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Improved Stabilization Conditions for Takagi-Sugeno Fuzzy Systems via Fuzzy Integral Lyapunov Functions

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***Abstract*— This paper presents new results concerning the design of state feedback controllers for continuous-time Takagi-Sugeno (T–S) fuzzy systems. The conditions, based on a line-integral fuzzy Lyapunov function, are specially suitable for T–S fuzzy systems where no information about the time-derivatives of the membership functions is available. The controller is designed through linear matrix inequalities in a two step procedure: at the first step, a stabilizing fuzzy controller is obtained for a relaxed frozen (i.e. time-invariant) T–S fuzzy system. This control gain is then used as an input data at the second step, that provides a stabilizing control law guaranteed by the line-integral Lyapunov function. An extension to cope with $\mathcal{H}_\infty$ guaranteed cost control of T–S fuzzy systems is also provided. Numerical examples illustrate the advantages of the proposed method when compared to other techniques available in the literature.**

I. Introduction

Takagi-Sugeno (T–S) fuzzy models [1] provide a convenient strategy for the study of nonlinear systems, by describing the local dynamics in different regions of the state space using linear models, that are combined to represent the overall system. For the stability analysis and control design of this class of systems, the Lyapunov stability theory has been widely applied, leading to optimization procedures based on Linear Matrix Inequalities (LMIs). LMIs are very attractive since they can be solved by algorithms with global convergence in polynomial time [2–4].

One of the first methods for control design applied to T–S fuzzy systems is the parallel distributed compensator [5], where the control law shares the same fuzzy rules and sets of the T–S system. The standard approach to cope with stability and stabilization issues in this context is based on a common quadratic Lyapunov function, that guarantees the closed-loop stability and other performance criteria. LMI conditions for state and output feedback control for T–S fuzzy systems in several different contexts can be found in [6]. It is well known, however, that LMI conditions based on a common Lyapunov matrix are in general conservative, particularly in the case of a large number of rules. Therefore, much effort has been devoted to obtain less conservative conditions.

Fuzzy Lyapunov functions appeared as a more general alternative to the use of a common quadratic Lyapunov function [7], being constructed as a fuzzy blending of multiple quadratic in the state functions, similarly to the way the T–S model is built. Several results based on fuzzy Lyapunov functions can be found for discrete-time T–S fuzzy systems [8–13]. On the other hand, fewer results have been reported for continuous-time T–S fuzzy systems [14–19]. One of the difficulties of handling fuzzy Lyapunov functions in continuous-time is the explicit presence of the time-derivative of the membership functions in the standard Lyapunov stability conditions. Some strategies to handle time-derivatives in the LMI conditions for stability appeared, for instance, by taking into account upper bounds [18, 20]. However, it can be very difficult to obtain such bounds in control design problems. Another alternative is to use a line-integral Lyapunov function, as proposed bin [15], that can be used only if the premise variables are the states. As a consequence, no information about the bounds on the time-derivatives of the membership functions is required and sufficient LMI conditions for stability can be obtained by simply imposing a particular structure to the Lyapunov matrix. Moreover, arbitrary variations of the membership functions are allowed. Although useful for analysis purposes, the synthesis conditions derived in [15] are bilinear matrix inequalities. This drawback has been surpassed in [17], where slack matrix variables are used to decouple the Lyapunov matrices from the system matrices, providing new degrees of freedom to the corresponding LMI problem for stability analysis and stabilization of T–S fuzzy systems. In order to guarantee the desired structure to the Lyapunov matrix, however, extra constraints have been imposed to the slack variables in [17], increasing the conservativeness of the conditions.

