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Global stabilization of the discrete-time double integrator using a saturated linear state feedback controller

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Abstract— **This paper further studies the discrete-time double integrator. Earlier it has been shown by us that there exists a class of locally stabilizing saturated linear state feedbacks which cause periodic behavior for certain initial conditions and hence the closed-loop system is then not globally asymptotically stable. In this paper, we show that all the remaining locally stabilizing saturated linear state feedbacks globally stabilize the discrete-time double integrator.**

I. Introduction

Linear systems subject to actuator saturation are ubiquitous and have been the subject of extensive study, see for instance two special issues, [1], [2], and references therein.

Internal stabilization for this class of systems has a long history. Let us briefly review the literature on linear systems subject to actuator saturation. [5], [9] established that, global stabilization of linear systems subject to actuator saturation can be achieved if and only if the linear system in the absence of actuator saturation is stabilizable, and has all its open-loop poles in the closed left-half plane for continuous-time linear systems and in the closed unit disc for discrete-time linear systems (equivalently, *asymptotically null controllable with bounded control*). In general, this requires nonlinear feedback control laws. However, for certain cases, global stabilization can be achieved by linear static state feedback control laws. For example, in both continuous-time and discrete-time settings, it is well known that there exists linear static state feedback control laws which globally stabilize neutrally stable linear systems subject to actuator saturation, see for instance [4]. Moreover, in continuous-time setting, there also exists linear static state feedback control laws which globally stabilize the system consisting of double-integrator, single-integrator, and neutrally stable dynamics, subject to actuator saturation, see for instance [6], [7]. Furthermore, in continuous-time setting, it is well known that a linear static state feedback law which locally stabilizes the double integrator also globally stabilizes the double integrator subject to actuator saturation. See for instance, [6], [3]. However, similar results have not yet been obtained for discrete-time.

In a previous conference paper [8], we considered the discrete-time equivalent of a double integrator subject to actuator saturation and showed that there are intrinsic differences between continuous- and discrete-time systems. In particular, a large class of linear state feedbacks, which achieve local stability for the discrete-time equivalent of the double integrator, fail to achieve global asymptotic stability. This property is established by explicitly constructing non-zero periodic solutions. The result in this earlier paper is therefore in direct contrast with continuous-time where local stability for the double integrator always implies global stability.

Although it was established earlier that there exists linear state feedbacks which achieve global stability, the objective of this paper is an attempt to completely classify all linear state feedbacks which result in global stability for the discrete-time equivalent of the double integrator.

II. Problem formulation and Review

Consider a discrete-time double integrator subject to actuator saturation described by

$$\begin{cases} x_1(k+1) = x_1(k) + x_2(k), \\ x_2(k+1) = x_2(k) + \sigma(u(k)), \end{cases} \tag{1}$$

where $\sigma(u(k))$ is the standard saturation function

$$\sigma(u(k)) = \operatorname{sgn}(u(k)) \min\{1, |u(k)|\},$$

and

$$u(k) = f_1 x_1(k) + f_2 x_2(k). \tag{2}$$

Let us first consider system (1) with a feedback control law (2) in the absence of actuator saturation. From Jury's test, we see that any feedback control law (2) where f_1 and f_2 satisfy the following conditions

$$\frac{1}{2} f_1 - 2 < f_2 < f_1 < 0, \tag{3}$$

stabilizes the system (1) in the absence of actuator saturation or, in other words, achieves local asymptotic stability.

Let us first recall our previous result from [8].

Theorem 1: If f_1 and f_2 satisfy Jury's condition (3) plus the following condition

$$f_2 > \frac{3}{2} f_1, \tag{4}$$

then the closed-loop system has non-zero periodic solutions for certain initial conditions and is therefore not globally asymptotically stable.

From Theorem 1, we see that in order to globally stabilize system (1) via a linear state feedback control law (2), it is

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necessary that the feedback gains f_1 and f_2 have to satisfy the following condition

$$\frac{1}{2}f_1 - 2 < f_2 < \frac{3}{2}f_1 < 0. \tag{5}$$

We will show that the condition (5) is also a sufficient condition in Section III.

