2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

LQR Performance Index Distribution with Uncertain Boundary Conditions

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***Abstract*—The track assignment problem in applications with large gaps in tracking measurements and uncertain boundary conditions is addressed as a Two Point Boundary Value Problem (TPBVP) using Hamiltonian formalisms. An L_2-norm analog Linear Quadratic Regulator (LQR) performance function metric is used to measure the trajectory cost, which may be interpreted as a control distance metric. Distributions of the performance function are determined by linearizing about the deterministic optimal nonlinear trajectory solution to the TPBVP and accounting for statistical variations in the boundary conditions. The performance function random variable under this treatment is found to have a quadratic form, and Pearson's Approximation is used to model it as a chi-squared random variable. Stochastic dominance is borrowed from mathematical finance and is used to rank statistical distributions in a metric sense. Analytical results and approximations are validated and an example of the approach utility is given. Finally conclusions and future work are discussed.**

I. Introduction and Background

In recent years there has been a significant increase in trackable on-orbit objects due to new launches, collisions, and improved sensor capabilities [1], [2], [3]. Object correlation for objects with maneuver execution capability has become increasingly complex, and operational methods to correlate objects have become more important, specifically for collision avoidance operations [4], [5].

Object track correlation has an extensive body of literature, particularly in continuous visible and radar tracking applications. Many approaches use statistical properties of sensors or known target qualities to minimize false alarms and the effects of clutter. Probabilistic Multi-Hypothesis Tracking (PMHT), Probabilistic Data Association Filter (PDAF), and Modified Gain Extended Kalman Filter (MGEKF) are representative of these algorithms (discussions and examples of their usage are in [6], [7], [8], [9]). Largely, these approaches update their correlations as new measurements are generated. An alternate class of problems to examine are those with large gaps in observation, such as on-orbit object tracking. In these scenarios the problem is to associate individual object tracks incorporating 5-15 minutes of observations (perhaps generated using some of the methods mentioned above) separated by observation gaps on the order of tens of minutes to days [10], [11].

One way to support candidate object pairing association is to propagate the initial track uncertainty (also introducing process noise) and compute the Kullback-Leibler Distance (KL-D) [12], the Battacharya Distance (B-D) [13], or the Mahalanobis Distance (M-D) [14] from the newly generated object track state and uncertainty.

Alternately, with an initial and final track for each candidate association pairing, each association may be addressed as a Two Point Boundary Value Problem (TPBVP) linking uncertain boundary conditions. In this approach the optimal connecting trajectory performance distribution may itself be used as a metric, and mutually exclusive track association pairings may be ranked against one another in a metric sense. Specifically, LQR-type costs (quadratic in state or fuel deviations from a nominal homogeneous trajectory) directly measure the effect of active maneuvers. Note that for space applications, a subclass of this problem concerned only with quadratic cost in fuel usage can be used [15]. This paper examines the full LQR performance index to expand upon previous efforts. This proposed method is fundamentally different from computing the statistical KL-D, B-D, or M-D from the objects' expected state distribution as it directly accounts for control usage. Computing the distribution of the LQR performance index has the additional advantage that the resulting probability distribution function may be used to support hypothesis testing to infer intentions or detect maneuvers.

The contribution of this paper is to generalize from quadratic control costs to full LQR trajectory costs. Effort is made to maintain applicability to general dynamical systems with LQR trajectory costs. Theory verification and operational use examples are given, then conclusions and future work are discussed.

II. Problem Definition

The problem is illustrated in Figure 1. One or more initial Uncorrelated Tracks (initial UCTs) must be 'paired' with one or more final Uncorrelated Tracks (final UCTs). Rather than computing the KL-D, B-D, or M-D between the expected homogeneous state distribution and the final UCTs, optimal connecting trajectories will be used to generate norm analog performance function distributions that will in turn be used to rank candidate UCT pairings.

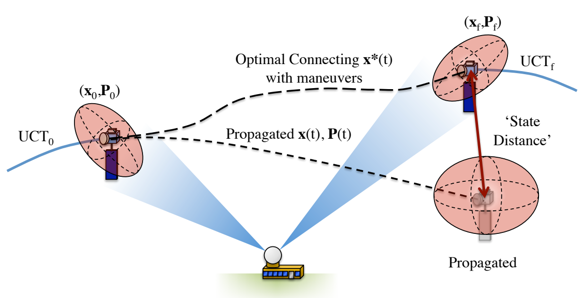

Fig. 1. Problem Illustration

Deterministic dynamics are considered in this effort. The dynamics are $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u},t)$, where $\mathbf{x} \in \mathbb{R}^n$ is the state, $\mathbf{u} \in \mathbb{R}^m$ is the control input, and $t \in [t_0, t_f]$ is the time. A performance function commonly found in Linear Quadratic Regulation (LQR) problems is used.

$$P = \int_{t_0}^{t_f} \frac{1}{2} \begin{bmatrix} \mathbf{x}(\tau) \\ \mathbf{u}(\tau) \end{bmatrix}^T \begin{bmatrix} \mathbf{Q} & \mathbf{N} \\ \mathbf{N}^T & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{x}(\tau) \\ \mathbf{u}(\tau) \end{bmatrix} d\tau \quad (1)$$

with $t \in [t_0, t_f]$, $P \in \mathbb{R}$, $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{Q} \in \mathbb{R}^{n\times n}$, $\mathbf{R} \in \mathbb{R}^{m\times m}$, and $\mathbf{N} \in \mathbb{R}^{m\times n}$. Presently it is only assumed that $\mathbf{Q} = \mathbf{Q}^T \geq 0$ and $\mathbf{R} = \mathbf{R}^T \geq 0$. Further conditions will be placed on $\mathbf{Q}$, $\mathbf{N}$, and $\mathbf{R}$ as necessary. There is no terminal state cost (typically written as $V(\mathbf{x}_f) = \mathbf{x}_f^T \mathbf{P} \mathbf{x}_f$), as the boundary conditions are considered equality constraints for this problem. Thus, this performance function is essentially a fixed-horizon LQR optimal control problem. Optionally, the cost function may be defined relative to a reference trajectory $(\mathbf{x}_r(t), \mathbf{u}_r(t))$ by choosing $\mathbf{x}(\tau) \to \mathbf{x}_r(\tau) - \mathbf{x}(\tau)$ and $\mathbf{u}(\tau) \to \mathbf{u}_r(\tau) - \mathbf{u}(\tau)$ in (1).