The aim of this paper is to provide less conservative LMI conditions for stabilizability of T–S fuzzy systems when no information about the time-derivative of the membership functions is available. For that, a generalized line-integral fuzzy Lyapunov function is proposed. The stabilizing state feedback control is obtained through LMI conditions solved in two steps, inspired by a strategy proposed in [21–25] for static output feedback control. At the first step, an auxiliary state feedback controller is obtained. This gain, which can be viewed as a stabilizing control law for the frozen (i.e. time-invariant) T–S fuzzy system, is then used at the second stage. A feasible solution at the second step provides a stabilizing state feedback control gain and a structured Lyapunov matrix assuring closed-loop stability, without imposing any constraint to the slack variables. An extension to cope with $\mathcal{H}_\infty$ control design for T–S fuzzy systems is also presented. Examples illustrate the efficacy of the proposed approach, that provides less conservative results for stabilizability and $\mathcal{H}_\infty$ control ofthe considered class of T–S fuzzy systems when compared to other conditions from

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the literature.

II. Problem Description

Consider the i-th rule of a continuous-time Takagi-Sugeno fuzzy model (see [6]), given by

$$\begin{aligned}&\mathcal{R}_i : \text{If } x_1(t) \text{ is } M_1^{\alpha_{i1}} \text{ and } \dots \text{ and } x_n(t) \text{ is } M_n^{\alpha_{in}}\\ &\quad\text{Then} \left\{ \begin{array}{l} \dot{x}(t) = A_i x(t) + B_i u(t) + E_i w(t) \\ y(t) = C_i x(t) + D_i u(t) + F_i w(t) \end{array}\right.\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^q$ is the controlled output, $w(t) \in \mathbb{R}^o$ is the exogenous input, $u(t) \in \mathbb{R}^m$ is the control input, and the linear subsystem matrices are $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $E_i \in \mathbb{R}^{n\times o}$, $C_i \in \mathbb{R}^{q\times n}$, $D_i \in \mathbb{R}^{q\times m}$ and $F_i \in \mathbb{R}^{q\times o}$. The states are chosen as premise variables, $M_j^{\alpha_{ij}}$ denotes a x_j-based fuzzy set used for the i-th fuzzy rule, where α_{ij} specifies which x_j-based fuzzy set is used in the i-th fuzzy rule. The total number of fuzzy rules is r. For instance, if $\alpha_{11} = \alpha_{21} = k$ then it means that in rules 1 and 2 the premise variable $x_1(t)$ belongs to the same fuzzy set, M_1^k.

The membership function of the i-th fuzzy rule is given by $h_i(x(t))$ and has the properties

$$\sum_{i=1}^{r} h_i(x(t)) = 1, \quad h_i(x(t)) \geq 0, \quad i = 1, \dots, r.$$

Therefore, the T–S fuzzy system may be represented by

$$\left\{ \begin{array}{l} \dot{x}(t) = A(h(x))x(t) + B(h(x))u(t) + E(h(x))w(t), \\ y(t) = C(h(x))x(t) + D(h(x))u(t) + F(h(x))w(t) \end{array}\right. \tag{2}$$

with

$$\begin{aligned}&(A, B, E, C, D, F)\left(h(x(t))\right) = \\ &\quad\sum_{i=1}^{r} h_i\left(x(t)\right)\left(A_i, B_i, E_i, C_i, D_i, F_i\right), \quad h(x(t)) \in \mathcal{U}\end{aligned}$$

where

$$\mathcal{U} = \Big\{ [\lambda_1 \cdots \lambda_r]' \in \mathbb{R}^r : \sum_{i=1}^{r} \lambda_i = 1,\ \lambda_i \geq 0 \Big\}. \tag{3}$$

The aim is to compute a state feedback control law

$$u(t) = K(h(x))x(t) \tag{4}$$

that stabilizes the T–S fuzzy model (2) independently of the rate of variation of the membership functions. For that, the same Lyapunov function considered in [15, 17] is used, i.e.

$$V(x) = 2\int_{\rho(0,x)} f(\psi) \cdot d\psi. \tag{5}$$

where $\rho(0, x)$ is a path from the origin to the present state, $(\cdot)$ stands for the inner product of vectors, ψ is a vector for the integral and $d\psi$ is an infinitesimal displacement. As in [15, 17], $f\left(x(t)\right)$ is a fuzzy vector, parameterized in the same way as the T–S fuzzy system (1), i.e.

