume that $\mathfrak{f} : \mathbb{R}^m \to \mathbb{R}^n$ is locally bounded. Then $\exists N_\mathfrak{f} \subset \mathbb{R}^m$, $u(N_\mathfrak{f}) = 0$ such that $\forall N \subset \mathbb{R}^m$, $u(N) = 0$,
$$K[\mathfrak{f}](x) = \text{co}\{\lim f(x_i) | x_i \to x, x_i \notin N_\mathfrak{f} \cup N\}.$$
2) Assume that $\mathfrak{f}, \mathfrak{g} : \mathbb{R}^m \to \mathbb{R}^n$ are locally bounded, then
$$K[\mathfrak{f} + \mathfrak{g}](x) \subseteq K[\mathfrak{f}](x) + K[\mathfrak{g}](x).$$
3) Let $\mathfrak{f} : \mathbb{R}^m \to \mathbb{R}^n$ be continuous, then
$$K[\mathfrak{f}](x) = \{\mathfrak{f}(x)\}. \tag{5}$$

Definition 2: [12] Consider a vector differential equation

$$\dot{x}(t) = \mathfrak{f}(t,x), \tag{6}$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^T$. A vector function $x(\cdot)$ is called a Filippov solution of (6) on $[t_0, t_1]$, where t_1 could be ∞, if $x(\cdot)$ is absolutely continuous and for almost all $t \in [t_0, t_1]$

$$\dot{x}(t) \in K[\mathfrak{f}](t,x).$$

The next lemma establishes mild conditions under which Filippov solutions exist.

Lemma 2: [13] Given (6), let $\mathfrak{f}(t,x)$ be measurable and locally essentially bounded, that is, bounded on a bounded neighborhood of every point excluding sets of measure zero. Then, for all $x_0 \in \mathbb{R}^n$, there exists a Filippov solution of (6) with the initial condition $x(0) = x_0$.

It follows from Lemma 2 that Filippov solutions for the system (4) exist because $f_i(t)$, $i = 1, \ldots, n$, and sgn$(\cdot)$ are measurable and bounded.

Let W be a locally Lipschitz function of x, where $x = [x_1, \ldots, x_n]^T$. The generalized gradient of the function W with respect to x_i (cf. [12]) is defined by

$$(\partial W)_i \triangleq \text{co}\{\lim_{j\to\infty} \frac{\partial W}{\partial x_i} | x_j' \to x_i, x_j' \notin \Omega_W \cup S\},$$

where Ω_W is the set of points where W fails to be differentiable and S is a set of zero measure that can be arbitrarily chosen to simplify the calculation. Then the generalized gradient of W is $\partial W \triangleq [(\partial W)_1, \ldots, (\partial W)_n]^T$.

The set-valued Lie derivative of W with respect to x, the trajectory of (6) , is defined as

$$\dot{\tilde{W}} \triangleq \cap_{\xi\in\partial W} \xi^T K[\mathfrak{f}].$$

We can use $\dot{\tilde{W}}$ to study the evolution of W along the Filippov solutions of (6), which is guaranteed by the following lemma.

Lemma 3: [14] Let $t \mapsto x(t)$ be a Filippov solution of system (6), and let $W : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz and regular function. Then $\dot{W}$ exists a.e. and

$$\dot{W} \in^{a.e.} \dot{\tilde{W}}.$$

By using Lemma 3, a Lyapunov stability theorem is established as follows.

Lemma 4: [14] Given (6), let $\mathfrak{f}(t,x)$ be locally essentially bounded and $0 \in K[\mathfrak{f}](t,0)$ in a region $Q \supset \{t | t_0 \le t < \infty\} \times \{x \in \mathbb{R}^n | \|x\| < r\}$, where $r > 0$. Also, let $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a regular function satisfying

$$V(t,0) = 0,$$

and

$$0 < V_1(\|x\|) \le V(t,x) \le V_2(\|x\|), \quad \text{for } x \ne 0$$

in Q for some class $\mathcal{K}$ functions V_1 and V_2. If in addition, there is a class $\mathcal{K}$ function $\omega(\cdot)$ in Q with the property

$$\dot{\tilde{V}}(t,x) \le -\omega(x) < 0, \quad \text{for } x \ne 0,$$

or there exists a constant $\omega > 0$ such that

$$\dot{\tilde{V}}(t,x) \le -\omega < 0, \quad \text{for } x \ne 0,$$

then the solution $x(t) \equiv 0$ is uniformly asymptotically stable.[1]

