2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Optimal Reachability Sets Using Generalized Independent Parameters

Marcus J. Holzinger
Aerospace Engineering Sciences
University of Colorado at Boulder

Daniel J. Scheeres
Aerospace Engineering Sciences
University of Colorado at Boulder

John Hauser
Electrical and Computer Engineering
University of Colorado at Boulder

***Abstract*—The problem of free-time optimal reachability set computation with alternate integral constraints is motivated and examined. Specific examples of such systems are optimal spacecraft, aircraft and automobile free-time, fuel-limited range computation. An alternate Hamilton Jacobi Bellman PDE formulation is derived using a Generalized Independent Parameter (GIP) associated with the integration constraint and GIP mapping function with respect to time is defined. Necessary conditions on the GIP mapping function are identified and discussed. Singular independent parameter mapping functions, often found in astrodynamics optimal control problems, are shown to be challenging to solve using a simple change of integration variable, motivating an approach to transform such problems before solving. Several short illustrations are used to emphasize theoretical cases of interest, and two simple fully-worked examples are given to demonstrate the potential utility of this approach.**

I. Introduction

When endeavoring to compute an optimal reachability set with an integral constraint while there are no temporal integral constraints ('free-time' problems) it is convenient to use a framework that leverages proven results for traditional time-constrained reachability. A particularly useful example of such an optimal reachability set is the range set, defined here as the set of reachable states in the state-space given a fuel or control integral constraint but no trajectory duration constraints.

For this paper, range set computation is motivated by the desire to compute fuel-optimal range sets for spacecraft applications. Optimal control policies for fuel-optimal thrust-constrained spacecraft trajectories include periods of zero-thrust, and directly making a change of variables from time to fuel introduces singularities.

Background literature in optimal reachability sets given time integral constraints (specifically integrating over $t \in [t_0, t_f]$) is extensive. If the optimal value function in the Dynamic Programing Equation (DPE) is not discontinuous it can be shown that the Hamilton Jacobi Bellman Partial Differential Equation (HJB PDE) must be satisfied along all optimal trajectories generating an optimal reachability set [1], [2], [3], [4], [5].

Analytical solutions to the HJB PDE are rarely found, however many methods may be used to approximately generate optimal reachability sets given an initial or final set. Two methods briefly discussed here are viscosity solution methods and trajectory based methods. In general, viscosity solution methods directly integrate the HJB PDE given an initial condition (set) over time, either forwards or backwards, and compute the zero-level sets of the resulting value function [6], [7], [8], [9], [10]. Alternately, rather than treating the reachability set as a viscosity solution to the HJB PDE, individual optimal trajectories or expansions about these trajectories may be used to sample and represent the reachability set surface [11], [12].

Alternative integration constraints of interest may include performance function costs, capital allocation limits, fuel mass, or control effort. Specific examples of the utility of such integral constraints are aircraft range, automobile range, and total energy constraints. High fidelity computations of fuel limited aircraft and/or launch vehicle range, electric motor angle/angle-rate reachability given total energy constraints, and fuel limited spacecraft orbit element range may be particularly useful. In cases where there is no a-priori time integral constraint (duration), the goal of this paper is to adapt the typical framework by which such optimal reachability sets may be computed using Generalized Independent Parameters (GIPs) and constraints and to identify and address problematic mappings.

The specific contributions of this paper are a) the introduction of a GIP and its mapping function with time, b) the derivation of the GIP Hamilton-Jacobi-Bellman (HJB) PDE using the newly defined GIP and associated mapping function, c) the necessary conditions on the GIP mapping function for a mapping to be invertible, and d) an approach to transform a class of problems with discontinuous GIP HJB PDEs to problems with continuous GIP HJB PDEs.

A motivating problem is briefly outlined and some fundamental problems in computing a maximum range set (fuel-limited reachability) are discussed in §II. In §III the GIP mapping function is introduced, followed by the detailed derivation of a GIP HJB PDE. Necessary conditions on the mapping function integrand are developed and discussed. A short verification of the adjoint state dynamics in the new independent parameter space and several illustrations of interest are given and discussed. §III closes with a Lemma detailing how ΔV integration space discontinuities may be circumvented through intelligent choice of state-space coordinates. Worked examples of the utility of the approach are given in §IV. Finally, conclusions and future work are discussed in §V.

II. Motivation

To properly motivate this paper the classical primer vector problem in astrodynamics [13] is first stated, then an attempt to transform it to a fuel-limited, minimum-fuel, free-time reachability problem involving the HJB PDE is made. For central-body motion, an acceleration-limited spacecraft minimizing fuel (ΔV) usage in the Two-Point Boundary Value Problem (TPBVP) may be written as a problem statement of the form

$$\begin{aligned}\inf_{\mathbf{u}\in U} \Delta V = \inf_{\mathbf{u}\in U} \|\mathbf{u}(t)\|_{L_1} = \inf_{\mathbf{u}\in U} \int_{t_0}^{t_f} \|\mathbf{u}(\tau)\|_2 d\tau \\ \text{s.t. } \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u}) = \begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ -\frac{\mu}{\|\mathbf{r}\|_2^3}\mathbf{r} + \mathbf{u} \end{bmatrix} \\ U = \{\mathbf{u}(t) | \ \|\mathbf{u}(t)\|_2 \le u_m\} \\ \mathbf{x}(t_0) = \mathbf{x}_0, \ \mathbf{x}(t_f) = \mathbf{x}_f\end{aligned}$$

where $\mathbf{r},\mathbf{v},\mathbf{u} \in \mathbb{R}^3$, the final time $t_f > t_0$ is the final time, $u_m > 0$ is the maximum deliverable acceleration, and $\mu > 0$ is a gravitational constant. The optimal control policy found using the Pontryagin Maximum Principle is

$$\mathbf{u}(t) = \begin{cases} -u_m \frac{\mathbf{p}_v(t)}{\|\mathbf{p}_v(t)\|_2} & \text{if } \|\mathbf{p}_v(t)\|_2 < 1 \\ \mathbf{0} & \text{if } \|\mathbf{p}_v(t)\|_2 \ge 1 \end{cases} \tag{1}$$

with $\mathbf{p}_v$ being the adjoint state associated with velocity $\mathbf{v}$. To write the problem above using ΔV constraints in a manner similar to the minimum-time reachability HJB PDE formulation, the dynamics must first be written with ΔV as the independent variable. The relationship between ΔV and time t is expressed in the performance function above, generating

$$\frac{d\mathbf{x}}{d\Delta V} = \frac{d\mathbf{x}}{dt}\frac{dt}{d\Delta V} = \frac{\mathbf{f}(\mathbf{x},\mathbf{u})}{\|\mathbf{u}\|_2}$$