It is well known that this form has the additional benefit that it has extrema for the same trajectories $(\mathbf{x}(t), \mathbf{u}(t))$ as the L_2-norm defined for $\mathbf{y}(t)^T = \left[\mathbf{x}(t)^T \mathbf{u}(t)^T\right]$ over the inner-product space generated by $\mathbf{Q}$, $\mathbf{N}$, and $\mathbf{R}$. A minima found using (1) is also a minima of the analogous L_2-norm. Of particular interest to the examples given in this paper is if $\mathbf{Q} = \mathbf{0}$, $\mathbf{N} = \mathbf{0}$, and $\mathbf{R} = \mathbb{I}$, then the control distance analog $\|\mathbf{u}\|_{L_2} = \sqrt{2P}$ is generated. An excellent discussion of norms and their properties is given by Naylor [16]. This definition makes particular sense for on-orbit track correlation problems where control usage minimization is a top operator priority. The modification made to the TPBVP for this analysis is that $\mathbf{x}_0$ and $\mathbf{x}_f$ are not known exactly, and are described as random vectors $\mathbf{X}_0$ and $\mathbf{X}_f$, respectively. The distributions of $\mathbf{X}_0$ and $\mathbf{X}_f$ are assumed to be Gaussian, with

$$\mathbb{E}[\mathbf{X}_0] = \mathbf{x}_0,\ \mathbb{E}\left[(\mathbf{X}_0 - \mathbf{x}_0)(\mathbf{X}_0 - \mathbf{x}_0)^T\right] = \mathbf{P}_0$$

$$\mathbb{E}[\mathbf{X}_f] = \mathbf{x}_f,\ \mathbb{E}\left[(\mathbf{X}_f - \mathbf{x}_f)(\mathbf{X}_f - \mathbf{x}_f)^T\right] = \mathbf{P}_f$$

Where $\mathbf{P}_0 \in \mathbb{S}_+^{n\times n}$ and $\mathbf{P}_f \in \mathbb{S}_+^{n\times n}$ are covariance matrices associated with means $\mathbf{x}_0$ and $\mathbf{x}_f$, respectively. Equivalently, this paper uses the notation $\mathbf{X}_0 \in N(\mathbf{x}_0, \mathbf{P}_0)$ and $\mathbf{X}_f \in N(\mathbf{x}_f, \mathbf{P}_f)$ to define the initial and final boundary conditions as being random vectors with normal (Gaussian) distributions.

III. Theory

The nominal optimal trajectory problem for a general nonlinear system is now solved using Hamiltonian formalisms [17], [18]. After this solution is found, variations in the boundary conditions are considered and their impact on the performance function evaluated.

For nonlinear systems with a performance function P the Hamiltonian $\mathcal{H}$ is

$$\mathcal{H} = \inf_{\mathbf{u}} \left[\frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{x}^T\mathbf{N}\mathbf{u} + \frac{1}{2}\mathbf{u}^T\mathbf{R}\mathbf{u} + \mathbf{p}^T\mathbf{f}(\mathbf{x},\mathbf{u},t)\right] \quad (2)$$

Finding the optimal control $\mathbf{u}^*$:

$$\mathbf{u}^* = \arg\inf_{\mathbf{u}} \left[\frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{x}^T\mathbf{N}\mathbf{u} + \frac{1}{2}\mathbf{u}^T\mathbf{R}\mathbf{u} + \mathbf{p}^T\mathbf{f}(\mathbf{x},\mathbf{u},t)\right]$$

Applying the first-order necessary condition of optimality for the control $\mathbf{u}$:

$$\frac{\partial\mathcal{H}}{\partial\mathbf{u}} = \mathbf{N}^T\mathbf{x} + \mathbf{R}\mathbf{u} + \frac{\partial\mathbf{f}}{\partial\mathbf{u}}^T\mathbf{p} = \mathbf{0} \quad (3)$$

which yields

$$\mathbf{u}^* = -\mathbf{R}^{-1}\left(\mathbf{N}^T\mathbf{x} + \frac{\partial\mathbf{f}}{\partial\mathbf{u}}^T\mathbf{p}\right) \quad (4)$$

with the second-order necessary condition $[\partial^2\mathcal{H}/\partial\mathbf{u}^2] \geq 0$ being satisfied if $\mathbf{R} > 0$. This ultimately requires that $\mathbf{R}$ be a symmetric positive definite matrix. The state and co-state dynamics are then

$$\frac{\partial\mathcal{H}}{\partial\mathbf{p}} = \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u}^*,t) \quad (5)$$

$$-\frac{\partial\mathcal{H}}{\partial\mathbf{x}} = \dot{\mathbf{p}} = -\mathbf{Q}\mathbf{x} - \mathbf{N}\mathbf{u}^* - \frac{\partial\mathbf{f}}{\partial\mathbf{x}}^T\mathbf{p} \quad (6)$$