Lemma 5 (Comparison Lemma [15]): Consider the scalar differential equation

$$\dot{\mathfrak{u}} = \mathfrak{f}(t,\mathfrak{u}), \quad \mathfrak{u}(t_0) = \mathfrak{u}_0$$

where $\mathfrak{f}(t,\mathfrak{u})$ is continuous in t and locally Lipschitz in $\mathfrak{u}$, for all $t \ge 0$ and all $\mathfrak{u} \in J \subset \mathbb{R}$. Let $[t_0, T)$ (T could be infinity) be the maximal interval of existence of the solution $\mathfrak{u}(t)$, and suppose that $\mathfrak{u}(t) \in J$ for all $t \in [t_0, T)$. Let $\mathfrak{v}(t)$ be a continuous function whose upper right-hand derivative $D^+\mathfrak{v}(t)$ satisfies the differential inequality

$$D^+\mathfrak{v}(t) \le \mathfrak{f}(t, \mathfrak{v}(t)), \quad \mathfrak{v}(t_0) \le \mathfrak{u}_0,$$

where $\mathfrak{v}(t) \in J$ for all $t \in [t_0, T)$. Then, $\mathfrak{v}(t) \le \mathfrak{u}(t)$ for all $t \in [t_0, T)$.

III. Stability Analysis

In this section, we analyze (4) under undirected and directed network topologies. Before moving on, we need the following lemmas.

Lemma 6: For the system (4), if the graph $\mathcal{G}(t)$ is undirected and $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$ for all $i, j = 1, \ldots, n$, then

$$\lim_{t\to\infty} |x_i(t) - \frac{1}{n}\sum_{j=1}^{n} r_j(t)| = 0,$$

for all $i = 1, \ldots, n$.

Proof: The proof is omitted due to the space limit. ■

We next present the first main result of this paper.

Theorem 1: For the system (4), if $\mathcal{G}$ is time invariant, undirected, and connected, and $\alpha > \bar{f}$, then $|x_i(t) - \frac{1}{n}\sum_{j=1}^{n} r_j(t)| \to 0$ in finite time for all $i = 1, \ldots, n$, and the convergence time is upper bounded by $\frac{1}{2(\alpha - \bar{f})}\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i} |x_i(0) - x_j(0)|$.

Proof: Define $e(t)$ as the column stack vector formed by all $x_i(t) - x_j(t)$, $(i,j) \in \mathcal{E}$. Consider the Lyapunov function candidate

$$V[e(t)] = \frac{1}{2}\|e(t)\|_1 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i} |x_i(t) - x_j(t)|. \quad (7)$$

Because $V[e(t)]$ is nonsmooth, the time derivative of $V[e(t)]$ is not defined at some time instants. We first show that the function $V[e(t)]$ is regular. To prove this, it suffices to show that the function $h(x) \triangleq |x|$, $x \in \mathbb{R}$, is regular at the discontinuous point 0. The right directional derivative of $|x|$ at 0 in the direction of $v \in \mathbb{R}$ is defined as

$$f'(0; v) = \lim_{h\to 0^+} \left.\frac{|x + hv| - |x|}{h}\right|_{x=0} = |v|.$$

The generalized directional derivative of $|x|$ at 0 in the direction of $v \in \mathbb{R}$ is defined as

$$\begin{aligned} f^o(0; v) &= \limsup_{y\to 0, h\to 0^+} \frac{|y + hv| - |y|}{h} \\ &= \lim_{\delta\to 0^+, \epsilon\to 0^+} \sup_{y\in B(0,\delta), h\in[0,\epsilon)} \frac{|y + hv| - |y|}{h} \\ &= \lim_{\delta\to 0^+, \epsilon\to 0^+} \frac{|hv|}{h} \\ &= |v|. \end{aligned} \quad (8)$$

Therefore, $|x|$ is a regular function.

[1]Here $\dot{\tilde{V}}(t,x) \le a$ means that for all $v \in \dot{\tilde{V}}(t,x)$, $v \le a$.