$$\begin{aligned}&\mathcal{R}_i : \text{ If } x_1(t) \text{ is } M_1^{\alpha_{i1}} \text{ and } \dots \text{ and } x_n(t) \text{ is } M_n^{\alpha_{in}}\\ &\text{Then} \quad f(x) = P_i x,\end{aligned} \tag{6}$$

for $i = 1, \dots, r$, where $P_i \in \mathbb{R}^{n\times n}$ is a positive definite symmetric matrix given by

$$P_i = \bar{P} + D_i, \tag{7}$$

with

$$\bar{P} = \begin{bmatrix} 0 & d_{12} & \cdots & d_{1n} \\ d_{12} & 0 & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{1n} & d_{2n} & \cdots & 0 \end{bmatrix}$$

$$D_i = \begin{bmatrix} d_{11}^{\alpha_{i1}} & 0 & \cdots & 0 \\ 0 & d_{22}^{\alpha_{i2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn}^{\alpha_{in}} \end{bmatrix}.$$

The diagonal elements of P_i can change according to the fuzzy sets in the If-Then rules. Thus, the overall function $f(x)$ can be rewritten as

$$f(x) = P(h(x))x, \qquad P(h(x)) = \sum_{i=1}^{r} h_i(x)P_i > 0. \tag{8}$$

The particular structure of P_i given by (7) has been exploited in [15, 17] in order to guarantee the desired properties of the Lyapunov function $V(x)$.

III. Main Results

The main results of the paper are presented in this section, that is, the conditions for the existence of a stabilizing state feedback control law and of an $\mathcal{H}_\infty$ guaranteed cost controller as in (4), for any bounds on the rate of variation of the membership functions. The method proposed is based on two steps formulated as LMI optimization problems. A feedback control law assuring closed-loop stability of a frozen T–S fuzzy system (time-invariant) is computed at the first step. The control gain matrices are then used at the second step, where only the Lyapunov matrix is constrained to be as in (7). Differently from [17], the slack variables are left free, reducing the conservativeness of the results.

Next theorem presents the synthesis conditions for the existence of a stabilizing state feedback control. In this case, consider the T–S fuzzy system given by (2) with $E(h(x)) = 0$ and $F(h(x)) = 0$ (i.e. no exogenous input).

Theorem 1: Let $\hat{K}_i = Z_i X^{-1}$, where $W_i = W_i' > 0$, X and Z_i, $i = 1, \dots, r$, are matrices that satisfy[1]

$$\begin{aligned}&\Lambda_{ii} < 0, \quad i = 1, \dots, r, \\ &\Lambda_{ij} + \Lambda_{ji} < 0, \quad i < j = 1, \dots, r,\end{aligned} \tag{9}$$

with

$$\Lambda_{ij} \triangleq \begin{bmatrix} A_i X + X' A_i' + B_i Z_j + Z_j' B_i' & \star \\ W_i - X + \beta(A_i X + B_i Z_j)' & -\beta(X + X') \end{bmatrix}$$

for a given scalar $\beta > 0$. If there exist symmetric matrices $P_i = P_i' > 0$, $i = 1, \dots, r$, with the structure given by (7),

[1]The symbol $\star$ means a symmetric block in the LMI.

matrices S_i, G_i, H_i and J_i, $i = 1, \dots, r$, such that

$$\begin{aligned}
&\Gamma_{iii} < 0, \quad i = 1, \dots, r \\
&\Gamma_{iij} + \Gamma_{iji} + \Gamma_{jii} < 0, \quad i \neq j, \quad i = 1, \dots, r \\
&\Gamma_{ijk} + \Gamma_{ikj} + \Gamma_{jik} + \Gamma_{jki} + \Gamma_{kij} + \Gamma_{kji} < 0, \\
&\qquad i < j < k = 1, \dots, r,
\end{aligned} \tag{10}$$