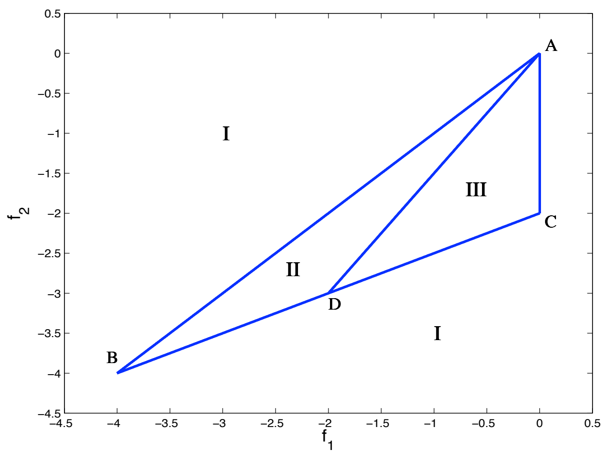

Fig. 1. Stability characteristics as a function of f_1 and f_2

Let us briefly summarize the results of our previous paper [8] and state the goal of this paper. The behavior of the closed-loop system can be illustrated by Figure 1. Note that in Figure 1, line AB is $f_2 = f_1$, line BC is $f_2 = \frac{1}{2}f_1 - 2$, line AD is $f_2 = \frac{3}{2}f_1$ and line AC is $f_1 = 0$. The Jury test establishes that whenever f_1 and f_2 take their values within the triangle ABC, the closed-loop system is locally asymptotically stable; otherwise unstable. However, in Region II (triangle ABD) the closed-loop system is not globally asymptotically stable since these controllers always yield non-zero periodic solutions as shown in [8]. In that paper it was, however, not established whether the closed-loop system is globally asymptotically stable in Region III (triangle ADC). Some preliminary results were obtained which showed that the closed-loop system is globally asymptotically stable in part of this Region III. The goal of this paper is to show that the closed-loop system is globally asymptotically stable in Region III. Consequently, the previous paper [8] and this paper complete the stability issues of the discrete-time double integrator via saturated linear state feedbacks.

III. Main Results

In this section, we present our main result.

Theorem 2: If f_1 and f_2 satisfy the Jury's condition (3) plus the following condition

$$f_2 < \tfrac{3}{2}f_1, \tag{6}$$

then the closed-loop system is globally asymptotically stable.

In order to prove Theorem 2, we need to establish asymptotic stability in the region III depicted in Figure 1. A basis transformation turns out to be useful for establishing this result. We define $y_1(k) = u(k)$ and $y_2(k) = f_1 x_2(k)$. The closed-loop system is then given by:

$$\begin{cases} y_1(k+1) = y_1(k) + y_2(k) + f_2\sigma(y_1(k)), \\ y_2(k+1) = y_2(k) + f_1\sigma(y_1(k)). \end{cases} \tag{7}$$

We sometimes denote:

$$y(k) = \begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix}$$

and y, y_1 or y_2 without explicitly indicating time will refer to $y(k)$, $y_1(k)$ or $y_2(k)$ respectively.

Let us recall the following lemma from [8].

Lemma 1: Consider system (1) with a feedback control law (2). With the basis transformation, $y_1(k) = u(k)$ and $y_2(k) = f_1 x_2(k)$, the closed-loop system is given by (7). The following Lyapunov candidate

$$V_k = 2y_1\sigma(y_1) - \sigma^2(y_1) - 2\sigma(y_1)y_2 - \tfrac{1}{f_1}y_2^2$$

establishes the global asymptotic stability of the closed-loop system (7) if the feedback gains f_1 and f_2 are in Region IV of Figure 2, that is, condition (5) is satisfied and

$$(f_2 - f_1 + 1)^2 - 1 < f_1. \tag{8}$$

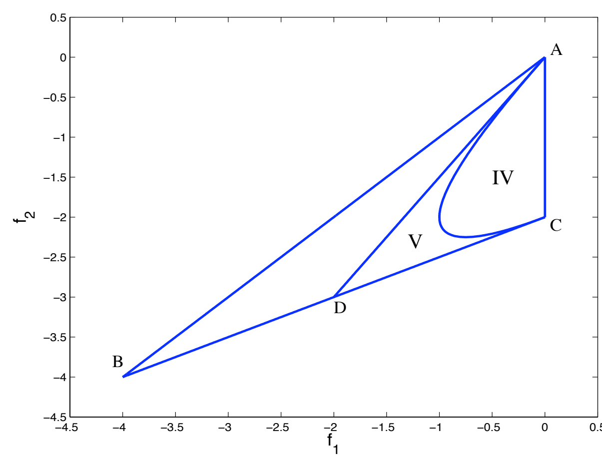

Fig. 2. Stability characteristics as a function of f_1 and f_2

Note that Region III has been split into two regions: Region IV and Region V, as depicted in Figure 2. Also note that Lemma 1 shows that the closed-loop system is globally asymptotically stable in Region IV, thus in order to prove Theorem 2, it remains to show that the closed-loop system is globally asymptotically stable in Region V.