We next show that the function $V(e)$ satisfies

$$V_1(\|e\|_2) \le V(e) \le V_2(\|e\|_2) \qquad (9)$$

for some class $\mathcal{K}$ functions $V_1(\cdot)$ and $V_2(\cdot)$. Because all norms on a finite-dimensional linear space are equivalent, it follows that there are positive constants a and b such that $a\|e\|_2 \le \|e\|_1 \le b\|e\|_2$. Therefore, it is obvious that (9) holds.

We can derive that

$$K[\sum_{j\in\mathcal{N}_i} \text{sgn}(x_j - x_i)] = \begin{cases} \{\#\mathcal{N}_i^+ - \#\mathcal{N}_i^-\}, \\ \text{if } x_i \ne x_j, \forall j \in \mathcal{N}_i, \\ \{\#\mathcal{N}_i^+ - \#\mathcal{N}_i^-\} \\ +[-\#\mathcal{N}_i^0, \#\mathcal{N}_i^0], \text{otherwise.} \end{cases}$$

Here, $\mathcal{N}_i^+$ is the set of neighbors of agent i with $x_j > x_i$, $j \in \mathcal{N}_i$, $\mathcal{N}_i^-$ is the set of neighbors of agent i with $x_j < x_i$, and $\mathcal{N}_i^0$ is the set of neighbors of agent i with $x_j = x_i$. In addition, it can be shown that the generalized gradient of $V(e)$ with respect to x_i is

$$(\partial V)_i = \begin{cases} \{\#\mathcal{N}_i^- - \#\mathcal{N}_i^+\}, \text{if } x_i \ne x_j, \forall j \in \mathcal{N}_i, \\ \{\#\mathcal{N}_i^- - \#\mathcal{N}_i^+\} + [-\#\mathcal{N}_i^0, \#\mathcal{N}_i^0], \\ \text{otherwise.} \end{cases} \qquad (10)$$

By using Lemma 1, the set-valued Lie derivative of $V(e)$ is calculated as follows

$$\dot{\tilde{V}} \subseteq \cap_{\xi\in\partial V}\xi^T \left\{ \begin{array}{l} K[f_1] + \alpha K\left[\sum_{j\in\mathcal{N}_1} \text{sgn}(x_j - x_1)\right] \\ \vdots \\ K[f_n] + \alpha K\left[\sum_{j\in\mathcal{N}_n} \text{sgn}(x_j - x_n)\right] \end{array} \right\},$$

where $\partial V \triangleq [(\partial V)_1, \dots, (\partial V)_n]^T$.

Define

$$(\dot{\tilde{V}})_i \triangleq \cap_{\xi_i\in(\partial V)_i}\xi_i \left(K[f_i] + \alpha K\left[\sum_{j\in\mathcal{N}_i} \text{sgn}(x_j - x_i)\right]\right).$$

To show that $\dot{\tilde{V}} < 0$, we distinguish between two cases. The first case is that $x_i \ne x_j$ for all $i = 1, \dots, n$, $j \in \mathcal{N}_i$. In this case, we have that

$$(\dot{\tilde{V}})_i = \cap_{\xi_i\in(\partial V)_i}\xi_i \left(K[f_i] + \alpha\{\#\mathcal{N}_i^+ - \#\mathcal{N}_i^-\}\right).$$

If $(\dot{\tilde{V}})_i \ne \emptyset$, suppose that $a_i \in (\dot{\tilde{V}})_i$ and $a_i^f \in K[f_i]$. From (10), we know that ξ_i has only one value which equals to $\#\mathcal{N}_i^- - \#\mathcal{N}_i^+$. Because $\sup_{t\to\infty} \|f_i(t)\|_\infty \le \bar{f}$, according to the definition of $K[f_i]$, we know that $K[f_i] \subseteq [-\bar{f}, \bar{f}]$, which indicates that $|a_i^f| \le \bar{f}$. Thus,

$$\begin{aligned} a_i &= (\#\mathcal{N}_i^- - \#\mathcal{N}_i^+)a_i^f - \alpha\left(\#\mathcal{N}_i^- - \#\mathcal{N}_i^+\right)^2 \\ &\le \left|\#\mathcal{N}_i^- - \#\mathcal{N}_i^+\right| \left(|a_i^f| - \alpha\left|\#\mathcal{N}_i^- - \#\mathcal{N}_i^+\right|\right). \end{aligned}$$

If $\#\mathcal{N}_i^+ - \#\mathcal{N}_i^- = 0$, then $a_i = 0$. If $\#\mathcal{N}_i^+ - \#\mathcal{N}_i^- \ne 0$, which implies that $\left|\#\mathcal{N}_i^- - \#\mathcal{N}_i^+\right|$ is an integer that is greater than or equal to one, we have

$$a_i \le \bar{f} - \alpha < 0.$$

If $e \ne 0$, there always exists a node $p \in \{1, \dots, n\}$ such that $\#\mathcal{N}_p^+ - \#\mathcal{N}_p^- \ne 0$, which immediately implies that $\dot{\tilde{V}} = \sum_{i=1}^n (\dot{\tilde{V}})_i \le \bar{f} - \alpha < 0$.