It is here that the motivation for this paper is encountered. The optimal control policy (1) specifically states that in the time domain there may be large periods where the spacecraft drifts along a homogeneous trajectory. In this case the state dynamics with respect to ΔV become undefined, causing the optimal control Hamiltonian and the HJB PDE itself to become undefined. Intuitively this happens because when the problem is integrated forward in the ΔV domain, dynamics occuring during homogeneous drift periods are not captured, generating discontinuities in the state, Hamiltonian, and HJB PDE.

To address this situation, the following section derives the GIP HJB PDE using a GIP mapping function and identifies cases where the mapping is poorly defined. At the end of the Theory section a method to transform cases such as the motivating problem into solvable problems is proposed.

III. Theory

To begin the derivation of the Hamilton Jacobi Bellman PDE using a GIP, a function $l(\mathbf{x},\mathbf{u},t)$ is defined such that

$$s = S(\mathbf{x}(\cdot),\mathbf{u}(\cdot),t) = \int_{t_0}^{t} l(\mathbf{x}(\tau),\mathbf{u}(\tau),\tau)d\tau + s_0 \tag{2}$$

with $\mathbf{x} \in \mathbb{R}^n$ called the state and $\mathbf{u} \in \mathbb{R}^m$ called the control input. The notation $\mathbf{x}(\cdot)$ and $\mathbf{u}(\cdot)$ denote the state and control trajectories. For now, it is required that $S(\mathbf{x}(\cdot),\mathbf{u}(\cdot),t) \in \mathbb{R}$ be invertible with respect to t over the interval $[t_0,t_f]$. Figure 1 depicts what such a function may look like and emphasizes the mapping from time t to the free variable s. Assuming then that

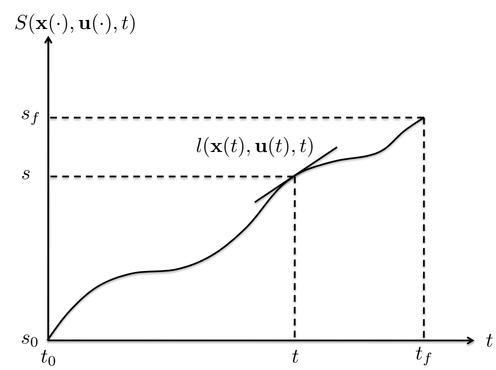

Fig. 1. Visualization of the fully invertible GIP mapping function $s = S(\mathbf{x}(\cdot),\mathbf{u}(\cdot),t)$

the inverse exists, the following definitions are made

$$\begin{aligned} s_0 &= S(\mathbf{x}(\cdot),\mathbf{u}(\cdot),t_0) \\ s_f &= S(\mathbf{x}(\cdot),\mathbf{u}(\cdot),t_f) \\ t_0 &= S^{-1}(\mathbf{x}(\cdot),\mathbf{u}(\cdot),s_0) = R(\mathbf{x}(\cdot),\mathbf{u}(\cdot),s_0) \\ t_f &= S^{-1}(\mathbf{x}(\cdot),\mathbf{u}(\cdot),s_f) = R(\mathbf{x}(\cdot),\mathbf{u}(\cdot),s_f) \end{aligned}$$

It is said then that the function S and it's inverse R are both one-to-one and onto between the intervals $[t_0,t_f]$

and $[s_0, s_f]$. Computing the time derivative of s:

$$\frac{ds}{dt} = \frac{d}{dt}S(\mathbf{x}(\cdot), \mathbf{u}(\cdot), t) = l(\mathbf{x}(t), \mathbf{u}(t), t)$$

Similarly, the derivative of time with respect to the alternate free variable is

$$\frac{dt}{ds} = \frac{d}{dt}R(\mathbf{x}(\cdot), \mathbf{u}(\cdot), s) = \frac{1}{l(\mathbf{x}(s), \mathbf{u}(s), s)}$$

The state variable $\mathbf{x}$ may be written in terms of the new free variable s as $\mathbf{x}(t(s)) = \mathbf{x}(R(\mathbf{x}(\cdot), \mathbf{x}(\cdot), s))$. The system dynamics may then be rewritten with respect to the new GIP $s \in [s_0, s_f]$:

$$\mathbf{x}' = \frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{dt}\frac{dt}{ds} = \frac{\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)}{l(\mathbf{x}(s), \mathbf{u}(s), s)}$$

Since $\mathbf{x}'$ still contains references to t, these must be replaced using the function $t = R(\mathbf{x}(\cdot), \mathbf{u}(\cdot), s) + t_0$:

$$\mathbf{x}' = \frac{d\mathbf{x}}{ds} = \frac{\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)}{l(\mathbf{x}(s), \mathbf{u}(s), s)} = \frac{\tilde{\mathbf{f}}(\mathbf{x}, \mathbf{u}, s)}{\tilde{l}(\mathbf{x}, \mathbf{u}, s)}$$

where the $\tilde{\cdot}$ notation indicates that a function $\cdot$ is written in terms of the new free variable s (for example the state equations of motion $\tilde{\mathbf{f}}(\cdot, \cdot, \cdot)$ are understood to be in terms of s, while $\mathbf{f}(\cdot, \cdot, \cdot)$ are in terms of t). The $\tilde{\cdot}$ notation is used through the remainder of the paper in the interest of brevity. Now that the dynamics have been entirely rewritten in terms of the GIP s, the relationship with the Hamilton Jacobi Bellman PDE may now be determined. Recall the Dynamic Programming Equation (DPE)

$$V(\mathbf{x}_0, t_0) = \operatorname*{opt}_{\mathbf{u} \in U}\left[\int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}, \mathbf{u}, \tau) d\tau + V(\mathbf{x}_f, t_f)\right] \quad (3)$$