Let us consider a Lyapunov candidate in the presence of saturation, which is based on the linearized system as follows:

$$V_k = 2y_1\sigma(y_1) - \sigma^2(y_1) + 2b\sigma(y_1)y_2 - \tfrac{1}{f_1}y_2^2, \tag{9}$$

where

$$b = \begin{cases} \frac{2}{f_2} & f_2^2 + 4f_1 \geq 0, \\ -\frac{f_2}{2f_1} & f_2^2 + 4f_1 < 0. \end{cases} \tag{10}$$

It is easy to verify that in the Region V of Figure 2 we have $b \in [-1, -0.5)$ while for $b = -1$ we get the Lyapunov function used in Lemma 1. We sometimes refer to the first case, when $f_2^2+4f_1 \geq 0$ as the real case since in that case the linearized system has real eigenvalues while the second case, when $f_2^2+4f_1<0$, is referred to as the complex case since in that case the linearized system has complex eigenvalues.

It is easy to see that the Lyapunov candidate (9) works for the linearized closed-loop system. In order to be a valid Lyapunov function, it is necessary that it must work when $\sigma(y_1)$ stays at 1 or at -1 in two consecutive time instants. It is easy to verify that in that case:

$$\Delta V=(2b-1)f_1+2f_2, \tag{11}$$

where $(\Delta V)(k)=V_{k+1}-V_k$, while $V_k=V(y(k))$. Thus, $\Delta V=(f_2^2+4f_1)/f_2+(f_2-f_1)<0$ in the real case while $\Delta V=f_2-f_1<0$ in the complex case.

Therefore, the Lyapunov candidate (9) has the required properties when $\sigma(y_1)$ is in saturation for two consecutive time instants or is out of saturation for two consecutive time instants. Note that for a continuous-time problem, we would be done, since y_1 is continuous. However, for discrete-time systems, y_1 obviously jumps from one time to the other and hence if $\sigma(y_1(k))$ saturates then it might well be that $\sigma(y_1(k+1))$ is out of saturation or conversely. This is intrinsically different from the continuous-time case. Thus, we have to show that the Lyapunov candidate (9) also decreases when y_1 jumps either into saturation or out of saturation. The traditional Lyapunov argument is to show that $V_{k+1}-V_k<0$ for all initial conditions. However, this approach does not work here. For the real case, if $f_2<-2$, there exist initial conditions, such that $V_{k+1}-V_k>0$. A similar problem can arise in the complex case. Thus, we need a different technique. The main idea is to show that V decreases over a specifically chosen number of time steps, and V is bounded in the interim. In order to proceed with this idea, we first choose suitable time instants k_i. The formal definition of k_i is given by:

Definition 1: $k_0=0$, and k_i is the smallest integer larger than k_{i-1}, such that either

- $|y_1(k_i)|<1$; or
- $y_1(k_i)y_1(k_i+1)<0$ and $|y_1(k_i+1)| \geq 1$.

In other words, k_i is defined as the first time instant $k>k_{i-1}$ where $y_1(k)$ either gets out of saturation, or where $y_1(k)$ switches sign. It is easily seen that k_i is well defined given k_{i-1} since the only way k_i would not be well defined is if $y_1(k)>1$ for all $k>k_{i-1}$ or if $y_1(k)<-1$ for all $k>k_{i-1}$. It is easily seen from the dynamics (7) that this is not possible. Instead of a classical Lyapunov design we will study whether V_{k_i} is decreasing as a function of i. Before we formally prove Theorem 2, we present two crucial lemmas.

Lemma 2: Let the Lyapunov candidate V be defined in (9) and assume the feedback gains f_1 and f_2 are in Region V of Figure 2, that is, (5) is satisfied and

$$(f_2-f_1+1)^2-1>f_1. \tag{12}$$

In that case, if $|y_1(k_i)|<1$ and $V_{k_{i-1}} \neq 0$, then

$$V_{k_i}-V_{k_{i-1}}<0.$$

Proof: Since the proof is very lengthy, for the readability, we give the proof in Section IV. ∎

Lemma 3: Let the Lyapunov candidate V be defined in (9) and assume the feedback gains f_1 and f_2 are in region V of Figure 2, that is, (5) is satisfied and

$$(f_2-f_1+1)^2-1>f_1.$$

In that case, if $|y_1(k_i)| \geq 1$ and $V_{k_{i-1}} \neq 0$, then

$$V_{k_i}-V_{k_{i-1}}<0 \quad \text{or} \quad V_{k_{i+1}}-V_{k_{i-1}}<0$$

Proof: Due to the space limitation, we have omitted the proof. ∎

Remark 1: Note that if the feedback gains f_1 and f_2 take their values inside the triangle ABD (Region II) in Figure 1, there actually exist initial conditions for which $V_{k_{i+1}}-V_{k_{i-1}}=0$ since $k_{i+1}-k_{i-1}$ is precisely the period of the periodic behavior as constructed in the proof of Theorem 1 in [8].