where

$$\Gamma_{ijk} \triangleq \begin{bmatrix} \Omega_{ijk} & \star & \star \\ \Psi_{ijk} & -G_i - G_i' & \star \\ \Phi_{ij} & B_j' G_i' & -H_i - H_i' \end{bmatrix}, \tag{11}$$

with

$$\begin{aligned}
\Omega_{ijk} &\triangleq A_i' S_j' + S_j A_i + \hat{K}_k' B_j' S_i' + S_i B_j \hat{K}_k \\
\Psi_{ijk} &\triangleq P_i - S_i' + G_j A_i + G_i B_j \hat{K}_k \\
\Phi_{ij} &\triangleq B_j' S_i' + J_i - H_i \hat{K}_j.
\end{aligned}$$

then

$$K(h(x)) = H(h(x))^{-1} J(h(x)) \tag{12}$$

with

$$H(h(x)) = \sum_{i=1}^{r} h_i(x) H_i, \quad J(h(x)) = \sum_{i=1}^{r} h_i(x) J_i$$

is a state feedback gain that stabilizes the T–S fuzzy system (2).

Proof: For simplicity, $K(h(x))$ is denoted $K(x)$ (the same for other matrices). First, note that

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{r} h_i(x) h_j(x) h_k(x) \Gamma_{ijk} = \Gamma(x)$$

and, if the LMIs (10) hold, $\Gamma(x) < 0$. Pre- and post-multiplying $\Gamma(x)$ by T and T' respectively, with

$$T = \begin{bmatrix} I & 0 & Y(x)' \\ 0 & I & 0 \end{bmatrix}$$

$$Y(x) = H(x)^{-1} J(x) - \hat{K}(x) \tag{13}$$

$$\hat{K}(x) = \sum_{i=1}^{r} h_i(x) Z_i X^{-1} \tag{14}$$

and Z_i, $i = 1, \dots, r$ and X obtained from the solution of (9), yields

$$\begin{bmatrix} \bar{A}(x)' S(x)' + S(x) \bar{A}(x) & \star \\ P(x) - S(x)' + G(x) \bar{A}(x) & -G(x) - G(x)' \end{bmatrix} < 0 \tag{15}$$

where $K(x)$ is given by (12) and

$$\bar{A}(x) \triangleq A(x) + B(x) K(x) \tag{16}$$

Note that (15) guarantees the stability of $\bar{A}(x)$, since pre-multiplying (15) by $\begin{bmatrix} I & \bar{A}(x)' \end{bmatrix}$ and post-multiplying by its transpose provides

$$\bar{A}(x)' P(x) + P(x) \bar{A}(x) < 0$$

with $P(x)$ as in (8). Applying the same congruence transformation T to $\Gamma(x)$ with $Y(x) = 0$ yields

$$\begin{bmatrix} \tilde{A}(x)' S(x)' + S(x) \tilde{A}(x) & \star \\ P(x) - S(x)' + G(x) \tilde{A}(x) & -G(x) - G(x)' \end{bmatrix} < 0.$$

with

$$\tilde{A}(x) \triangleq A(x) + B(x) \hat{K}(x), \tag{17}$$

which, by its turn, is a stability condition for $\tilde{A}(x)$ with $P(x)$ as in (8). In fact, pre-multiplying the above inequality by $\begin{bmatrix} I & \tilde{A}(x)' \end{bmatrix}$ and post-multiplying by its transpose yields

$$\tilde{A}(x)' P(x) + P(x) \tilde{A}(x) < 0. \tag{18}$$

■

Note that $\hat{K}(x)$ obtained from LMIs (9) does not assure the closed-loop stability of the T–S fuzzy system if the second part of Theorem 1 does not provide a feasible solution. In fact, if the LMIs are verified, then

$$\sum_{i=1}^{r} \sum_{j=1}^{r} h_i(x) h_j(x) \Lambda_{ij} = \Lambda(x) < 0$$

By congruence, pre-multiplying $\Lambda(x)$ by $\begin{bmatrix} I & \tilde{A}(x) \end{bmatrix}$ and post-multiplying by its transpose yields

$$\tilde{A}(x) W(x) + W(x) \tilde{A}(x)' < 0, \quad W(x) > 0$$

which only assures that the real part of the eigenvalues of $\tilde{A}(x)$, for frozen values of x, is negative. In other words, $\dot{V}(x) < 0$ cannot be guaranteed because there is no constraint in the Lyapunov matrix $W(x)$ above.