where the 'opt' operation may be either an infimum or supremum. The pair $(\mathbf{x}^*, \mathbf{u}^*)$ denote the optimum trajectory found using either the infimum or supremum on the DPE shown in (3). Under the assumption that $V(\mathbf{x}, t)$ is sufficiently smooth, along an optimal trajectory it can be shown that

$$\frac{dV}{dt} = -\mathcal{L}(\mathbf{x}^*, \mathbf{u}^*, t)$$

Thus, to change the independent parameter in the DPE (3), the rate of change of the Lagrangian must first be determined:

$$\frac{dV}{ds} = \frac{dV}{dt}\frac{dt}{ds} = -\frac{\mathcal{L}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)}{l(\mathbf{x}^*(s), \mathbf{u}^*(s), s)} = -\frac{\tilde{\mathcal{L}}(\mathbf{x}^*, \mathbf{u}^*, s)}{\tilde{l}(\mathbf{x}^*, \mathbf{u}^*, s)} \quad (4)$$

Now, the DPE using the new independent parameter s is written as

$$\tilde{V}(\mathbf{x}_0, s_0) = \operatorname*{opt}_{\mathbf{u} \in U}\left[\int_{s_0}^{s_f} \left(\frac{\tilde{\mathcal{L}}(\mathbf{x}, \mathbf{u}, v)}{\tilde{l}(\mathbf{x}, \mathbf{u}, v)}\right) dv + \tilde{V}(\mathbf{x}_f, s_f)\right] \quad (5)$$

Following a standard HJB PDE derivation [5], the initial boundary conditions are chosen as $\mathbf{x}_0 = \mathbf{x}$ and $s_0 = s$ and the final boundary conditions $\mathbf{x}_f$ and s_f are chosen such that $\mathbf{x}_f = \mathbf{x} + \delta\mathbf{x}$ and $s_f = s + \delta s$. The modifiend DPE (5) is then written as

$$\tilde{V}(\mathbf{x}, s) = \operatorname*{opt}_{\mathbf{u} \in U}\left[\int_{s}^{s+\delta s} \left(\frac{\tilde{\mathcal{L}}(\mathbf{x}, \mathbf{u}, v)}{\tilde{l}(\mathbf{x}, \mathbf{u}, v)}\right) dv + \tilde{V}(\mathbf{x} + \delta\mathbf{x}, s + \delta s)\right]$$

The incremental state change $\delta\mathbf{x}$ from an incremental free variable change δs may be written as

$$\delta\mathbf{x} = \frac{d\mathbf{x}}{ds}\delta s + O(\delta s^2) = \frac{\tilde{\mathbf{f}}(\mathbf{x}, \mathbf{u}, s)}{\tilde{l}(\mathbf{x}, \mathbf{u}, s)}\delta s + O(\delta s^2)$$

The Taylor series expansion of the value function at $s_f = s + \delta s$ becomes

$$\begin{aligned} &\tilde{V}(\mathbf{x} + \delta\mathbf{x}, s + \delta s) \\ &= \tilde{V}\left(\mathbf{x} + \frac{\tilde{\mathbf{f}}(\mathbf{x}, \mathbf{u}, s)}{\tilde{l}(\mathbf{x}, \mathbf{u}, s)}\delta s + O(\delta s^2), s + \delta s\right) \\ &= \tilde{V}(\mathbf{x}, s) + \frac{\partial \tilde{V}}{\partial \mathbf{x}}^T \frac{\tilde{\mathbf{f}}(\mathbf{x}, \mathbf{u}, s)}{\tilde{l}(\mathbf{x}, \mathbf{u}, s)}\delta s + \frac{\partial \tilde{V}}{\partial s}\delta s + O(\delta s^2) \end{aligned}$$

Similarly, the Lagrangian may be written as

$$\int_{s}^{s+\delta s} \left(\frac{\tilde{\mathcal{L}}(\mathbf{x}, \mathbf{u}, v)}{\tilde{l}(\mathbf{x}, \mathbf{u}, v)}\right) dv = \frac{\tilde{\mathcal{L}}(\mathbf{x}, \mathbf{u}, s)}{\tilde{l}(\mathbf{x}, \mathbf{u}, s)}\delta s + O(\delta s^2)$$

Substituting these relationships into the GIP DPE produces

$$\tilde{V}(\mathbf{x}, s) = \operatorname*{opt}_{\mathbf{u} \in U}\left[\frac{\tilde{\mathcal{L}}(\mathbf{x}, \mathbf{u}, s)}{\tilde{l}(\mathbf{x}, \mathbf{u}, s)}\delta s + \tilde{V}(\mathbf{x}, s) + \frac{\partial \tilde{V}}{\partial \mathbf{x}}^T \frac{\tilde{\mathbf{f}}(\mathbf{x}, \mathbf{u}, s)}{\tilde{l}(\mathbf{x}, \mathbf{u}, s)}\delta s + \frac{\partial \tilde{V}}{\partial s}\delta s + O(\delta s^2)\right]$$

Subtracting $\tilde{V}(\mathbf{x}, s)$ from both sides, ignoring terms of $O(\delta s^2)$ or higher, and dividing both sides by δs generates the GIP HJB PDE:

$$\frac{\partial \tilde{V}}{\partial s} + \operatorname*{opt}_{\mathbf{u} \in U}\left[\frac{\tilde{\mathcal{L}}(\mathbf{x}, \mathbf{u}, s)}{\tilde{l}(\mathbf{x}, \mathbf{u}, s)} + \frac{\partial \tilde{V}}{\partial \mathbf{x}}^T \frac{\tilde{\mathbf{f}}(\mathbf{x}, \mathbf{u}, s)}{\tilde{l}(\mathbf{x}, \mathbf{u}, s)}\right] = 0 \quad (6)$$