Proof of Theorem 2: We know that the system is locally asymptotically stable from Jury's test. It remains to show global attractivity of the origin. If (8) is satisfied, Lemma 1 guarantees global asymptotic stability. Therefore, we only need to consider the case where (12) is satisfied in addition to (5).

We first note that (9) can be rewritten as

$$V(y)=2\sigma(y_1)\left[y_1-\sigma(y_1)\right]+\left[\sigma(y_1)+by_2\right]^2-\left(b^2+\tfrac{1}{f_1}\right)y_2^2.$$

It is easy to show that $b^2+\frac{1}{f_1} \leq 0$. This immediately implies $V(y)>0$ if $y \neq 0$ unless $b^2+\frac{1}{f_1}=0$, or equivalently, $f_2^2+4f_1=0$. However for the latter case, it is easily verified that $V(y(k))=0$ implies that $V(y(k+1))=0$ and that $y(k) \to 0$ as $k \to \infty$.

Next, we note that y is bounded since $V(y)$ is bounded. Moreover, given a M there exists a K such that $\|y_{k_i}\|<M$ implies that $k_{i+1}-k_i<K$.

Lemma 2 and Lemma 3 imply that either $V_{k_{i+1}}-V_{k_i}<0$ or $V_{k_{i+2}}-V_{k_i}<0$ if $V_{k_i} \neq 0$. This results in a sequence $\{\bar{k}_i\}$ such that $V_{\bar{k}_{i+1}}<V_{\bar{k}_i}$ for all i. This clearly implies that $V_{\bar{k}_i}$ is bounded and hence $\bar{k}_{i+1}-\bar{k}_i$ is bounded as well. This implies that $V_{\bar{k}_i} \to 0$ as $i \to \infty$.

If $b^2+\frac{1}{f_1} \neq 0$, then local asymptotic stability implies that if $V_{\bar{k}_i}$ is small enough for some i then $y(k) \to 0$ as $k \to \infty$ and therefore we have global attractivity. For the case that $b^2+\frac{1}{f_1}=0$, that is, $f_2^2+4f_1=0$, global attractivity follows by using a slightly modified version of LaSalle's invariance principle. ∎

IV. Proof of Lemma 2

We first note that (3), (6) and (12) imply that b as defined in (10) satisfies $b \in (-1,-\frac{2}{3}]$ in the real case and $b \in (-1,-\frac{3}{4}]$ in the complex case.

For simplicity we denote $y_1(k_{i-1})$ and $y_2(k_{i-1})$ by y_1 and y_2 respectively while $y_1(k_i)$ and $y_2(k_i)$ are denoted by $\tilde{y}_1$ and $\tilde{y}_2$ respectively. We will prove the Lyapunov function will decay for two cases:

- case 1: $y_1 \geq 1$ and $\tilde{y}_1 \in [-1,1]$,
- case 2: $y_1 \in [-1,1]$ and $\tilde{y}_1 \in [-1,1]$.

Without loss of generality we only consider $y_1 \geq 1$ (the other case where $y_1 \leq -1$ is completely symmetric). Let us first consider case 1.

Proof of Lemma 2 with $y_1 \geq 1$: In the case where $y_1 \geq 1$, we have

$$\tilde{y}_1 = y_1 + ky_2 + e_1,$$
$$\tilde{y}_2 = y_2 - (k-2)f_1,$$

where we denote $k = k_i - k_{i-1}$ while

$$e_1 = f_2 + (k-1)(f_1 - f_2) - \tfrac{f_1}{2}(k-1)(k-2).$$

We will prove the Lyapunov function defined in (9) will decay if $y_1 \geq 1$ and $\tilde{y}_1 \in [-1,1]$. In doing this, we ignore the other constraints which follow from the definition of k_i, namely that $y_1(k_{i-1}+j) \leq -1$ for $j = 1,\dots,k-1$. However, if the Lyapunov function always decays without these constraints then it will definitely still decay when these additional constraints are imposed. We get

$$V_{k_i} - V_{k_{i-1}} = \tilde{y}_1^2 + 2b\tilde{y}_1\tilde{y}_2 - \frac{1}{f_1}\tilde{y}_2^2 - 2y_1 + 1 - 2by_2 + \frac{1}{f_1}y_2^2.$$

This can be rewritten completely in terms of $\tilde{y}_1$ and y_1 as:

$$\begin{aligned} V_{k_i} - V_{k_{i-1}} = {} & (1 + 2\tfrac{b}{k})\tilde{y}_1^2 + \left[-2\tfrac{b}{k}(\tilde{y}_1 - 1) - 4\tfrac{k-1}{k}\right] y_1 \\ & + \left[-2\tfrac{b}{k}e_1 + 2(2-k)bf_1 + 2 - 2(2+b)\tfrac{1}{k}\right]\tilde{y}_1 \\ & - 2e_1 + 2(2+b)\tfrac{1}{k}e_1 - (k-2)^2 f_1 + 1. \end{aligned}$$