The second case is that there exists $j \in \mathcal{N}_i$ such that $x_i = x_j$. If $\dot{\tilde{V}}_i \ne \emptyset$, suppose that $a_i \in \dot{\tilde{V}}_i$ and $a_i^f \in K[f_i]$. Then we know that $\forall \xi_i \in (\partial V)_i$,

$$a_i = \xi_i(a_i^f + \alpha v_i),$$

where

$$v_i \in \{\#\mathcal{N}_i^+ - \#\mathcal{N}_i^-\} + [-\#\mathcal{N}_i^0, \#\mathcal{N}_i^0].$$

Choose

$$\xi_i = -v_i \in \{\#\mathcal{N}_i^- - \#\mathcal{N}_i^+\} + [-\#\mathcal{N}_i^0, \#\mathcal{N}_i^0].$$

Thus, we have that

$$a_i = -v_i a_i^f - \alpha v_i^2 \le |v_i|(\bar{f} - \alpha|v_i|).$$

If $v_i = 0$, then $a_i = 0$. If $v_i \ne 0$, it follows from Proposition 2.2.9 in [12] that $|v_i| \ge 1$, which implies that

$$a_i \le \bar{f} - \alpha < 0. \qquad (11)$$

If $e \ne 0$, there always exists a node $p \in \{1, \dots, n\}$ such that $v_p \ne 0$. Thus, we can conclude that

$$\dot{\tilde{V}} \le \bar{f} - \alpha < 0, \text{ for } e \ne 0.$$

It follows from Lemma 4 that $e(t) \to 0$ as $t \to \infty$, which implies that $|x_i(t) - x_j(t)| \to 0$, $\forall (i, j) \in \mathcal{E}$, as $t \to \infty$. Because $\mathcal{G}$ is connected, we know that $|x_i(t) - x_j(t)| \to 0$, $\forall i, j = 1, \dots, n$, as $t \to \infty$. It follows from Lemma 6 that $|x_i(t) - \frac{1}{n}\sum_{j=1}^n r_j(t)| \to 0$, $\forall i = 1, \dots, n$, as $t \to \infty$.

From the Lebesgue's criterion for the Riemann integrability, we know that a function on a compact interval is Riemann integrable if and only if it is bounded and the set of its points of discontinuity has measure zero [16]. Write $V[e(t)]$ as $V(t)$ for simplicity. Therefore, although the time-derivative $\dot{V}(t)$ is discontinuous at some time instants, it is Riemann integrable. Then, we have that

$$V(t+h) - V(t) = \int_t^{t+h} \dot{V}(\tau)d\tau \le (\bar{f} - \alpha)h \qquad (12)$$

with $h > 0$. The upper right-hand derivative of the function V is given by

$$D^+V \triangleq \limsup_{h\to 0^+} \frac{V(t+h) - V(t)}{h}. \qquad (13)$$

One of the properties of $\limsup$ is that if $z_k \le x_k$ for each $k = 1, 2, \dots$, then $\limsup_{k\to\infty} z_k \le \limsup_{k\to\infty} x_k$ [16]. Thus, from (12) and (13), we know that

$$D^+V \le \bar{f} - \alpha.$$

Then it follows from Lemma 5 that

$$V(t) \le V(0) - (\alpha - \bar{f})t,$$

which indicates that $|x_i(t) - \frac{1}{n}\sum_i r_i(t)| \to 0$ in finite time, and the convergence time is upper bounded by $\frac{V[e(0)]}{\alpha - \bar{f}}$. ■

The first result for a switching topology is stated in the following.

Theorem 2: For the system (4), if the proximity-based graph $\mathcal{G}(t)$ is switching, undirected, and connected for all $t \geq 0$, and $\alpha > \bar{f}$, then $|x_i(t) - \frac{1}{n}\sum_j r_j(t)| \to 0$ in finite time for all $i = 1, \ldots, n$.