It can be shown that the second term in (6) has many of the special properties of Hamiltonians in classical mechanics [14]. Further, Pontryagin's Maximum Principle tells us that the gradient $\partial V / \partial \mathbf{x}$ satisfies all of the properties of an adjoint variable $\mathbf{p}$ in a Hamiltonian system [5]. The Optimal Control Hamiltonian (OCH) may then be written as

$$\tilde{\mathcal{H}}(\mathbf{x}, \mathbf{p}, \mathbf{u}, s) = \operatorname*{opt}_{\mathbf{u} \in U}\left[\frac{\tilde{\mathcal{L}}(\mathbf{x}, \mathbf{u}, s)}{\tilde{l}(\mathbf{x}, \mathbf{u}, s)} + \mathbf{p}^T \frac{\tilde{\mathbf{f}}(\mathbf{x}, \mathbf{u}, s)}{\tilde{l}(\mathbf{x}, \mathbf{u}, s)}\right] \quad (7)$$

where the adjoint variable $\mathbf{p} = \partial \tilde{V}/\partial \mathbf{x}$. Note that none of the steps make any second-order optimality assumptions, so the 'opt' argument may be replaced with an optimization argument as desired ('min,' 'max,' 'infimum', 'supremum,' etc.). To derive the equations of motion of the adjoint in the s-domain, an approach similar to that in classical mechanics [13], [14], [15] is used. Variations in the performance index

$$\tilde{P} = \int_{s_0}^{s_f} \frac{\tilde{\mathcal{L}}(\mathbf{x}, \mathbf{u}, v)}{\tilde{l}(\mathbf{x}, \mathbf{u}, v)} dv$$

are examined as a starting point. Recalling the definition of the Hamiltonian using the new independent parameter (7), the performance index may be re-written as

$$\tilde{P} = \int_{s_0}^{s_f} \left[\tilde{\mathcal{H}}(\mathbf{x}, \mathbf{p}, \mathbf{u}, v) - \mathbf{p}^T \mathbf{x}'\right] dv$$

Taking the first variation of $\tilde{P}$ about optimal trajectories (and thereby assuming that $\partial \tilde{\mathcal{H}}/\partial \mathbf{u} = \mathbf{0}$) generates

$$\delta \tilde{P} = \int_{s_0}^{s_f} \left[\frac{\partial \tilde{\mathcal{H}}}{\partial \mathbf{x}} \delta \mathbf{x} + \frac{\partial \tilde{\mathcal{H}}}{\partial \mathbf{p}} \delta \mathbf{p} - \mathbf{p}^T \delta \mathbf{x}' - \mathbf{x}'^T \delta \mathbf{p}\right] ds$$

Using integration by parts the second to last term in the integrand may be re-written:

$$\int_{s_0}^{s_f} \mathbf{p}^T \delta \mathbf{x}' dv = \mathbf{p}^T \delta \mathbf{x}' \Big|_{s_0}^{s_f} - \int_{s_0}^{s_f} \mathbf{p}'^T \delta \mathbf{x} dv$$

Substituting this relation, observing that the problem is essentially a Two-Point Boundary Value Problem ($\delta \mathbf{x}(s_0) = \delta \mathbf{x}(s_f) = \mathbf{0}$) and simplifying yields

$$\delta \tilde{P} = \int_{s_0}^{s_f} \left[\left(\frac{\partial \tilde{\mathcal{H}}}{\partial \mathbf{x}} + \mathbf{p}'\right) \delta \mathbf{x} + \left(\frac{\partial \tilde{\mathcal{H}}}{\partial \mathbf{p}} - \mathbf{x}'\right) \delta \mathbf{p}\right] dv$$

Requiring stationarity ($\delta \tilde{P} = 0$) for arbitrary variations $\delta \mathbf{x}$ and $\delta \mathbf{p}$ requires

$$\mathbf{x}' = \frac{d\mathbf{x}}{ds} = \frac{\partial \tilde{\mathcal{H}}}{\partial \mathbf{p}} \tag{8}$$

and

$$\mathbf{p}' = \frac{d\mathbf{p}}{ds} = -\frac{\partial \tilde{\mathcal{H}}}{\partial \mathbf{x}} \tag{9}$$

Equations (8) and (9) precisely mirror the classical results for time-based trajectory optimization.

Remark III.1. ***Initial Conditions on the HJB PDE***
The assumptions of the HJB PDE derivation involved supposing an initial condition $V(\boldsymbol{x}_0, t_0)$ or $\tilde{V}(\boldsymbol{x}_0, s_0)$ in the GIP Dynamic Programing Equation (5). By convention, the zero-level sets of $V(\boldsymbol{x}_0, s_0)$ are used to define the boundary of the reachable set, as when $V(\boldsymbol{x}, s) = 0$, it may equivalently be said that the Performance-to-go $V(\boldsymbol{x}, s)$ is zero.

Thus far it has been required that the GIP mapping function $s = S(\mathbf{x}(\cdot), \mathbf{u}(\cdot), t)$ be invertible, and that $t = R(\mathbf{x}(\cdot), \mathbf{u}(\cdot), s)$ is its inverse (S and R satisfy the identity relation $t = R(\mathbf{x}(\cdot), \mathbf{u}(\cdot), S(\mathbf{x}(\cdot), \mathbf{u}(\cdot), t))$ for the domain $t \in [t_0, t_f]$ and range $s \in [s_0, s_f]$) [16]. Several cases for the value of $l(\mathbf{x}, \mathbf{u}, t)$ are now investigated with the aim of identifying specific requirements on the parameter mapping function $S(\mathbf{x}(\cdot), \mathbf{u}(\cdot), t)$ (S):