We need to show this is negative for all $y_1 \geq 1$ and all $\tilde{y}_1 \in [-1,1]$. However, this is a linear function of y_1 whose coefficient is negative and hence $V_{k_i} - V_{k_{i-1}}$ is maximal for $y_1 = 1$. Thus, we have

$$\begin{aligned} V_{k_i} - V_{k_{i-1}} \leq {} & (1 + 2\tfrac{b}{k})\tilde{y}_1^2 + [-2\tfrac{b}{k}e_1 + 2(2-k)bf_1 + 2 \\ & - 4(1+b)\tfrac{1}{k}]\tilde{y}_1 - 2e_1 + 2(2+b)\tfrac{1}{k}e_1 \\ & + 2\tfrac{b}{k} - 4\tfrac{k-1}{k} - (k-2)^2 f_1 + 1. \end{aligned} \quad (13)$$

The upper bound is a quadratic function which we need to maximize. Clearly, the sign of the quadratic term is crucial here. For $k = 1$ the coefficient of the quadratic term is negative and the maximum is obtained by setting the derivative equal to zero (if we ignore that $\tilde{y}_1 \in [-1,1]$). Thus, we obtain for $k = 1$:

$$\begin{aligned} V_{k_i} - V_{k_{i-1}} \leq {} & (1+2b)\tilde{y}_1^2 + [2b(f_1 - f_2) - 2(1+2b)]\tilde{y}_1 \\ & + 2b + 1 - f_1 + 2(1+b)f_2. \end{aligned} \quad (14)$$

For the real case ($f_2^2 + 4f_1 > 0$), since $b = 2f_2^{-1}$ and using (3), we obtain from (14) that

$$V_{k_i} - V_{k_{i-1}} \leq (f_2^2 + 4f_1)(4 + 2f_2 - f_1)f_2^{-1}/(f_2 + 4) < 0.$$

For the complex case ($f_2^2 + 4f_1 < 0$), since $b = -f_2 f_1^{-1}/2$ and again using (3), we obtain from (14) that

$$V_{k_i} - V_{k_{i-1}} \leq (f_2 - f_1)(f_2^2 + 4f_1)f_1^{-1}/4 < 0.$$

Next, we return to the case where $k > 1$. In that case, the upper bound (13) has a quadratic term with a positive coefficient. Therefore, the maximum is attained on the boundary, that is, $\tilde{y}_1 = 1$ or $\tilde{y}_1 = -1$. For $\tilde{y}_1 = 1$ we obtain:

$$\begin{aligned} V_{k_i} - V_{k_{i-1}} & \leq (k-2)\left[-2bf_1 - \tfrac{4}{k}(f_2 - f_1) - 3f_1 + 2f_2\right] \\ & \leq (k-2)[-2bf_1 - 2(f_2 - f_1) - 3f_1 + 2f_2] \\ & \leq (2-k)(2b+1)f_1 \leq 0, \end{aligned}$$

where we used $k \geq 2$ and $b < -0.5$. Note that the upper bound is negative unless $k = 2$ in which case it is easy to verify that the decay equals zero only if $\tilde{y}_1 = 1$ and $y_1 = 1$. The latter is inconsistent with $k = 2$ since we then get

$$y(k_{i-1}+1) = y_1 + y_2 + f_2 = 1 - \tfrac{1}{2}f_1 \geq 1$$

while we should have $y(k_{i-1}+1) \leq -1$. Next, we need to investigate the other boundary $\tilde{y}_1 = -1$. We get

$$V_{k_i} - V_{k_{i-1}} \leq [2b - (k-2)]\left[(3f_1 - 2f_2) - \tfrac{4}{k}(f_1 - f_2 - 1)\right] < 0 \quad (15)$$

The first inequality is a simple rewriting of our upper bound for $\tilde{y}_1 = -1$. The second inequality is more subtle. It is easy to see that $2b - (k-2) < 0$ for $k \geq 2$. If $f_1 - f_2 - 1 \leq 0$, we immediately find that the expression is negative since we know from (6) that $3f_1 - 2f_2 > 0$. On the other hand if $f_1 - f_2 - 1 > 0$, we find that

$$\begin{aligned} (3f_1 - 2f_2) - \tfrac{4}{k}(f_1 - f_2 - 1) & \geq (3f_1 - 2f_2) - 2(f_1 - f_2 - 1) \\ & = f_1 + 2 > 0 \end{aligned}$$

and the inequality (15) is satisfied. The fact that $f_1 > -2$ follows from (5). ■