Proof: Define

$$V_{ij}(x_i - x_j) \triangleq \begin{cases} |x_i - x_j|, & \text{if } |x_i - x_j| \leq R, \\ R, & \text{otherwise.} \end{cases}$$

Let e be the column stack vector of all $x_i - x_j$, $(i,j) \in \mathcal{E}$. Define $V(e) \triangleq \frac{1}{2}\sum_{i=1}^n \sum_{j \neq i} V_{ij}(x_i - x_j)$. We can prove that all V_{ij} are regular functions. We can also show that $V_1(\|e\|) \leq V(e) \leq V_2(\|e\|)$ by using similar arguments to those in Theorem 1, where $V_1(\cdot)$ and $V_2(\cdot)$ are some class $\mathcal{K}$ functions.

Moreover, we can derive that

$$K[\sum_{j \in \mathcal{N}_i} \text{sgn}(x_j - x_i)] = \begin{cases} \{\#\mathcal{N}_i^+ - \#\mathcal{N}_i^-\} + [-\#\mathcal{N}_i^0, \#\mathcal{N}_i^0], \\ \text{if agent } i \text{ is not undergoing a switching,} \\ \{\#\mathcal{N}_i^+ - \#\mathcal{N}_i^-\} + [-\#\mathcal{N}_i^0, \#\mathcal{N}_i^0] \\ +[-\#\mathcal{N}_i^s, \#\mathcal{N}_i^s], \qquad \text{otherwise,} \end{cases}$$

where $\mathcal{N}_i^s$ is the set of neighbors of agent i that are undergoing switchings. Similarly, we can show that the generalized gradient of $V(e)$ with respect to x_i is

$$(\partial V)_i = \begin{cases} \{\#\mathcal{N}_i^- - \#\mathcal{N}_i^+\} + [-\#\mathcal{N}_i^0, \#\mathcal{N}_i^0], \\ \text{if agent } i \text{ is not undergoing a switching,} \\ \{\#\mathcal{N}_i^- - \#\mathcal{N}_i^+\} + [-\#\mathcal{N}_i^0, \#\mathcal{N}_i^0] \\ +[-\#\mathcal{N}_i^s, \#\mathcal{N}_i^s], \qquad \text{otherwise.} \end{cases}$$

If all agents in the team are not undergoing switchings, we have already proved in Theorem 1 that $\dot{\tilde{V}} \leq \bar{f} - \alpha < 0$ for $e \neq 0$. Let $(\dot{\tilde{V}})_i$ be defined as in (11). If agent i undergoes a switching and $(\dot{\tilde{V}})_i \neq \emptyset$, suppose that $a_i \in (\dot{\tilde{V}})_i$ and $a_i^f \in K[f_i]$. Then we know that $\forall \xi_i \in \{\#\mathcal{N}_i^- - \#\mathcal{N}_i^+\} + [-\#\mathcal{N}_i^0, \#\mathcal{N}_i^0] + [-\#\mathcal{N}_i^s, \#\mathcal{N}_i^s]$,

$$a_i = \xi_i(a_i^f + \alpha v_i),$$

where $v_i \in \{\#\mathcal{N}_i^+ - \#\mathcal{N}_i^-\} + [-\#\mathcal{N}_i^0, \#\mathcal{N}_i^0] + [-\#\mathcal{N}_i^s, \#\mathcal{N}_i^s]$. Choose $\xi_i = -v_i$. We immediately have

$$a_i = -v_i a_i^f - \alpha v_i^2 \leq |v_i|(\bar{f} - \alpha|v_i|).$$

Again, by using Proposition 2.2.9 of [13], we know that $|v_i| \geq 1$ if $v_i \neq 0$. It follows that

$$a_i \leq \bar{f} - \alpha < 0, \text{ for } v_i \neq 0.$$

Therefore, we have

$$\dot{\tilde{V}} = \sum_i (\dot{\tilde{V}})_i \leq \bar{f} - \alpha < 0.$$

From Proposition 4 in [17], it follows that $|x_i(t) - \frac{1}{n}\sum_i r_i(t)| \to 0$ in finite time. ■

Theorem 3: For the system (4), let $t \mapsto \sigma(t) : \mathbb{R}^+ \to \mathcal{I}_c$ be a switching signal. If $\alpha > (n-1)\bar{f}$, then $|x_i(t) - \frac{1}{n}\sum_j r_j(t)| \to 0$ as $t \to \infty$ for all $i = 1, \ldots, n$.