1) $l(\mathbf{x}, \mathbf{u}, t) = \pm\infty$ (but not $= 0$) over $t \in [t_0, t_f]$. The slope l has become vertical, meaning that an infinitesimal change in time t will cause a finite (or even infinite) change in the GIP s. The function S is said to be *left-invertible*, and there exists a unique onto mapping R from $s \in [s_0, s_f]$ to $t \in [t_0, t_f]$, but the reverse mapping is not one-to-one. Figure 2 illustrates this case. Note that when this happens, the HJB PDE (6) reduces to $\partial \tilde{V}/\partial s = 0$, implying that the value function is constant and the reach set does not progress in such regions. Similarly, $\mathbf{x}' \to \mathbf{0}$ and $\mathbf{p}' \to \mathbf{0}$, meaning that the dynamics cease to propagate while $l(\mathbf{x}, \mathbf{u}, t) = \pm\infty$.
2) $l(\mathbf{x}, \mathbf{u}, t) = 0$ (but not $= \pm\infty$) over $t \in [t_0, t_f]$. The slope of l is completely horizontal, requiring a finite change in time t to induce an infinitesimal change in the independent parameter s. The function S is said to be *right-invertible*, and there exists a unique onto mapping S from $t \in [t_0, t_f]$ to $s \in [s_0, s_f]$, but the reverse mapping R is not one-to-one. Figure 3 illustrates this case. When this situation occurs the HJB PDE (6), the Hamiltonian (7), and the state/adjoint dynamics (8) and (9) become undefined and the value function $\tilde{V}(\mathbf{x}, s)$ experiences a discontinuity, violating one of the assumptions of the HJB PDE derivation. This is the case the motivating problem in §II emphasizes.
3) $l(\mathbf{x}, \mathbf{u}, t) = \pm\infty$ and $l(\mathbf{x}, \mathbf{u}, t) = 0$ over $t \in [t_0, t_f]$. The function experiences discontinuities on both the domain $t \in [t_0, t_f]$ and the range $s \in [s_0, s_f]$, and is neither left- nor right-invertible. Figure 4 illustrates this case.
4) $-\infty < l(\mathbf{x}, \mathbf{u}, t) < \infty$ over $t \in [t_0, t_f]$. The S has both a positive and negative slope over its domain $t \in [t_0, t_f]$. The function S is right-invertible, as the mapping from $t \in [t_0, t_f]$ is onto, though the reverse mapping is not onto. Figure 5 illustrates this case.

These cases motivate the following remarks

Remark III.2. ***Parameter Mapping Function Integrand Constraints***
For the mapping $S(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), t)$ to be invertible, the parameter mapping function integrand $l(\boldsymbol{x}, \boldsymbol{u}, t)$ must

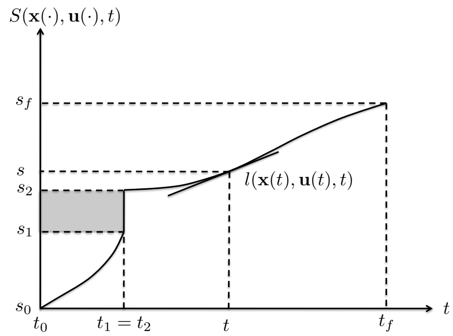

Fig. 2. Case 1: $l(\mathbf{x},\mathbf{u},t) = \pm\infty$

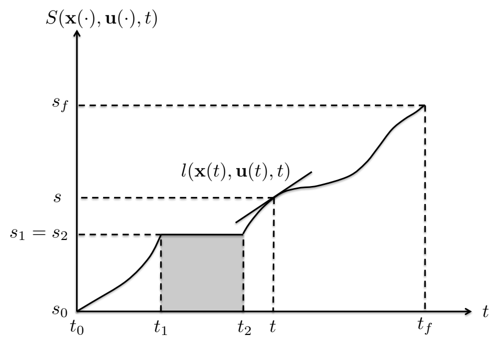

Fig. 3. Case 2: $l(\mathbf{x},\mathbf{u},t) = 0$

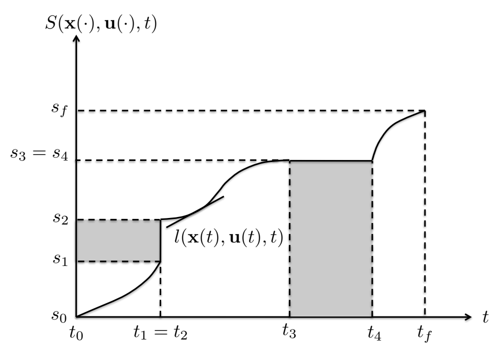

Fig. 4. Case 3: $l(\mathbf{x},\mathbf{u},t) = \pm\infty$ and $l(\mathbf{x},\mathbf{u},t) = 0$

satisfy either of the following inequalities:

$$0 < l(\boldsymbol{x},\boldsymbol{u},t) < \infty,\ \forall t \in [t_0, t_f] \tag{10}$$

or

$$-\infty < l(\boldsymbol{x},\boldsymbol{u},t) < 0,\ \forall t \in [t_0, t_f] \tag{11}$$

Requiring either (10) or (11) ensures that the mapping $s = S(\boldsymbol{x}(\cdot),\boldsymbol{u}(\cdot),t)$ is one-to-one and onto.

Remark III.3. ***Ideal Integration Domain***
If the GIP mapping function S is both left- and right-invertible, then both the GIP HJB PDE (6) in s and the classical HJB PDE in t are suitable settings in which to solve optimal control problems. If S is only right-invertible, then the traditional HJB PDE propagated in t may be a more convenient domain for solving optimal control problems. If S is only left-invertible, then the GIP HJB PDE (6) in s may be the most convenient domain in which to solve optimal control problems. A change in coordinates and/or control variables may significantly alter the invertibility of the GIP mapping function.

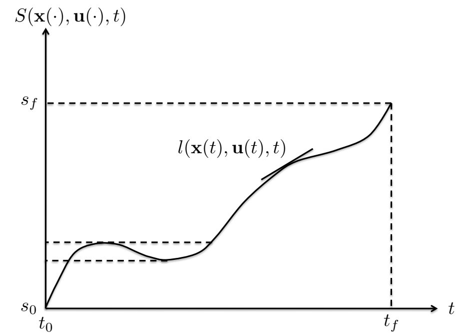

Fig. 5. Case 4: $-\infty < l(\mathbf{x},\mathbf{u},t) < \infty$

Remark III.4. ***Reachability with Integration Equality Constraints***
Given a reachability problem with an integration constraint of the form

$$\int_{t_0}^{t_f} c(\boldsymbol{x},\boldsymbol{u},\tau)d\tau + c_0 - c_f = 0$$

where $t_f > t_0$ is unspecified, the constraint integrand $c(\boldsymbol{x},\boldsymbol{u},t)$ may be used as the parameter mapping integrand $l(\boldsymbol{x},\boldsymbol{u},t)$, and the corresponding value function $\tilde{V}(\boldsymbol{x},s)$ computing using the GIP HJB PDE

$$\frac{\partial \tilde{V}}{\partial s} + \underset{\boldsymbol{u}\in U}{opt}\left[\frac{\tilde{\mathcal{L}}(\boldsymbol{x},\boldsymbol{u},s)}{\tilde{c}(\boldsymbol{x},\boldsymbol{u},s)} + \frac{\partial \tilde{V}}{\partial \boldsymbol{x}}^T \frac{\tilde{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u},s)}{\tilde{c}(\boldsymbol{x},\boldsymbol{u},s)}\right] = 0$$

by integrating over the parameter constraints $s \in [s_0, s_f] = [c_0, c_f]$.
Proof: *Substituting $c(\boldsymbol{x},\boldsymbol{u},t) = l(\boldsymbol{x},\boldsymbol{u},t)$ and defining $\tilde{c}(\boldsymbol{x},\boldsymbol{u},t) = c(\boldsymbol{x},\boldsymbol{u},R(\boldsymbol{x},\boldsymbol{u},t))$ into (6) directly generates the above result.* □