Next, we need to study case 2, that is,

$$y_1 \in [-1,1] \quad \text{and} \quad \tilde{y}_1 \in [-1,1]. \quad (16)$$

The proof is split into two cases: the real case ($f_2^2 + 4f_1 \geq 0$) and the complex case ($f_2^2 + 4f_1 < 0$). Due to the space limitation, we only present the proof for the real case:

Proof of Lemma 2 with $y_1 \in [-1,1]$ and $f_2^2 + 4f_1 \geq 0$: In this case, we have

$$\tilde{y}_1 = d_4 y_1 + ky_2 + e_4 \quad (17)$$
$$\tilde{y}_2 = f_1 y_1 + y_2 - (k-1)f_1, \quad (18)$$

where we denote $k = k_i - k_{i-1}$ and

$$d_4 = 1 + f_2 + (k-1)f_1$$
$$e_4 = -(k-1)\left(f_2 - f_1 + \tfrac{1}{2}f_1 k\right).$$

Given (16), we find that:

$$V_{k_i} - V_{k_{i-1}} = \tilde{y}_1^2 + 2b\tilde{y}_1\tilde{y}_2 - \tfrac{1}{f_1}\tilde{y}_2^2 - y_1^2 - 2by_1y_2 + \tfrac{1}{f_1}y_2^2.$$

We can eliminate $\tilde{y}_2$ and y_2 from the above expression by using (17) and (18):

$$\begin{aligned} V_{k_i} - V_{k_{i-1}} = {} & \tilde{y}_1^2 + 2b\tilde{y}_1[f_1y_1 + \tfrac{1}{k}(\tilde{y}_1 - d_4y_1 - e_4) - \\ & (k-1)f_1] - \tfrac{1}{f_1}\left[f_1y_1 + \tfrac{1}{k}(\tilde{y}_1 - d_4y_1 - e_4) - (k-1)f_1\right]^2 \\ & - y_1^2 - 2\tfrac{b}{k}y_1[\tilde{y}_1 - d_4y_1 - e_4] + \tfrac{1}{k^2 f_1}[\tilde{y}_1 - d_4y_1 - e_4]^2. \end{aligned} \quad (19)$$

Our objective is now to prove that (19) is negative. We first note that for $k=1$ we only need to study the unsaturated linear system and it is easy to verify that (19) is negative provided $y(k_{i-1})\neq 0$. For $k=2$ we will show that (19) is negative for all

$$-1\le y_1\le 1,\qquad -1\le \tilde{y}_1\le -1-(1+f_2-f_1)(y_1+1)-f_1 \tag{20}$$

where the upper bound for $\tilde{y}_1$ follows from the constraint that $y_1(k_i-1)\le -1$. For $k>2$ we consider all

$$-1\le y_1\le 1,\qquad -1\le \tilde{y}_1\le -1 \tag{21}$$

and we ignore all other constraints which follow from the definition of k_i namely that $y_1(k_{i-1}+j)\le -1$ for $j=1,\dots,k-1$.

The quadratic term in $\tilde{y}_1$ in (19) is equal to

$$1+2b\tfrac{1}{k}$$

which is positive for $k\ge 2$ since $b>-1$. Therefore, we know (19) is maximal in a boundary point, that is, for $k=2$,

$$\tilde{y}_1=-1 \text{ or } \tilde{y}_1=-1-(1+f_2-f_1)(y_1+1)-f_1,$$

and for $k>2$,

$$\tilde{y}_1=-1 \text{ or } \tilde{y}_1=1.$$

For the lower bound $\tilde{y}_1=-1$ we do not need to distinguish between $k=2$ and $k>2$ and we obtain that

$$\begin{aligned}V_{k_i}-V_{k_{i-1}}=1-2bf_1\left[y_1-(k-1)\right]\\ -\tfrac{2b}{k}(1+y_1)\left[-1-d_4y_1-e_4\right]\\ -\tfrac{2}{k}\left[y_1-(k-1)\right]\left[-1-d_4y_1-e_4\right]\\ -f_1\left[y_1-(k-1)\right]^2-y_1^2\end{aligned} \tag{22}$$

and we need to show this expression is negative. The expression has the form:

$$\bar{a}y_1^2+\bar{b}y_1+\bar{c}. \tag{23}$$

Here we have:

$$\bar{a}=(2b+1)f_1-1+\tfrac{2}{k}(b+1)(1+f_2-f_1), \tag{24}$$

$$\begin{aligned}\bar{b}=-(1+b)f_1k+[(1+b)(3f_1-2f_2)-2(1+f_2-f_1)]\\ +\tfrac{4}{k}(1+b)(1+f_2-f_1),\end{aligned} \tag{25}$$

$$\begin{aligned}\bar{c}=[(1+b)f_1-(3f_1-2f_2)]k\\ +[1-(2b+1)f_1+(b+1)(3f_1-2f_2)-2(1+f_2-f_1)]\\ +\tfrac{2}{k}(1+b)(1+f_2-f_1).\end{aligned} \tag{26}$$