Because these equations are nonlinear an analytical solution may not exist. Now a nominal optimal solution $(\mathbf{x}_n(t), \mathbf{u}_n(t))$ connecting the nominal boundary conditions $\mathbf{x}_0$ and $\mathbf{x}_f$ is assumed (which may be numerical), and is expressed as

$$\mathbf{x}_n(t) = \phi_x(t; \mathbf{x}_0, \mathbf{p}_0, t_0) \tag{7}$$

$$\mathbf{p}_n(t) = \phi_p(t; \mathbf{x}_0, \mathbf{p}_0, t_0) \tag{8}$$

for times up to and including t_f. In a manner similar to that of linear systems, the effect of random initial and final states $\mathbf{X}_0$ and $\mathbf{X}_f$ are considered. If the trajectory space surrounding the nominal optimal trajectory $(\mathbf{x}_n(t), \mathbf{p}_n(t))$ is linearized, then linear variations in the boundary conditions may be directly examined. Taking the Taylor series expansion of $\mathbf{x}_n(t)$ and $\mathbf{p}_n(t)$ with respect to variations in $\mathbf{x}_0$ and $\mathbf{p}_0$:

$$\mathbf{x}_n(t) + \delta\mathbf{x}(t) = \phi_x(t; \mathbf{x}_0, \mathbf{p}_0, t_0) + \frac{\partial \phi_x}{\partial \mathbf{x}_0}\delta\mathbf{x}_0 + \frac{\partial \phi_x}{\partial \mathbf{p}_0}\delta\mathbf{p}_0 + \text{H.O.T.}$$

$$\mathbf{p}_n(t) + \delta\mathbf{p}(t) = \phi_p(t; \mathbf{x}_0, \mathbf{p}_0, t_0) + \frac{\partial \phi_p}{\partial \mathbf{x}_0}\delta\mathbf{x}_0 + \frac{\partial \phi_p}{\partial \mathbf{p}_0}\delta\mathbf{p}_0 + \text{H.O.T.}$$

Keeping only first order terms and realizing that $\mathbf{x}(t) - \phi_x(t; \mathbf{x}_0, \mathbf{p}_0, t_0) = \mathbf{0}$ and $\mathbf{p}(t) - \phi_p(t; \mathbf{x}_0, \mathbf{p}_0, t_0) = \mathbf{0}$, the state transition matrix is produced:

$$\begin{bmatrix} \delta\mathbf{x}(t) \\ \delta\mathbf{p}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{xx}(t, t_0) & \mathbf{\Phi}_{xp}(t, t_0) \\ \mathbf{\Phi}_{px}(t, t_0) & \mathbf{\Phi}_{pp}(t, t_0) \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_0 \\ \delta\mathbf{p}_0 \end{bmatrix} \tag{9}$$

The state transition matrix $\mathbf{\Phi}(t, t_0)$ can be solved using the differential matrix equation $\dot{\mathbf{\Phi}}(t, t_0) = \mathbf{A}_n\mathbf{\Phi}(t, t_0)$ where $\mathbf{A}_n$ is the gradient of the full state and adjoint dynamics with respect to bot the state and adjoint and the initial condition $\mathbf{\Phi}(t_0, t_0) = \mathbb{I}_{2n\times 2n}$. So far the Theory section has closely followed [19], however at this point the approach and application in this paper diverges. Evaluating (9) at $t = t_f$ where both $\delta\mathbf{x}_0$ and $\delta\mathbf{x}_f$ are assumed to be known, $\delta\mathbf{p}_0$ can be computed:

$$\delta\mathbf{p}_0 = \mathbf{\Phi}_{xp}(t_f, t_0)^{-1} \begin{bmatrix} -\mathbf{\Phi}_{xx}(t_f, t_0) & \mathbb{I} \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_0 \\ \delta\mathbf{x}_f \end{bmatrix} \tag{10}$$

The matrix partition inverse $\mathbf{\Phi}_{xp}(t_f, t_0)^{-1}$ is shown to exist for some systems [20] (particularly Clohessy-Wiltshire dynamics). Here it is assumed that $\mathbf{\Phi}_{xp}(t_f, t_0)^{-1}$ exists. Also, for $t_0 < t \le t_f$, the state and co-state variation $\delta\mathbf{x}(t)$ $\delta\mathbf{p}(t)$ may be found by solving (9) using (10) and produce (17) and (18). Which are re-written as

$$\delta\mathbf{x}(t) = \kappa(t, t_0) \begin{bmatrix} \delta\mathbf{x}_0 \\ \delta\mathbf{x}_f \end{bmatrix} \tag{13}$$

$$\delta\mathbf{p}(t) = \mathbf{\Lambda}(t, t_0) \begin{bmatrix} \delta\mathbf{x}_0 \\ \delta\mathbf{x}_f \end{bmatrix} \tag{14}$$

Note that some of the state transition matrix portions are computed over the interval $[t_0, t]$ while others are computed over $[t_0, t_f]$. The functions $\kappa(t, t_0)$ and $\mathbf{\Lambda}(t, t_0)$ map variations in the initial and final states to variations in the state $\delta\mathbf{x}(t)$ and the co-state $\delta\mathbf{p}(t)$ at time t. Since the optimal control is defined in terms of the state and co-state, linear variations in the control are written as

$$\mathbf{u}^*(t) = \mathbf{u}_n(t) + \delta\mathbf{u}(t) \approx -\mathbf{R}^{-1}\left(\mathbf{N}^T(\mathbf{x}_n(t) + \delta\mathbf{x}(t)) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}^T(\mathbf{p}_n(t) + \delta\mathbf{x}(t))\right)$$