On the other hand, if LMIs (10) hold for some $P(x)$ satisfying (8), then both $\hat{K}(x)$ given by (14) and $K(x)$ given by (12) are stabilizing state feedback control gains for the T–S fuzzy system. The main interest of using the two-step procedure proposed in Theorem 1 is the facility of imposing the desired structure to the Lyapunov matrix $P(x)$ while searching at the same time for the stabilizing control gain $K(x)$. Moreover, structural constraints could also be imposed to the control law itself, as for instance decentralization (i.e. $K(x)$ block diagonal). It is clear that, for a given $\hat{K}(x)$, one could use directly inequality (18) to search for a $P(x)$ as in (8), but the existence of slack variables in the second step of Theorem 1 facilitates this task.

The advantage of the proposed two steps procedure becomes more clear when a performance criterion is used, such as the $\mathcal{H}_\infty$ guaranteed cost control for T–S fuzzy systems, as presented in next theorem.

Theorem 2: Let $\hat{K}_i = Z_i X^{-1}$, where $W_i = W_i' > 0$, X, Z_i and M_i, $i = 1, \dots, r$, are matrices that satisfy

$$\begin{aligned}
&\Xi_{ii} < 0, \quad i = 1, \dots, r, \\
&\Xi_{ij} + \Xi_{ji} < 0, \quad i < j = 1, \dots, r,
\end{aligned} \tag{19}$$

with

$$\Xi_{ij} \triangleq \begin{bmatrix} -A_iX - X'A_i' - B_iZ_j - Z_j'B_i' & \star & \star & \star \\ W_i + X - \beta(A_iX + B_iZ_j)' & \beta(X + X') & \star & \star \\ -C_iX - D_iZ_j & -\beta(C_iX + D_iZ_j) & -\gamma^2 I & \star \\ -M_j'E_i' & 0 & -M_j'F_i' & I + M_i + M_i' \end{bmatrix}$$

for a given scalar $\beta > 0$. If there exits symmetric matrices $P_i = P_i' > 0$, $i = 1, \dots, r$, with the structure given by (7), matrices S_i, G_i, H_i, J_i and Q_i, $i = 1, \dots, r$, and a scalar $\gamma > 0$ such that

$$\begin{aligned} &\Upsilon_{iii} < 0, \quad i = 1, \dots, r \\ &\Upsilon_{iij} + \Upsilon_{iji} + \Upsilon_{jii} < 0, \quad i \neq j, \quad i = 1, \dots, r \\ &\Upsilon_{ijk} + \Upsilon_{ikj} + \Upsilon_{jik} + \Upsilon_{jki} + \Upsilon_{kij} + \Upsilon_{kji} < 0, \\ &\qquad i < j < k = 1, \dots, r, \end{aligned} \tag{20}$$

where Υ_{ijk} is given by (21), then $K(h(x))$ given by (12) is a stabilizing state feedback gain for the closed-loop T–S fuzzy system (2) with an $\mathcal{H}_\infty$ guaranteed cost given by γ.