We note that $\bar{a}<0$. Since if $1+f_2-f_1\le 0$, we have

$$\bar{a}\le(2b+1)f_1-1\le(2b+1)(-\tfrac{1}{4}f_2^2)-1=-(\tfrac{1}{2}f_2+1)^2<0,$$

where we used that $2b+1<0$, the fact that in the real case $f_2^2+4f_1\ge 0$ and the definition of b. If $1+f_2-f_1>0$, we then obtain that $\bar{a}$ is maximal for $k=2$ and we obtain:

$$\begin{aligned}\bar{a}\le bf_1+(1+b)f_2+b\le b(-\tfrac{1}{4}f_2^2)+(1+b)f_2+b\\ =\tfrac{1}{2f_2}(f_2+2)^2<0.\end{aligned}$$

We need to verify that (23) is negative for all $y_1\in[-1,1]$. We first verify it is negative at the boundary points. We get for $y_1=-1$ that (23) equals:

$$\bar{a}-\bar{b}+\bar{c}=[2(1+b)f_1-(3f_1-2f_2)]k<0.$$

In the proof of Lemma 2 with $y_1\ge 1$ we already established that for $y_1=1$ we have:

$$V_{k_i}-V_{k_{i-1}}<0.$$

Finally, (23) may attain its maximum in the interior where

$$y_1^*=-\frac{\bar{b}}{2\bar{a}}\quad\text{with}\quad\left|\frac{\bar{b}}{2\bar{a}}\right|<1,$$

but then the maximum is less than $\bar{c}-\bar{a}$ and we get

$$\bar{c}-\bar{a}=[(1+b)f_1-(3f_1-2f_2)]k+(3-b)f_1-(2b+4)f_2.$$

Note that the above expression is a linear function of k whose coefficient is negative since $b>-1$ and $3f_1-2f_2>0$ and hence it is maximal for $k=2$. Thus, we get

$$\begin{aligned}\bar{c}-\bar{a}\le(b-1)f_1-4&=\tfrac{1}{f_2}\left[2f_1-f_2(f_1+4)\right]\\ &\le\tfrac{1}{f_2}\left[2f_1+2(f_1+4)\right]\\ &=\tfrac{4}{f_2}(f_1+2)<0.\end{aligned}$$

In other words, for $\tilde{y}_1=-1$ (19) is negative.

It remains to check whether (19) is negative for the upper bound for $\tilde{y}_1$. Unfortunately, here we have to distinguish between $k=2$ and $k>2$. For $k=2$ we have

$$\tilde{y}_1=-1-(1+f_2-f_1)(y_1+1)-f_1$$

for the upper bound. We obtain that

$$V_{k_i}-V_{k_{i-1}}=\hat{a}y_1^2+\hat{b}y_1+\hat{c}. \tag{27}$$

Here we have:

$$\begin{aligned}\hat{a}&=(1+2b)(f_1-f_2-1)^2-(f_1+1)+2(1+b)(f_2+1),\\ \hat{b}&=-2(1+b)(f_1-f_2-1)(f_2+2)-2b(f_1-f_2-1)(1+f_1)\\ &\quad+2(1+b)+2(f_1-f_2-1),\\ \hat{c}&=(f_2+2)^2+2b(f_2+2)(f_1+1)+2(f_1-f_2-1)\\ &\quad+2f_2-3f_1.\end{aligned}$$

Using $-4f_1<f_2^2$, we get:

$$\hat{a}<(1+2b)(f_1-f_2-1)^2+\tfrac{1}{4f_2}(f_2+4)(f_2+2)^2<0$$

where we used that $1+2b<0$ and $-4<f_2<-2$. Therefore the maximum is attained on the boundary or in the interior. Next we show that $-\frac{\hat{b}}{2\hat{a}}\ge 1$. Since $\hat{a}<0$, it is equivalent to show that $\hat{b}+2\hat{a}>0$. With some algebra, we get:

$$\hat{b}+2\hat{a}=2(2f_2-f_1+4)[-(f_1-f_2-1)+1](1+\tfrac{2}{f_2})>0.$$

Thus, for case $k=2$, $V_{k_i}-V_{k_{i-1}}$ in (27) is maximal when $y_1=1$, and the maximum is $\hat{a}+\hat{b}+\hat{c}$. Next, we show that this is indeed negative. We get:

$$\begin{aligned}\hat{a}+\hat{b}+\hat{c}&=4(f_2-\tfrac{1}{2}f_1+2)(f_2+\tfrac{4}{f_2}-\tfrac{1}{2}f_1+3)\\ &<4(f_2-\tfrac{1}{2}f_1+2)(-\tfrac{1}{2}f_1-1)<0.\end{aligned}$$