Because $\mathbf{u}_n(t) = -\mathbf{R}^{-1}(\mathbf{N}^T\mathbf{x}_n(t) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}^T\mathbf{p}_n(t))$, the linear variation in the control is:

$$\delta\mathbf{u}(t) = -\mathbf{R}^{-1}\left(\mathbf{N}^T\kappa(t, t_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}^T\Lambda(t, t_0)\right) \begin{bmatrix} \delta\mathbf{x}_0 \\ \delta\mathbf{x}_f \end{bmatrix} \tag{15}$$

Recall that $\partial\mathbf{f}/\partial\mathbf{u}$ is evaluated along the nominal optimal trajectory $(\mathbf{x}_n(t), \mathbf{u}_n(t))$. Returning now to the performance function P defined in (1) and substituting $\mathbf{u}^*(\tau) = \mathbf{u}_n(\tau) + \delta\mathbf{u}(\tau)$, the performance function becomes

$$P = \int_{t_0}^{t_f} \frac{1}{2} \begin{bmatrix} \mathbf{x}_n + \delta\mathbf{x} \\ \mathbf{u}_n + \delta\mathbf{u} \end{bmatrix}^T \begin{bmatrix} \mathbf{Q} & \mathbf{N} \\ \mathbf{N}^T & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{x}_n + \delta\mathbf{x} \\ \mathbf{u}_n + \delta\mathbf{u} \end{bmatrix} d\tau \tag{16}$$

The state and optimal control variations can be written in terms of the boundary condition variations as

$$\begin{bmatrix} \delta\mathbf{x} \\ \delta\mathbf{u} \end{bmatrix} = \begin{bmatrix} \kappa(t, t_0) \\ -\mathbf{R}^{-1}\left(\mathbf{N}^T\kappa(t, t_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}^T\Lambda(t, t_0)\right) \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_0 \\ \delta\mathbf{x}_f \end{bmatrix}$$

Which for ease of notation is defined as

$$\mathbf{W}(t, t_0) = \begin{bmatrix} \kappa(t, t_0) \\ -\mathbf{R}^{-1}\left(\mathbf{N}^T\kappa(t, t_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}^T\Lambda(t, t_0)\right) \end{bmatrix}$$

Expanding (16) P may be written as

$$\begin{aligned} P(t) = &\int_{t_0}^{t} \frac{1}{2} \begin{bmatrix} \mathbf{x}_n \\ \mathbf{u}_n \end{bmatrix}^T \begin{bmatrix} \mathbf{Q} & \mathbf{N} \\ \mathbf{N}^T & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{x}_n \\ \mathbf{u}_n \end{bmatrix} d\tau \\ &+ \int_{t_0}^{t} \begin{bmatrix} \mathbf{x}_n \\ \mathbf{u}_n \end{bmatrix}^T \begin{bmatrix} \mathbf{Q} & \mathbf{N} \\ \mathbf{N}^T & \mathbf{R} \end{bmatrix} \mathbf{W}(\tau, 0)\delta\mathbf{z}d\tau \\ &+ \int_{t_0}^{t} \frac{1}{2}\delta\mathbf{z}^T\mathbf{W}(\tau, 0)^T \begin{bmatrix} \mathbf{Q} & \mathbf{N} \\ \mathbf{N}^T & \mathbf{R} \end{bmatrix} \mathbf{W}(\tau, 0)\delta\mathbf{z}d\tau \end{aligned}$$

over the interval $t \in [t_0, t_f]$ where $\delta\mathbf{z}^T = \left[\delta\mathbf{x}_0^T \ \delta\mathbf{x}_f^T\right] \in \mathbb{R}^{2n}$. The variable $\delta\mathbf{z}$ does not depend on τ, so the

$$\delta \mathbf{x}(t) = \begin{bmatrix} \mathbf{\Phi}_{xx}(t,t_0) - \mathbf{\Phi}_{xp}(t,t_0)\mathbf{\Phi}_{xp}(t_f,t_0)^{\dagger}\mathbf{\Phi}_{xx}(t_f,t_0) & \mathbf{\Phi}_{xp}(t,t_0)\mathbf{\Phi}_{xp}(t_f,t_0)^{-1} \end{bmatrix} \begin{bmatrix} \delta \mathbf{x}_0 \\ \delta \mathbf{x}_f \end{bmatrix} \tag{17}$$

$$\delta \mathbf{p}(t) = \begin{bmatrix} \mathbf{\Phi}_{px}(t,t_0) - \mathbf{\Phi}_{pp}(t,t_0)\mathbf{\Phi}_{xp}(t_f,t_0)^{\dagger}\mathbf{\Phi}_{xx}(t_f,t_0) & \mathbf{\Phi}_{pp}(t,t_0)\mathbf{\Phi}_{xp}(t_f,t_0)^{-1} \end{bmatrix} \begin{bmatrix} \delta \mathbf{x}_0 \\ \delta \mathbf{x}_f \end{bmatrix} \tag{18}$$

following definitions are made:

$$P_n(t) = \frac{1}{2}\int_{t_0}^{t} \begin{bmatrix} \mathbf{x}_n \\ \mathbf{u}_n \end{bmatrix}^T \begin{bmatrix} \mathbf{Q} & \mathbf{N} \\ \mathbf{N}^T & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{x}_n \\ \mathbf{u}_n \end{bmatrix} d\tau, \tag{17}$$

$$\omega(t,t_0) = \int_{t_0}^{t} \mathbf{W}(\tau,0)^T \begin{bmatrix} \mathbf{Q} & \mathbf{N} \\ \mathbf{N}^T & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{x}_n \\ \mathbf{u}_n \end{bmatrix} d\tau, \tag{18}$$

and

$$\mathbf{\Omega}(t,t_0) = \frac{1}{2}\int_{t_0}^{t_f} \mathbf{W}(\tau,0)^T \begin{bmatrix} \mathbf{Q} & \mathbf{N} \\ \mathbf{N}^T & \mathbf{R} \end{bmatrix} \mathbf{W}(\tau,0) d\tau. \tag{19}$$