Proof: Similarly to the proof of Theorem 1, observe that, if the LMIs (20) hold,

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} h_i(x)h_j(x)h_k(x)\Upsilon_{ijk} = \Upsilon(x) < 0.$$

Multiplying $\Upsilon(x)$ on the left by T_2 and on the right by T_2', with

$$T_2 = \begin{bmatrix} I & 0 & 0 & 0 & Y(x)' \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix}$$

and $Y(x)$ as in (13), and considering $(I - Q(x))'(I - Q(x)) \geq 0$, implying that $-Q(x)'Q(x) \leq I - Q(x) - Q(x)'$, one has

$$\begin{bmatrix} S(x)\bar{A}(x) + \bar{A}(x)'S(x)' & P(x) - S(x) + \bar{A}(x)'G(x)' & S(x)E(x) & \bar{C}(x)'Q(x) \\ \star & -G(x) - G(x)' & G(x)E(x) & 0 \\ \star & \star & -\gamma^2 I & F(x)'Q(x) \\ \star & \star & \star & -Q(x)'Q(x) \end{bmatrix} < 0, \tag{22}$$

with $K(x) = H(x)^{-1}J(x)$ as in (12), $\bar{A}(x)$ as in (16) and $\bar{C}(x) = C(x) + D(x)K(x)$. The multiplication of (22) on the right by T_3 and on the left by T_3', with

$$T_3 = \begin{bmatrix} I & 0 & 0 \\ \bar{A}(x) & E(x) & 0 \\ 0 & I & 0 \\ 0 & 0 & Q(x)^{-1} \end{bmatrix},$$

yields the bounded real lemma [2] with $P(x)$ as in (8) implying that $\dot{V}(x) + y'y - \gamma^2 w'w < 0$. Thus, the state feedback gain $K(x)$ stabilizes the T–S fuzzy system (2) with an $\mathcal{H}_\infty$ guaranteed cost given by γ. ■

Observe that, if the LMIs (19) hold, then

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(x)h_j(x)\Xi_{ij} = \Xi(x) < 0$$

The congruence transformation

$$T_4(x) = \begin{bmatrix} I & \tilde{A}(x) & 0 & E(x) \\ 0 & \tilde{C}(x) & I & F(x) \end{bmatrix},$$

with $\tilde{A}(x)$ as in (17) and $\tilde{C}(x) = C(x) + D(x)\hat{K}(x)$ applied to $\Xi(x)$, with a Schur complement, yields

$$\begin{bmatrix} \tilde{A}(x)W(x) + W(x)\tilde{A}(x)' & \star & \star \\ \tilde{C}(x)W(x) & -\gamma^2 I & \star \\ E(x)' & F(x)' & -I \end{bmatrix} < 0 \tag{23}$$

The above condition also can be recognized as the bounded real lemma [2], but in this case $W(x)$ has no structural constraints, implying that $A(x)$ has negative real part of its eigenvalues and a bound given by γ is assured to the closed-loop $\mathcal{H}_\infty$ norm of the T–S fuzzy system only if the membership functions remain frozen. On the other hand, similarly to the stabilization case, if a solution $P(x)$ as in (8) is obtained through the LMIs of Theorem 2, both $\hat{K}(x)$ given by (14) and $K(x)$ given by (12) obtained from Theorem 2 could be used as stabilizing control gains assuring the $\mathcal{H}_\infty$ guaranteed cost γ. Again, the bounded real lemma condition (23) could be used directly to test if $\hat{K}(x)$ obtained from (19) admits a Lyapunov matrix as in (7), but the slack variables introduced in the second step of the theorem help to provide smaller values of γ, as illustrated in the numerical experiments.

Theorems 1 and 2 provide a two-steps design method for T–S fuzzy systems where the control gain can be completely dissociated from the Lyapunov matrix assuring the desired closed-loop conditions. Therefore, structural constraints can be imposed independently to the gain and to the Lyapunov matrix. The method requires, at the first step, an LMI depending on a scalar β to be solved. A line search can be used, or simply a set of given values of β. Another important remark is that any stabilizing control gain $\hat{K}(x)$ could be used, even with structures more complex than the one given by (14). Note also that more sophisticated relaxations could be used to improve the accuracy of the conditions of Theorems 1 and 2, as proposed for instance in [26–30], at the price of a higher computational effort.