Remark III.4 briefly discusses the primary means by which free-time reachability problems with an integral constraint may be approached using the GIP HJB PDE framework. Several short illustrations are now given to explore some choices of the GIP mapping function integrand $l(\mathbf{x},\mathbf{u},t)$.

ILLUSTRATION III.1: *Unity Parameter Mapping*
Choosing $l(\mathbf{x},\mathbf{u},t) = 1$ gives $s = t$. Substituting into (6)

yields

$$\frac{\partial V}{\partial t} + \underset{\mathbf{u}\in U}{\text{opt}}\left[\mathcal{L}(\mathbf{x},\mathbf{u},t) + \frac{\partial V}{\partial \mathbf{x}}^T \mathbf{f}(\mathbf{x},\mathbf{u},t)\right] = 0$$

which is the traditional HJB realization. This GIP mapping is similar to that shown in Figure 1.

ILLUSTRATION III.2: *Negative Time Reachability* Choosing $\mathcal{L}(\mathbf{x},\mathbf{u},t) = 0$ and $l(\mathbf{x},\mathbf{u},t) = -1$ produces $s = -t$. Substituting into (6) yields

$$\frac{\partial V}{\partial t} + \underset{\mathbf{u}\in U}{\text{opt}}\left[-\frac{\partial V}{\partial \mathbf{x}}^T \mathbf{f}(\mathbf{x},\mathbf{u},t)\right] = 0$$

which is the traditional reachability HJB realization with negative time dynamics. Negative time dynamics are used in backward reachability problems. This GIP mapping is nearly the same as shown in Figure 1, but with a negative slope.

ILLUSTRATION III.3: *Performance is the Independent Parameter* In the case that the integrand $l(\mathbf{x},\mathbf{u},t)$ of the mapping function $S(\mathbf{x}(\cdot),\mathbf{u}(\cdot),t)$ is the Lagrangian of the performance function being optimized,

$$l(\mathbf{x},\mathbf{u},t) = \mathcal{L}(\mathbf{x},\mathbf{u},t)$$

This corresponds to a situation in which the independent parameter s is synonymous with the total performance $\mathcal{P}$, as

$$\mathcal{P} = s = S(\mathbf{x}(\cdot),\mathbf{u}(\cdot),t) = \int_{t_0}^{t} \mathcal{L}(\mathbf{x}(\tau),\mathbf{u}(\tau),\tau)d\tau$$

The GIP HJB PDE (6) is reduced to

$$\frac{\partial \tilde{V}}{\partial s} + 1 + \underset{\mathbf{u}\in U}{\text{opt}}\left[\frac{\partial \tilde{V}}{\partial \mathbf{x}}^T \frac{\tilde{\mathbf{f}}(\mathbf{x},\mathbf{u},s)}{\tilde{\mathcal{L}}(\mathbf{x},\mathbf{u},s)}\right] = 0 \qquad (12)$$

Note that if the Lagrangian is chosen to be identically zero but the original mapping function $l(\mathbf{x},\mathbf{u},t) = \mathcal{L}(\mathbf{x},\mathbf{u},t)$ is kept, then the 1 is eliminated and (12) appears exactly as a traditional time-based optimal reachability formulation. However in this instance it is with respect to the performance function $\mathcal{P}$ that the HJB PDE is integrated. This situation is precisely the one found in the motivation problem introduced in §II.

Now that sufficient discussion and illustration has been given to the GIP HJB PDE, a method by which a sublcass of problems with GIP mapping functions shown in Case 2 ($l(\mathbf{x}(t),\mathbf{u}(t),t) = 0$) may be solved is introduced. Importantly this approach applies to the motivating primer vector problem.

Lemma III.1. ***Solving Optimal Reachability Problems with ΔV Mapping Functions*** *Given an optimal reachability problem with control appearing as an acceleration and a GIP mapping function $l(\boldsymbol{u}(t)) = \|\boldsymbol{u}(t)\|_2 = 0$ over $t \in [t_0, t_f]$ where t_f is unspecified, the optimal reachability problem may instead by solved by writing the dynamics in terms of the $n-1$ constants of motion $\boldsymbol{k}(\boldsymbol{x},t) \in \mathbb{R}^{n-1}$, considering the original control acceleration $\boldsymbol{u}$ to be impulsive, and using the n^{th} time-varying state $K(\boldsymbol{x},t) \in \mathbb{R}$ as an additional control parameter.*
Proof: *The time-dynamics of the $n-1$ constants of motion may be written as*

$$\dot{\boldsymbol{k}} = \boldsymbol{g}(\boldsymbol{k}, K(\boldsymbol{k},t), t)\boldsymbol{u}$$

The corresponding GIP HJB PDE (6) is:

$$\frac{\partial \tilde{V}}{\partial s} + \underset{K,\boldsymbol{u}\in U}{opt}\left[\frac{\partial \tilde{V}}{\partial \boldsymbol{k}}^T \tilde{g}(\boldsymbol{k},K,s)\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|_2}\right] = 0$$

The HJB PDE is well defined over $s \in [s_0, s_f]$, as when $\|\boldsymbol{u}(t)\|_2 = 0$ the constants of motion $\boldsymbol{k}$ have absolutely no dynamics ($d\boldsymbol{k}/dt = \boldsymbol{0}$). Conversely, as the optimal trajectory is propagated in the s-domain the constants of motion are never stationary ($d\boldsymbol{k}/ds \neq \boldsymbol{0}$). □

Lemma III.1 provides a method by which the impulsive variant of the fuel-constrained, free time optimal reachability problem may be solved. A detailed application of this Lemma to on-orbit range may be found in [17] and a simple worked example is given in the next section.