Note that P_n is the performance of the nominal optimal trajectory $(\mathbf{x}_n(t), \mathbf{p}_n(t))$. The performance index $P(t)$ over the time interval $t \in [t_0, t_f]$ is

$$P(t) = P_n(t) + \omega(t,t_0)^T \delta \mathbf{z} + \delta \mathbf{z}^T \mathbf{\Omega}(t,t_0) \delta \mathbf{z}$$

The approximation of $P(t)$ in the linear space about $(\mathbf{x}_n(t), \mathbf{p}_n(t))$ is a quadratic form in terms of the boundary condition variations, $\delta \mathbf{z}$. From §II, $\delta \mathbf{z}$ may be treated as the realization of a Gaussian random vector $\delta \mathbf{Z} \in N(\mathbf{0}, \mathbf{P}_z)$, where

$$\mathbb{E}\left[\delta \mathbf{Z} \delta \mathbf{Z}^T\right] = \mathbf{P}_z = \begin{bmatrix} \mathbf{P}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_f \end{bmatrix} \tag{20}$$

Rewriting the performance function as a scalar random variable the following form is obtained:

$$P(t) = P_n(t) + \omega(t,t_0)^T \delta \mathbf{Z} + \delta \mathbf{Z}^T \mathbf{\Omega}(t,t_0) \delta \mathbf{Z} \tag{21}$$

The quadratic form shown in (21) is an intuitive result, as if the variations in the boundary conditions are reduced to zero, $P(t) = P_n(t)$. Similarly, if the variance in the boundary conditions is large, one would expect the possible values of P to increase. Typically one is interested in the final performance function value $P(t_f)$, however the development given above is valid for $t \in [t_0, t_f]$. Now, the question of computing the PDF of $P(t)$ arises.

Because (21) has a quadratic form, existing theory involving quadratic forms of normal random variables may be applied. To compute the PDF of $P(t)$, the following definitions (found using similar transformations as those described in [21]) are made:

$$\underline{P}_n(t) = P_n(t) - \frac{1}{4}\omega(t,t_0)^T \mathbf{\Omega}(t,t_0)^{\dagger} \omega(t,t_0) \tag{22}$$

$$\mu_X(t) = \frac{1}{2}\mathbf{\Omega}(t,t_0)^{\dagger}\omega(t,t_0) \tag{23}$$

$$\mathbf{B}\mathbf{B}^T = \mathbf{P}_z \tag{24}$$

$$\mathbf{b} = \mathbf{T}^T \mathbf{B}^{-T} \mu_X(t) \tag{25}$$

$$\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{2n} \end{bmatrix} = \mathbf{T}^T \mathbf{B}^T \Omega(t,t_0) \mathbf{B} \mathbf{T} \tag{26}$$

The matrix $\mathbf{B}$ is a matrix square root of $\mathbf{P}_z$ (or any decomposition such that $\mathbf{B}\mathbf{B}^T = \mathbf{P}_z$). Also, the matrix $\mathbf{T} \in \mathbb{R}^{2n \times 2n}$ is an orthonormal transformation matrix such that $\mathbf{T}^T \mathbf{B}^T \Omega(t,t_0) \mathbf{B}\mathbf{T}$ is diagonalized. These definitions transform P to

$$P(t) = \underline{P}_n(t) + \sum_{i=1}^{2n} \lambda_i(t) \left(U_i + b_i(t)\right)^2 \tag{27}$$

Many methods to approximate the distribution of $P(t)$ exist. One such approximation that benefits from ease of computation is Pearson's Approximation [21], where the first three moments of the distribution of $P(t)$ are matched and modeled using a chi-squared distribution. The approximation is given as

$$P(t) \approx \frac{\theta_3(t)}{\theta_2(t)} \chi^2_{v(t)} - \frac{\theta_2(t)^2}{\theta_3(t)} + \theta_1(t) + \underline{P}_n(t) \tag{28}$$

where

$$\theta_s(t) = \sum_{j=1}^{2n} \lambda_j(t)^s \left(1 + s b_j(t)^2\right), \ s = 1,2,3$$

and the degree of freedom $v(t)$ is defined as

$$v(t) = \frac{\theta_2(t)^3}{\theta_3(t)^2}$$

The approximation (28) is valid over $P \in [0, \infty)$.

Remark III.1. ***Sums of Performance Functions*** *When it is desirable to add several individual performance functions P_j, $j = 1, \ldots, q$, then the composite total value of all of the performance functions may be described as*

$$P_c(t) = \sum_{j=1}^{q} P_j(t)$$
$$= \sum_{j=1}^{q} \left(\underline{P}_{n,j}(t) + \sum_{i=1}^{2n} \lambda_{i,j}(t) \left(U_i + b_{i,j}(t)\right)^2 \right)$$

where $\lambda_{i,j}(t)$ is the i^{th} eigenvalue of performance function j and $b_{i,j}(t)$ is the i^{th} non-centrality parameter of performance function j. Alternately, if

$$\lambda_c(t)^T = \begin{bmatrix} \lambda_1(t)^T & \cdots & \lambda_q(t)^T \end{bmatrix}$$
$$\boldsymbol{b}_c(t)^T = \begin{bmatrix} \boldsymbol{b}_1(t)^T & \cdots & \boldsymbol{b}_q(t)^T \end{bmatrix}$$
$$\underline{P}_{n,c}(t) = \sum_{j=1}^{q} \underline{P}_{n,j}(t)$$

and all performance functions have boundary conditions of dimension $2n$, then the sum of the performance functions P_c may be written as

$$P_c(t) = \underline{P}_{n,c}(t) + \sum_{i=1}^{2nq} \lambda_{c,i}(t)\left(U_i + b_{c,i}(t)\right)^2 \tag{29}$$

Remark III.2. ***Alternate Performance Functions***
The derivation of the performance function (21) and its PDF used the performance function definition given in (1), though in practice other performance functions may be used with equal utility.