$$\Upsilon_{ijk} \triangleq \begin{bmatrix} A_i'S_j' + S_jA_i + \hat{K}_k'B_i'S_j' + S_jB_i\hat{K}_k & \star & \star & \star & \star \\ P_i - S_i' + G_jA_i + G_jB_i\hat{K}_k & -G_i - G_i' & \star & \star & \star \\ E_i'S_j' & E_i'G_j' & -\gamma^2 I & \star & \star \\ Q_j'(C_i + D_i\hat{K}_k) & 0 & Q_j'F_i & I - Q_i - Q_i' & \star \\ B_i'S_j' + J_i - H_i\hat{K}_j & B_i'G_j' & 0 & D_i'Q_j & -H_i - H_i' \end{bmatrix} \quad (21)$$

IV. Numerical Examples

The aim is to compare the proposed approach with other techniques from the literature in cases where the time-derivative bounds of the membership functions are not known or cannot be estimated. Since the numerical complexity associated to an LMI optimization problem can be estimated from the number V of scalar variables and the number L of LMI rows, V and L are given for the different methods. In the experiments, the scalar variable β has been chosen in the set $\{1, 0.1, 0.01, 0.001, 10^{-6}\}$. The results were obtained using YALMIP [31] and SeDuMi [4] within MATLAB on a 3.00 GHz IntelCore2Duo processor 2.00GB RAM computer.

Example 1: Consider the T–S fuzzy system with the following rules:

$$\mathcal{R}_1 : \text{If } x_2(t) \text{ is } M_2^1 \text{ then } \dot{x}(t) = A_1x(t) + B_1u(t)$$
$$\mathcal{R}_2 : \text{If } x_2(t) \text{ is } M_2^2 \text{ then } \dot{x}(t) = A_2x(t) + B_2u(t)$$

where, for given non-negative constants a and b,

$$A_1 = \begin{bmatrix} 3.6 & -1.6 \\ 6.2 & -4.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -a & -1.6 \\ 6.2 & -4.3 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} -0.45 \\ -3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -b \\ -3 \end{bmatrix}.$$

According to (7), the Lyapunov matrices to be searched for in Theorem 1 are

$$P_1 = \begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22}^1 \end{bmatrix} \text{ and } \quad P_2 = \begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22}^2 \end{bmatrix}.$$

The stabilizability was investigated for several values of a and b, and the maximum values obtained are shown in Table I with the associated numerical complexity. Compared to [8, Theorem 11] and [17, Theorem 7], taking the values of a as reference, the conditions of Theorem 1 provide stabilizing state feedback control laws for the highest values of b (thus assuring stabilizability for the entire interval $[0, b]$). The approaches based on a common Lyapunov function, such as the one in [8], are feasible for $b = 1.07$ as maximum value, independently of the value of a.

TABLE I

Maximum values of b for the stabilizability of the T–S fuzzy system of Example 1 using Theorem 1 (denoted by T1), [8, Theorem 11] and [17, Theorem 7]. V is the number of scalar variables and L is the number of LMI rows.

Method	a					L	V	Time (s)
	0	5	10	15	20			
[8, T11]	1.07	1.07	1.07	1.07	1.07	26	15	0.03
[17, T7]	1.00	1.20	1.30	1.20	1.10	20	17	0.03
T1	2.02	4.13	6.23	7.97	7.97	24	26	0.08