Proof: Let $\tilde{x}_i(t) \triangleq x_i(t) - \frac{1}{n}\sum_{j=1}^n x_j(t)$. Consider the Lyapunov function candidate

$$V(t) \triangleq \frac{1}{2}\|\tilde{x}(t)\|^2 = \frac{1}{2}\|x(t) - \frac{1}{n}x^T(t)\mathbf{1}\mathbf{1}\|^2. \quad (14)$$

Because the function $V(t)$ is continuously differentiable, we know that the generalized gradient of $V(t)$ with respect to $x_i(t)$ is a singleton. In particular, we have

$$(\partial V)_i = \{\frac{\partial V}{\partial x_i}\} = \{x_i - \frac{1}{n}x^T\mathbf{1}\}.$$

In the following, we calculate the set-valued Lie derivative of the function $V(t)$. If $\dot{\tilde{V}}(t) \neq \emptyset$, suppose that $a \in \dot{\tilde{V}}(t)$ and $a_i^f \in K[f_i]$. By the definition, we know that

$$a = \sum_{i=1}^n (x_i - \frac{1}{n}x^T\mathbf{1})v_i$$

with $v_i \in K[u_i]$. If $\forall i = 1, \ldots, n$, $x_j \neq x_i$ for all $j \in \mathcal{N}_i(t)$, then $K[u_i]$ is

$$K[u_i] = \{a_i^f + \alpha \sum_{j \in \mathcal{N}_i(t)} \text{sgn}(x_j - x_i)\}.$$

Define $\dot{\tilde{V}}_s \triangleq \{\sum_{i=1}^n (x_i - \frac{1}{n}x^T\mathbf{1})[a_i^f + \alpha\sum_{j \in \mathcal{N}_i(t)} \text{sgn}(x_j - x_i)]\}$. Then we have

$$\begin{aligned} \dot{\tilde{V}}_s &= \sum_{i=1}^n (x_i - \frac{1}{n}x^T\mathbf{1})a_i^f - \alpha\sum_{i=1}^n (x_i - \frac{1}{n}x^T\mathbf{1}) \\ &\quad \left\{\sum_{j \in \mathcal{N}_i(t)} \text{sgn}\left[(x_i - \frac{1}{n}x^T\mathbf{1}) - (x_j - \frac{1}{n}x^T\mathbf{1})\right]\right\}. \end{aligned}$$

We can derive that

$$\begin{aligned} &\sum_{i=1}^n (x_i - \frac{1}{n}x^T\mathbf{1})a_i^f \\ \leq\ &\frac{\bar{f}}{n}\sum_{i=1}^n \sum_{j \neq i} |x_i - x_j| \leq \bar{f} \max_{i=1,\ldots,n}\{\sum_{j \neq i} |x_i - x_j|\}. \end{aligned} \quad (15)$$

Since each graph is connected, we know that $\forall i \neq j$, $i, j = 1, \ldots, n$,

$$|x_i - x_j| \leq \frac{1}{2}\sum_{(i,j) \in \mathcal{E}(t)} |x_i - x_j|. \quad (16)$$

Then, (15) and (16) lead to

$$\sum_{i=1}^n (x_i - \frac{1}{n}x^T\mathbf{1})a_i^f \leq \frac{\bar{f}}{2}(n-1)\sum_{(i,j) \in \mathcal{E}(t)} |x_i - x_j|.$$

In addition, we have $\sum_{i=1}^n (x_i - \frac{1}{n}x^T\mathbf{1})\sum_{j \in \mathcal{N}_i(t)} \text{sgn}[(x_i - \frac{1}{n}x^T\mathbf{1}) - (x_j - \frac{1}{n}x^T\mathbf{1})] = \frac{1}{2}\sum_{(i,j) \in \mathcal{E}(t)} |x_i - x_j|$. Therefore, we know that

$$\dot{\tilde{V}}_s \leq \frac{\bar{f}}{2}(n-1)\sum_{(i,j) \in \mathcal{E}(t)} |x_i - x_j| - \frac{\alpha}{2}\sum_{(i,j) \in \mathcal{E}(t)} |x_i - x_j|.$$

Because $\alpha > \bar{f}(n-1)$, we know that $\dot{\tilde{V}}_s < 0$ for $\tilde{x} \neq 0$.