Remark III.3. ***Systems with Linear Dynamics***
A result of the first order calculus of variations expansion of $\boldsymbol{x}_n(t)$ and $\boldsymbol{p}_n(t)$ is that if the governing dynamics of the system are linear - the first-order Taylor series expansion is exact. In this case (21) is also exact.

The task of sensibly ranking probability distributions is confounded by the fact that many distributions have positive finite densities over their entire intervals, which often have significant overlap. Thus, there is a finite positive probability that a random variable X with a 'smaller' distribution will have a larger realized value x than a realized random variable y with a 'larger' distribution. Stochastic dominance provides a framework with which one may sensibly rank random variable distributions. The following definition and results are summarized from Meucci [22].

Definition III.1. ***Order-q Dominance:***
The distribution $f_A(p)$ is said to ***order-q dominate*** *the distribution $f_B(p)$ if, for all $p \in [0, \infty]$, the following inequality holds:*

$$\mathcal{I}^q\left[f_A(p)\right] \geq \mathcal{I}^q\left[f_B(p)\right] \tag{30}$$

where the operator $\mathcal{I}[\cdot]$ is the integration operator over $p \in [0, \infty]$. If $q = 1$ ($F_A(p) \geq F_B(p)$), distribution $f_A(p)$ is said to weakly dominate $f_B(p)$.

Brief inspection of (30) shows us that order q dominance implies order $q+1$ dominance. Starting with $q = 0$, this gives the following result

$$\text{0-dom} \Rightarrow \text{1-dom} \Rightarrow \cdots \Rightarrow \text{q-dom}$$

Since 0-dom does not typically occur, the first order dominance that can reasonably be expected is $q = 1$. Order $q = 1$ dominance is equivalent to the cumulative distribution function (CDF) of the distribution of $N_a \in f_A(p)$ being strictly less that the CDF of the random variable $N_b \in f_B(p)$. Orders higher than $q = 1$ do not always have a clear intuitive meaning. In general, there is no guarantee that there exists an order q such that any two distributions may be ranked.

Remark III.4. ***Gaussian Approximation***
If the boundary condition variations are 'sufficiently small' (such that the term $\delta\mathbf{Z}^T\boldsymbol{\Omega}(t,t_0)\delta\mathbf{Z}$ is negligible), then the approximate performance function P is approximately

$$P(t) = P_n(t) + \omega(t,t_0)^T\delta\mathbf{Z} \tag{31}$$

Since $\mathbb{E}\left[\delta\mathbf{Z}\right] = \boldsymbol{0}$,

$$\mathbb{E}\left[P\right] = P_n \tag{32}$$

Similarly, the variance of the performance function reduces to

$$\begin{aligned} &\sigma_P(t)^2 \\ &= \mathbb{E}\left[\left(P_n(t) + \omega(t,t_0)^T\delta\mathbf{Z}\right)\left(P_n(t) + \omega(t,t_0)^T\delta\mathbf{Z}\right)^T\right] \\ &= \omega(t,t_0)^T\boldsymbol{P}_z\omega(t,t_0) \end{aligned} \tag{33}$$

More simply, P may be considered a scalar Gaussian random variable $P(t) \in N(P_n(t), \omega(t,t_0)^T\boldsymbol{P}_z\omega(t,t_0))$. This simplification greatly eases computational burdens, as once P_n and $\sigma_P(t)^2$ have been computed, the PDF is an analytic function [21]:

$$f_{P,approx}(p) = \frac{1}{\sqrt{2\pi\sigma_P^2}}\exp\left(-\frac{(p-P_n)^2}{2\sigma_P^2}\right) \tag{34}$$

Deciding whether $\delta\mathbf{Z}^T\boldsymbol{\Omega}(t,t_0)\delta\mathbf{Z}$ is negligible depends very much on the situation. A good rule of thumb is to ensure that $f_{P,approx}(p)$ does not have significant probability density when $p < 0$. This can be achieved by requiring that $m\sigma_P(t) < P_n(t)$, where m is at least 2.

IV. Validation and Demonstration of Results

A simulation was written in MATLAB using the Clohessy-Wiltshire (CW) dynamics which describe the relative motion in a rotating Hill frame of an on-orbit object to a circular reference orbit. The CW dynamic equations are written as

$$\begin{aligned} \ddot{r} &= 3n^2r + 2n\dot{s} + u_r \\ \ddot{s} &= -2n\dot{r} + u_s \\ \ddot{w} &= -n^2w + u_w \end{aligned} \tag{35}$$

The coordinates r, s, and w represent motion in the radial, along-track, and cross-track directions respectively and form a right-handed cartesian coordinate system. To compute the control distance L_2-norm analog using the framework presented in this paper the following variable selections are made: $\mathbf{Q} = \mathbf{0}_{6\times6}$, $\mathbf{N} = \mathbf{0}_{6\times3}$, and $\mathbf{R} = \mathbb{I}_{3\times3}$.

The following two subsections describe the validation and utility demonstration of results presented in §III of this paper.