IV. WORKED EXAMPLES

Two straightforward analytical examples are given here. The first generates the aircraft range equation result using the GIP HJB PDE, while the other applies Lemma III.1 to the linear 2^{nd}-order oscillator.

A. Aircraft Range

The aircraft range equation developed by Louis Breguet is a classic formula that generates a nominal aircraft range based on a specified final mass m_f (dry mass) and initial mass m_0 (wet mass) such that $m_0 = m_f + \Delta m$. For steady-state level flight, the one-dimensional equation of motion is

$$\dot{r} = \sqrt{\frac{2mg}{\rho C_L S}} \qquad (13)$$

where r is distance, m is the current mass at time $t \in [t_0, t_f]$, C_L is the coefficient of lift, and S is the nominal surface area of the lifting surface. To apply the GIP HJB PDE to computing the range of an aircraft as a reachability problem, a GIP mapping is first found, then the resulting GIP HJB PDE is simplified and solved. Aircraft are limited in the mass of fuel they carry, making mass an ideal GIP. The mass GIP mapping function is chosen to be

$$\Delta m = m_f - m_0 = \int_{t_0}^{t_f} \frac{dm}{d\tau} d\tau$$

For most aircraft systems, $\dot{m}$ may be modeled as proportional to the propulsion system thrust, $\dot{m} = -kF_t$. Substituting the steady-state thrust $F_t = (C_D/C_L)mg$ generates $\dot{m} = -k(C_D/C_L)mg$. The GIP mapping function then becomes

$$m(t) = \int_{t_0}^{t} \left(-k\frac{C_D}{C_L}m(\tau)g\right) d\tau + m_0$$

This function is a one-to-one and onto mapping from $[t_0, t_f]$ to $[m_0, m_f]$ as long as $m > 0 \ \forall t \in [t_0, t_f]$, making the optimal trajectory in mass recoverable in terms of time (and back to mass as well). The equations of motion defined in (13) are scalar and do not involve control. Thus without loss of generality $\tilde{\mathcal{L}}(\mathbf{x}, \mathbf{u}, s) = 0$ and the range set is dependent only on propagation of the system dynamics. The GIP HJB for the steady-state, constant altitude aircraft case is written after some simplification as

$$\frac{\partial \tilde{V}}{\partial m} + \frac{\partial \tilde{V}}{\partial r}\left(-\frac{1}{kmg}\sqrt{\frac{2C_L mg}{\rho C_D^2 S}}\right) = 0 \qquad (14)$$

with the boundary condition $r_0 = 0$ and integration over m_0 to m_f. The objective in solving this PDE is to compute $\tilde{V}(r, m)$ over $[m_0, m_f]$. The values of r where $\tilde{V}(r, m) = 0$ over $[m_0, m_f]$ are the ranges for those aircraft masses. Equation (14) may be solved by separation. Integrating with respect to m over $[m_0, m_f]$ yields the classical Breguet equation

$$r = \frac{2}{k}\sqrt{\frac{2}{\rho g S}\frac{C_L}{C_D^2}}\left(\sqrt{m_0} - \sqrt{m_f}\right)$$

If the initial condition $\tilde{V}(r_0 = 0, m_0) = 0$ is imposed, the equivalent value function solution to the PDE (14) $\tilde{V}(r, m)$ is

$$\tilde{V}(r, m) = r - \frac{2}{k}\sqrt{\frac{2}{\rho g S}\frac{C_L}{C_D^2}}\left(\sqrt{m_0} - \sqrt{m}\right) \qquad (15)$$

for $r \in [0, \infty)$ and $m \in [m_0, m_f]$. Using the GIP HJB PDE (mass rather than time) produces the same result as the more straightforward change of integration variables in the original derivation by Breguet.

B. Amplitude Range

In this example the maximum free-time, control-limited range (reachability) of the classic 2^{nd}-order oscillator amplitude will be solved using the approach outlined in Lemma III.1. A 2^{nd}-order linear oscillator $\ddot{x} = -\omega_n^2 x + u$ may be written in terms of integrals of motion A and θ:

$$\dot{A} = f_A(A, \theta, u) = \frac{\cos\theta}{\omega_n}u \qquad (16)$$

$$\dot{\theta} = f_\theta(A, \theta, u) = \omega_n - \frac{\sin\theta}{A}u \qquad (17)$$

where A is the amplitude of the oscillation in position coordinates, θ is the angle x and $\dot{x}$ in the phase space, ω_n is the natural frequency, and u is an acceleration input. Choosing the GIP to be ΔV requires the GIP mapping function integrand $l(x, \dot{x}, u, t)$ to be defined such that

$$\Delta V = V_f - V_0 = \int_{t_0}^{t_f} |u(\tau)| d\tau$$

In this case, if the traditional coordinates x and $\dot{x}$ are used the mapping function becomes problematic if $u(\tau) = 0$ (see Case 2 in §III). Because A is constant in the absence of control there is no discontinuity. The phase angle θ however is not constant in the absence of control. Fortunately θ appears in terms of $\sin$ and $\cos$ operations, and if the input $u(\tau)$ is allowed to be impulsive, we may treat θ as a control parameter. Thus, to compute the time-free, control-limited range of the amplitude A, the goal is now to find θ and u that satisfy the GIP HJB PDE. These assumptions have transformed the mapping function such that the problem satisfies Remark III.2. For this example the GIP HJB PDE becomes

$$\frac{\partial \tilde{V}}{\partial \Delta V} + \sup_{\theta \in [0, 2\pi), u}\left[\frac{\partial \tilde{V}}{\partial A}\frac{f(A, \theta, u)}{|u|}\right] = 0$$

The Hamiltonian is then written as

$$\mathcal{H} = \sup_{\theta \in [0, 2\pi), u}\left[p_A \frac{\cos\theta}{\omega_n}\operatorname{sgn}(u)\right] \qquad (18)$$

Upon inspection, the optimal control policy $(\theta^*, \operatorname{sgn}(u^*))$ is

$$(\theta^*, \operatorname{sgn}(u^*)) = \begin{cases} (0, 1) \text{ or } (\pi, -1) \text{ if } p_A \geq 0 \\ (\pi, 1) \text{ or } (0, -1) \text{ if } p_A < 0 \end{cases} \qquad (19)$$