In the following, we will show that $\dot{\tilde{V}}_s = \dot{\tilde{V}}$ for almost every $t \in \mathbb{R}^+$ at the discontinuous points, that is, the points at which there exists $i \in \{1, \ldots, n\}$, $j \in \mathcal{N}_i$ such that $x_j = x_i$, except for the point $x_1 = x_2 = \cdots = x_n$. Let P denote the set of all discontinuous points. Then it suffices to show that for all $p \in P \backslash \{x | x_1 = x_2 = \cdots = x_n\}$, the system cannot stay at p for a time interval whose length is greater than 0. Define $M \triangleq \{k = 1, \ldots, n, |x_k = \max_j x_j\}$. Because each graph is connected, there must be a node $k \in M$ such that node k has a neighbor in $\{1, \ldots, n\} \backslash M$. Because

$$a_k^f + \alpha \sum_{j \in \mathcal{N}_k(t)} \text{sgn}(x_j - x_k) \leq \bar{f} - \alpha < 0, \qquad (17)$$

we know that the system cannot stay at p during a time interval. Therefore, we have $\dot{\tilde{V}} =^{a.e.} \dot{\tilde{V}}_s < 0$ for all $t \in \mathbb{R}^+$, which indicates that $\|\tilde{x}(t)\| \to 0$ as $t \to \infty$, which implies that $|x_i(t) - x_j(t)| \to 0$ as $t \to \infty$ for all $i, j = 1, \ldots, n$. Therefore, according to Lemma 6, we know that $|x_i(t) - \frac{1}{n} \sum_i r_i(t)| \to 0$ as $t \to \infty$ for all $i = 1, 2, \ldots, n$. ■

Theorem 4: For the system (4), if $\mathcal{G}(t)$ is directed and has a directed spanning tree at each time instant, and $\alpha > (n-1)\bar{f}$, then $|x_i(t) - x_j(t)| \to 0$ as $t \to \infty$ for all $i, j = 1, \ldots, n$. In particular, if $k[\mathcal{G}(t)] = 0$ for all $t \geq 0$, then $|x_i(t) - \frac{1}{n} \sum_j r_j(t)| \to 0$ as $t \to \infty$ for all $i = 1, \ldots, n$.

Proof: Consider the same Lyapunov function candidate as defined by (14). Notice that (16) still holds because $\mathcal{G}(t)$ has a directed spanning tree. In addition, we can show that

$$\sum_{i=1}^n x_i(t) = \sum_{i=1}^n r_i(t) + \alpha \int_0^t \sum_{(i,j) \in U(t)} \text{sgn}[x_j(\tau) - x_i(\tau)] d\tau.$$

Therefore, by using similar arguments to those in the proof of Theorem 3, we can obtain that

$$|x_i(t) - \frac{1}{n} \sum_{j=1}^n x_j(t)| \to 0$$

as $t \to \infty$, which implies that $|x_i(t) - x_j(t)| \to 0$ as $t \to \infty$ for all $i, j = 1, \ldots, n$. Note that $\frac{1}{n} \sum_{j=1}^n x_j(t) = \frac{1}{n} \sum_{i=1}^n r_i(t) + \frac{\alpha}{n} \int_0^t \sum_{(i,j) \in U} \text{sgn}[x_j(\tau) - x_i(\tau)] d\tau$. In particular, if $k[\mathcal{G}(t)] = 0$ for all $t \geq 0$, we know that $|x_i(t) - \frac{1}{n} \sum_{j=1}^n r_j(t)| \to 0$ as $t \to \infty$. ■

IV. Conclusions

We have presented in this paper a simple but appealing distributed control algorithm for a team of agents to solve the average tracking problem, which could find applications in various fields including mobile sensor networks, synchronization of oscillators, distributed estimation, decision making, or optimization. In the algorithm, each agent has a reference signal or measurement whose derivative is assumed to be bounded, and updates its states based on the information received from its neighbors and its reference signal. Our analysis started with undirected connected graphs. In this part, we have shown that the average tracking problem can be solved if the control gain is properly chosen. Then the result was extended to switching graphs. We have also considered the case of directed graphs.

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