A. *Performance Distribution Validation*

To validate that the theoretical results accurately represent the true distribution of the performance function a scenario with a single initial and final UCT is examined. The mean and covariance of UCT_0 and UCT_f are described in Table I. The initial and final covariances

TABLE I
VALIDATION TEST CASE BOUNDARY CONDITIONS AND ASSOCIATED UNCERTAINTY

Coordinate	UCT_0 $\mathbf{x}_0$	UCT_0 σ_0	UCT_f $\mathbf{x}_f$	UCT_f σ_f
Radial - r (m)	0	1	0	1
Along-Track - s (m)	-100	1	100	1
Cross-Track - w (m)	0	1	50	1
Radial - $\dot{r}$ (m/s)	0	0.05	0	0.05
Along-Track - $\dot{s}$ (m/s)	0	0.05	0	0.05
Cross-Track - $\dot{w}$ (m/s)	0	0.05	0	0.05

$\mathbf{P}_0$ and $\mathbf{P}_f$ for each coordinate $\mathbf{x}_0$ and $\mathbf{x}_f$ are formed as a diagonal covariance matrix of the form

$$\mathbf{P} = \text{diag}\left(\begin{bmatrix} \sigma_r^2 & \sigma_s^2 & \sigma_w^2 & \sigma_{\dot{r}}^2 & \sigma_{\dot{w}}^2 & \sigma_{\dot{w}}^2 \end{bmatrix}\right)$$

The time interval under consideration is $[t_0, t_f] = [0, 0.25]$ orbits (the orbit period chosen is 90 minutes). Four methods are used to determine the validity of the theory developed in this paper:

1) *Monte Carlo Simulation ('Truth'):* Individual realizations of the distribution of $\mathbf{X}_0 \in N(\mathbf{x}_0, \mathbf{P}_0)$ and $\mathbf{X}_0 \in N(\mathbf{x}_0, \mathbf{P}_0)$ are randomly generated and simulated. 10,000 simulations are run, their control distance is determined, and the composite numerical PDF and CDF is computed using a histogram approach with 100 bins.
2) *Small $\delta\mathbf{Z}$ Approximation:* The Gaussian approximation (34) for very small $\delta\mathbf{Z}$ discussed in Remark III.4 is generated and the corresponding PDF and CDF analytically determined.
3) *Sampled Distribution:* The known boundary condition distribution of $\delta\mathbf{Z} \in N(\mathbf{0}, \mathbf{P}_z)$ is realized and the corresponding control distance realization computed according to (21). 10,000 realizations are generated and the corresponding PDF and CDF approximations are determined (again with 100 bins).
4) *Pearson's Approximation:* Given computed values for $\omega(t_f, t_0)$ and $\mathbf{\Omega}(t_f, t_0)$, Pearson's Approximation is used to match the first three moments of the true distribution and model the control distance random variable as a chi-squared distribution.

The Monte Carlo (method 1) results are considered 'Truth' for the purposes of this verification, as it includes all nonlinearities and does not make any local linearization assumptions. It bears mentioning that despite the fact that the dynamics used for this verification are linear, the optimal control law (4) is decidedly nonlinear. Further, the approach in this paper is appropriate for systems with nonlinear dynamics provided the trajectory deviations are of a reasonable size. Figures 2 and 3 show the verification results of the PDF and CDF using methods 1-4 outlined above.

The small $\delta\mathbf{Z}$ approximation of the distribution agrees nicely with the Monte Carlo results. It is clear that not all of the PDF or CDF is captured, as the Monte Carlo PDF has a 'long tail' and absolutely zero probability for $P < 0$, where the Gaussian approximation has neither attribute. The sampled distribution found by directly generating $\delta\mathbf{Z} \in N(\mathbf{0}, \mathbf{P}_z)$ and computing each corresponding realization p matched the Monte Carlo results very closely. As expected, $\mathbb{P}[P < 0] = 0$, and the sampled distribution exhibited a 'long tail' very similar to the Monte Carlo simulations. Pearson's Approximation matches the Monte Carlo results quite well, as seen in Figures 2 and 3.

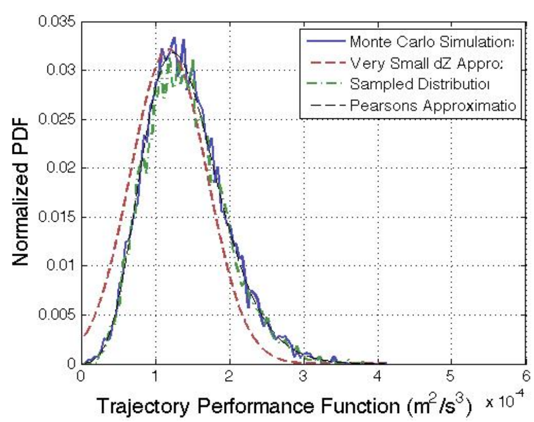

Fig. 2. Computed validation PDFs

B. *Example: Proximity Operations Object Matching*

This example demonstrates how two initial and final UCTs may be associated with one another. Two initial

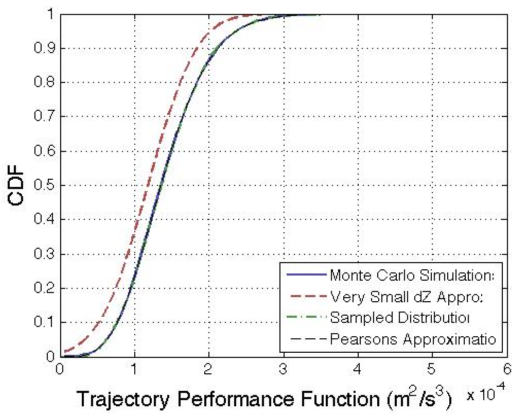

Fig. 3. Computed validation CDFs

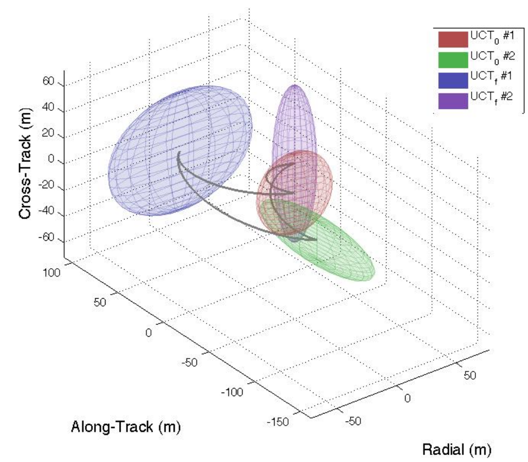

Fig. 4. Visualization of 3σ uncertain boundary conditions and nominal candidate connecting trajectories