The result in (19) is very intuitive. If $p_A \geq 0$, $(\theta^*, \operatorname{sgn}(u^*))$ will increase A ($\dot{A} > 0$). Conversely, if $p_A < 0$, $(\theta^*, \operatorname{sgn}(u^*))$ will decrease A ($\dot{A} < 0$). Since the objective is to maximize the range of A it is now assumed that $p_A > 0$ (the minimum case can be found by choosing $p_A < 0$). The GIP HJB PDE with the optimal control input becomes

$$\frac{\partial \tilde{V}}{d\Delta V} + \frac{\partial \tilde{V}}{dA}\frac{1}{\omega_n} = 0$$

This PDE is separable and can be re-written as

$$dA = -\frac{1}{\omega_n}d\Delta V \Rightarrow \int_{A_0}^{A} dA = \frac{1}{\omega_n}\int_{0}^{\Delta V} d\Delta V$$

yielding the value function $\tilde{V}(A, \Delta V)$:

$$\tilde{V}(A, \Delta V) = A - A_0 - \frac{1}{\omega_n}\Delta V \tag{20}$$

Recalling that reachability sets are defined as level sets of $\tilde{V}(A, \Delta V) = 0$ generates

$$A = A_0 + \frac{1}{w}\Delta V$$

which is the maximum time-independent amplitude range given an integral constraint on the control effort u (ΔV). To verify this solution partials of $\tilde{V}(A, \Delta V)$ are evaluated and compared with the GIP HJB PDE:

$$\frac{\partial \tilde{V}}{\partial \Delta V} + \frac{\partial \tilde{V}}{\partial A}\frac{\cos(0)}{\omega_n}\,\mathrm{sgn}(1) = \frac{1}{\omega_n} - \frac{1}{\omega_n} = 0$$

which satisfies the GIP HJB PDE.

V. Conclusion & Future Work

Fuel-optimal control for systems affine in control, specifically spacecraft, is used to motivate a rigorous examination of how optimal reachability sets using General Independent Parameters (GIPs) may be generated and how specific singular mappings may be circumvented. The mapping function between time and an arbitrary GIP is defined and the transformation of the time-dynamics into the dynamics with respect to a new GIP is given. Starting with the Dynamic Programming Equation the necessary conditions of optimality along optimal trajectories are combined with the GIP mapping function definition to derive the GIP HJB PDE using the new independent parameter. Necessary conditions for left, right, and full invertibility of the GIP mapping function are given and a discussion of convenient independent parameters is given. The dynamics of the adjoint state using the new GIP is shown to mirror those of the time-based adjoint dynamics. A method is introduced by which the mapping singularity in the primer vector problem may be circumvented using constants of integration and impulsive assumptions. Both the classical Breugot aircraft range equation and the 2^{nd}-order oscillator amplitude range equation are developed as simple worked examples of the GIP HJB PDE reachability applications.

Acknowledgments

This work was supported by AFOSR Grant No. FA9550-08-1-0460.

References

[1] P. Varaiya, "Reach set computation using optimal control," Proceedings of the KIT Workshop on Verification of Hybrid Systems, pages 377-383, Grenoble, France, 1998.

[2] A. B. Kurzhanski, P. Varaiya, "Dynamic Optimization for Reachability Problems", Journal of Optimization Theory and Applications, Vol. 108, No. 2, pp 227-251, February 2001.

[3] J. Lygeros, "On Reachability and Minimum Cost Optimal Control," Automatica 40, 2004.

[4] A. A. Kurzhanskiy, I. M. Mitchell, P. Varaiya, "Optimization Techniques for State-Constraineed Control and Obstacle Problems," Communicated by G. Leitmann, 2006.

[5] W. H. Fleming, H. M. Soner, "Controlled Markov Processes and Viscosity Solutions, 2^{nd} Ed." New York, Springer, 2006

[6] M. G. Crandall, L. C. Evans, and P. L. Lions, "Some properties of viscosity solutions of Hamilton-Jacobi equations," Transactions of the American Mathematical Society, Vol. 282, No. 2, April 1984.

[7] S. Osher, J. Sethian, "Fronts propagating with curvature-dependent speed:Algorithms based on Hamilton-Jacobi formulations," Journal of Computational Physics, 79, 12-49.

[8] J. A. Sethian, "Level set methods: Evolving interfaces in geometry, fluid mechanics, computer vision, and materials science." New York: Cambridge University Press.

[9] A. M. Bayen, C. J. Tomlin, "A construction procedure using characteristics for viscosity solutions of the Hamilton-Jacobi equation," IEEE Conference on Decision and Control, Orlando, 2001.

[10] I. M. Mitchell, "A Toolbox of Level Set Methods," UBC Department of Computer Science Technical Report TR-2007-11, June 2007.

[11] C. L. Navasca, A. J. Krener, "Solution of Hamilton Jacobi Bellman Equations," IEEE Conference on Decision and Control, Sydney, 2000.

[12] C. Navasca, A. J. Krener, "Patchy Solutions of Hamilton Jacobi Bellman Partial Differential Equations," In A. Chiuso, A. Ferrante and S. Pinzoni, eds, "Modeling, Estimation and Control, Lecture Notes in Control and Information Sciences," 364, pp. 251-270, 2007.

[13] D. F. Lawden, "Analytical Methods of Optimization," Dover Publications Inc., Mineola, NY, 2003.

[14] R. F. Stengel, "Optimal Control and Estimation," Dover Publications Inc., Mineola, 1994.

[15] D. T. Greenwood, "Classical Dynamics," Dover Publications Inc., Mineola, 1977.

[16] A. W. Naylor, G. R. Sell, "Linear Operator Theory in Engineering and Science," Applied Mathematical Sciences 40, Springer-Verlag, New York, NY, 2000.

[17] M. J. Holzinger, D. J. Scheeres, R. S. Erwin, "On-Orbit Range Computation Using Gauss' Variational Equations for Mean Orbit Elements with J_2 Perturbations," 21^{st} AAS/AIAA Space Flight Mechanics Meeting, New Orleans, Louisiana.