The following step is to check the upper bound $\tilde{y}=1$ for $k>2$. We obtain that

$$\begin{aligned}V_{k_i}-V_{k_{i-1}} &= 1+2bf_1\left[y_1-(k-1)\right] \\ &+\tfrac{2b}{k}(1-y_1)\left[1-d_4y_1-e_4\right]-\tfrac{2}{k}\left[y_1-(k-1)\right]\left[1-d_4y_1-e_4\right] \\ &-f_1\left[y_1-(k-1)\right]^2-y_1^2. \quad (28)\end{aligned}$$

and we need to show this expression is negative. The expression has the form:

$$\tilde{a}y_1^2+\tilde{b}y_1+\tilde{c}. \quad (29)$$

Here we have:

$$\begin{aligned}\tilde{a} &= (2b+1)f_1-1+\tfrac{2}{k}(b+1)(1+f_2-f_1), \\ \tilde{b} &= -(b+1)kf_1+b(3f_1-2f_2)-2-4f_2+5f_1 \\ &\quad +\tfrac{1}{k}(-4b-4f_1+4f_2), \\ \tilde{c} &= k(-bf_1+2f_2-2f_1)+3+4f_1-4f_2+b(2f_2-f_1) \\ &\quad +\tfrac{2}{k}(b-1)(1-f_2+f_1).\end{aligned}$$

Since $\tilde{a}=\bar{a}$ we already showed that $\tilde{a}$ is negative. In the proof of Lemma 2 with $y_1\geq 1$ we already established that for $y_1=1$ we have:

$$V_{k_i}-V_{k_{i-1}}<0.$$

On the other hand for $y_1=-1$ we have:

$$\tilde{a}-\tilde{b}+\tilde{c}=k(2f_2-f_1)+2b(2f_2-f_1)+4+\frac{8b}{k}.$$

For $k=3$ we get:

$$\begin{aligned}6f_2-3f_1+12-\tfrac{4}{f_2}f_1+\tfrac{16}{3f_2} &= 6f_2-(3+\tfrac{4}{f_2})f_1+12+\tfrac{16}{3f_2} \\ &< 6f_2-2f_1+12+\tfrac{16}{3f_2},\end{aligned}$$

This upper bound equals:

$$\tfrac{2}{3}(2f_2-3f_1)+\tfrac{1}{3f_2}(f_2+2)(14f_2+8)<0,$$

where we used that $2f_2-3f_1<0$ from (6) and $f_2<-2$. For $k>3$ we have:

$$\begin{aligned}\tilde{a}-\tilde{b}+\tilde{c} &< k(2f_2-f_1)+2b(2f_2-f_1)+4 \\ &\leq 4(2f_2-f_1)+2b(2f_2-f_1)+4 \\ &= 8f_2-4(1+\tfrac{1}{f_2})f_1+12 \\ &< 8f_2-\tfrac{8}{3}f_1+12<0.\end{aligned}$$

It remains to show that the maximum of (29) is also negative if (29) attains its maximum in the interior. As before, we note that the maximum is less than $\tilde{c}-\tilde{a}$ and we get

$$\begin{aligned}\tilde{c}-\tilde{a} &= k[-(b+2)f_1+2f_2]+3(1-b)f_1+2(b-2)f_2 \\ &\quad +4+\tfrac{4}{k}[b(f_1-f_2)-1].\end{aligned}$$

For $k=3$ we get:

$$\tilde{c}-\tilde{a}=-(\tfrac{28}{3f_2}+3)f_1+2f_2+4<\tfrac{1}{9}f_1+2f_2+4<0$$

while for $k>3$ we get:

$$\begin{aligned}\tilde{c}-\tilde{a} &< 4[-(b+2)f_1+2f_2]+3(1-b)f_1+2(b-2)f_2+4 \\ &= -(7b+5)f_1+4(f_2+2).\end{aligned}$$

If $7b+5<0$ then this expression is negative since $f_1<0$ and $f_2<-2$. On the other hand, if $7b+5>0$ then

$$\begin{aligned}-(7b+5)f_1+4f_2+8 &< \tfrac{28}{f_2}+4f_2+18<\tfrac{22}{f_2}+4f_2+16 \\ &= \tfrac{2}{f_2}\left(2(f_2+2)^2+3\right)<0\end{aligned}$$

since $f_1\in(-2,0)$ and $f_2\in(-3,-2)$. This completes the proof of Lemma 2 with $y_1\in[-1,1]$ and $f_2^2+4f_1\geq 0$. ■

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