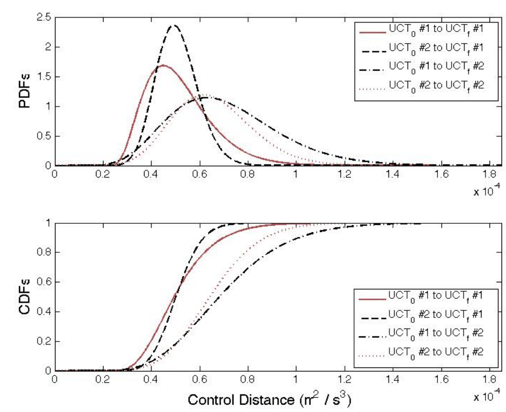

Fig. 5. Computed PDF and CDF approximations for each candidate UCT association

and final UCTs are described in Table II and Table III. The initial and final UCTs are also visualized in Figure 4 along with their nominal optimal connecting trajectories ($\mathbf{x}_n$, $\mathbf{u}_n$). The time interval and weightings chosen for $\mathbf{Q}$, $\mathbf{N}$, and $\mathbf{R}$ are the same as in the validation subsection.

TABLE II
EXAMPLE INITIAL UCT STATE AND COVARIANCE

Coord.	UCT_0 1 $\mathbf{x}_0$	UCT_0 1 σ_0	UCT_0 2 $\mathbf{x}_0$	UCT_0 2 σ_0
r (m)	0	10	0	5
s (m)	-50	5	-75	20
w (m)	25	10	0	5
$\dot{r}$ (m/s)	0	0.010	0	0.010
$\dot{s}$ (m/s)	0	0.005	0	0.010
$\dot{w}$ (m/s)	0	0.010	0	0.020

TABLE III
EXAMPLE FINAL UCT STATE AND COVARIANCE

Coord.	UCT_f 1 $\mathbf{x}_0$	UCT_f 1 σ_0	UCT_f 2 $\mathbf{x}_f$	UCT_f 2 σ_f
r (m)	0	20	50	5
s (m)	75	5	15	5
w (m)	0	15	0	20
$\dot{r}$ (m/s)	0	0.010	0	0.005
$\dot{s}$ (m/s)	0	0.005	0	0.003
$\dot{w}$ (m/s)	0	0.005	0	0.005

The corresponding distributions of the neighboring optimal trajectories for each UCT pairing are then computed. Their PDFs and CDFs are shown in Figure 5. Because there are only two initial and final UCTs to associate, there are two distinct cases: Case 1 occurs if UCT_0 1 connects to UCT_f 1 (and UCT_0 2 connects to UCT_f 2), while Case 2 occurs when UCT_0 1 connects to UCT_f 2 (and UCT_0 1 connects to UCT_f 2). These cases are summarized in Table IV.

TABLE IV
OBJECT ASSOCIATION CASE DESCRIPTIONS

Case	Associations
1	UCT_0 1 → UCT_f 1 UCT_0 2 → UCT_f 2
2	UCT_0 1 → UCT_f 2 UCT_0 2 → UCT_f 1

Figure 6 plots the CDFs of the sums of the performance functions calculated as discussed in Remark III.1 for each case. By inspection of Figure 6 it is clear that application of stochastic dominance suggests that Case 1 is 1^{st}-order dominant over Case 2. Reducing boundary condition uncertainty can emphasize this dominance.

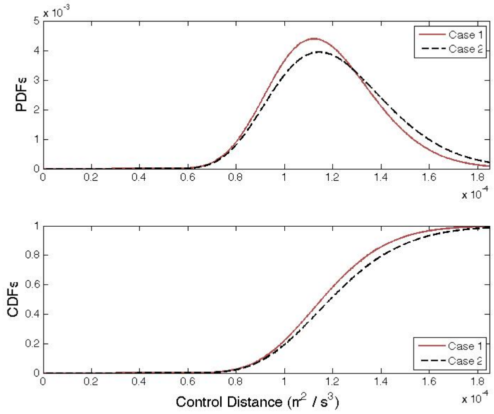

Fig. 6. Computed CDF approximations of the performance function sums for Case 1 and Case 2

V. CONCLUSIONS AND FUTURE WORK

A general LQR trajectory cost is defined and control distance metric analog is proposed as an intuitive measure with which to rank candidate UCT associations. The space about the nominal optimal connecting trajectories between UCTs is linearized and statistical variations in the boundary conditions are used to compute corresponding variations in the control distance metric. Pearson's Approximation is used to model the first three moments of the true distribution as a chi-squared distribution. Stochastic dominance is borrowed from mathematical finance as a mechanism to rank probability distributions of candidate UCT pair control distances.

The theory results are validated using the Clohessy-Wiltshire equations of relative on-orbit motion. Validation results indicate that Pearson's Approximation adequately models the true performance function probability distribution. The example given demonstrates the utility of the approach in this paper to candidate UCT pairing evaluation, particularly in the presence of large overlapping uncertainties and small relative distances of objects. Future work includes detailed comparisons with other correlation metrics (KL-D, B-D, and M-D) and rigorously accounting for uncertain dynamics.

ACKNOWLEDGMENTS

This work was supported by AFOSR Grant No. FA9550-08-1-0460.

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