Example 2: In this numerical experiment, a database of continuous-time systems stabilizable by a constant state feedback gain but not quadratically stabilizable (i.e. the closed-loop system does not admit a constant Lyapunov matrix) was generated for $n = 2$ (two states), $m = 1$ (one control input), $r = \{2, 4\}$ (number of rules) and different structure constraints in $P(x)$ given by (8), indicated by d_{jj}^i when the entry can vary or d_{jj} when it is fixed. One hundred systems have been generated for each case and the first stage of Theorem 1 provided a valid $\hat{K}(x)$ for all the cases. Table II shows the number of systems that were stabilized for each condition, with different structure constraints for P_i in (7): (d_{11}^i, d_{22}) – $x_1(t)$ is the premise variable; (d_{11}, d_{22}^i) – $x_2(t)$ is the premise variable; and (d_{11}^i, d_{22}^i) – $x_1(t)$ and $x_2(t)$ are the premise variables. It can be noted that the results obtained with the proposed method are considerably better than the ones from [17]. This is mainly due to the fact that the slack variables need to be constrained in [17]. Moreover, more degrees of freedom in $P(x)$ imply more restrictions in the method from [17]. On the other hand, the proposed approach uses unconstrained slack variables. Observe also that the condition with slack variables of the second step of Theorem 1 is more relaxed than the standard Lyapunov inequality (18), providing better results.

TABLE II

Number of systems stabilized through Theorem 1 (T1), by solving directly (18) with $P(x)$ as in (8) and through [17, Theorem 7]; r is the number of rules of the T–S model.

Structure	r	T1	(18)	[17, T7]
(d_{11}^i, d_{22})	2	57	44	17
	4	70	59	30
(d_{11}, d_{22}^i)	2	59	46	14
	4	66	57	31
(d_{11}^i, d_{22}^i)	4	100	98	13

Example 3: Consider the T–S fuzzy system given by (2) with the same matrices A_1, A_2, B_1 and B_2 as in Example 1, $(a, b) = (15, 1.06)$, and the following matrices

$$E_1 = \begin{bmatrix} 0.1 \\ 0.001 \end{bmatrix}, \; E_2 = \begin{bmatrix} -0.1 \\ -0.083 \end{bmatrix}, \; C_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 0.108 & 0 \\ 0 & 0 \end{bmatrix}, \; D_1 = \begin{bmatrix} 0.1 \\ 0.05 \end{bmatrix}, \; D_2 = \begin{bmatrix} -0.1 \\ -0.05 \end{bmatrix},$$
$$F_1' = \begin{bmatrix} 0 & 0.1 \end{bmatrix}, \; F_2' = \begin{bmatrix} 0 & -0.1 \end{bmatrix}.$$

Table III shows the $\mathcal{H}_\infty$ guaranteed costs obtained with Theorem 2 when compared to the quadratic stabilizability based approach from $[30]_{(g,d)}$ (g is the degree of the slack variable polynomials and d is the relaxation level used in

[30]), [26], [27] and [32]. The numbers L (LMI rows) and V (scalar variables), and the computational time are also given. In this example, using directly the bounded real lemma (23) with the Lyapunov matrix structured as in (7) for the closed-loop system and with the gain $\hat{K}(x)$ from the first stage of Theorem 2, the guaranteed $\mathcal{H}_\infty$ cost was 1.82, higher than the one obtained from the second step of Theorem 2.

TABLE III

$\mathcal{H}_\infty$ GUARANTEED COSTS OBTAINED WITH THEOREM 2 (DENOTED BY T2), [26, THEOREM 5], [27, THEOREM 2], [32, PROBLEM $\mathcal{H}_{\infty 1}$] AND WITH [30, THEOREM 3]$_{(g,d)}$. V IS THE NUMBER OF SCALAR VARIABLES AND L IS THE NUMBER OF LMI ROWS.

Method	γ	L	V	Time (s)
[26, T5]	2.97	32	23	0.14
[27, T2]	2.97	20	18	0.17
[32, H1]	2.97	42	185	0.84
[30, T3]$_{(2,0)}$	2.97	30	71	0.21
T2	0.10	36	35	0.26

V. CONCLUSION

A new method for the $\mathcal{H}_\infty$ control design of continuous-time T–S fuzzy systems where no information about the time-derivative of the membership functions is available has been proposed. The strategy is based on a two-steps LMI procedure that has the ability of handling separately the constraints on the Lyapunov matrix from the control gain. As illustrated by numerical examples, the proposed approach provides less conservative results when compared to other techniques from the literature.

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