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Global Stabilization of a Chain of Integrators with Input Saturation and Disturbances

Shreekant Gayaka and Bin Yao

Abstract— **In this paper, the problem of global stabilization for a chain of integrators in presence of input saturation and disturbances is solved. A novel and elegant approach to solve this problem, in absence of disturbances, was proposed by Teel [1] using saturation functions and coordinate transformation. With Teel's work as foundation, many results have been proposed to improve the performance of tracking/stabilizing controllers for chain of integrators. However, in presence of disturbances, the coordinate transformation can considerably shrink the region where the controller is unsaturated. In this work, we present a modified backstepping like approach to solve the global stabilization problem which does not rely on coordinate transformation. Comparative studies performed using a third order integrator chain proves the effectiveness of the proposed scheme.**

I. Introduction

Controller design in presence of input saturation is a theoretically challenging problem with deep practical implications. Among various approaches for dealing with input magnitude saturation, anti-windup schemes, model predictive control, and nested saturation functions are most popular. In anti-windup based schemes ([2], [3], [4]), a controller is first designed without any regard to the actuator limits, and then a modification is introduced to minimize the adverse effects of saturation. Model predictive control (MPC), which involves solving an open-loop optimization problem at each step, is adept at dealing with hard constraints and is fast becoming a useful tool in dealing with saturation problems [5], [6]. The main challenge for MPC based techniques is to address modeling uncertainties, which are inherent to any realistic system model. Teel introduced two of the most widely used tools for control of systems with input saturation – the nested saturation functions [1] and the small-gain theorem [7]. The nested saturation function approach was first proposed by Teel in 1992 [1] to solve the problem of global stabilization of a chain of integrators with input saturation. The first step of the design involved a coordinate transformation, which would transform the system to a feedforward form. In the second step, saturation functions were used to construct a nested control law in terms of the transformed coordinates to achieve global stabilization. Subsequently, this approach has been extended to various classes of nonlinear systems in feedforward form under various assumptions (see [8]). Anti-wind up schemes [2], low gain designs [9], [10] and quadratic programming based approaches were also used to solve this problem. However, as noted in [11], it is either difficult to obtain a stability proof for such techniques, or they are too expensive computationally to be practical. The authors would like to mention that the literature on saturated control is rich enough that an exhaustive review of all techniques is beyond the scope of this paper.

For systems which are chain of integrators type, many ingenious modifications to Teel's original design have been proposed to improve the transient response and robustness of the controller [12], [11], [13]. But, as all of these designs were based on [1], they also inherited the limitations of that approach. Particularly, as will be shown in the next section, in presence of bounded input disturbance the region where the controller is unsaturated shrinks drastically. In the worst case, it may even become theoretically impossible to guarantee the stability of such controllers. This is to be expected as all the transformed coordinates depend on the last state of the integrator chain, the dynamics of which includes the input disturbances. Thus, the input disturbances affect the dynamics of *every* state in the transformed coordinates. This implies that the parameters for *all* saturation functions need to be chosen conservatively to accommodate the effect of input disturbances. It should be now evident that the design conservativeness in presence of input disturbances can be overcome and the tuning of controller parameters can be made more transparent if the controller design can be carried out directly based on the physical states of the actual system instead of the fictitious transformed states in [1]. Along this line, in [14], globally stable backstepping based designs were proposed for the second order linear motor system in presence of uncertainties, and experimental results showed the effectiveness of the designs. However, the controller design suffers from the same problems as the conventional backstepping designs – "explosion" of terms with increasing system order. Thus, aside from the complexity issue of the resulting controller, extension of the approach [14] to higher order systems proved to be very difficult. In this work, we build on the framework proposed by Teel, but lay more emphasis on the design of saturation functions such that a modified backstepping like approach can be used to solve this problem. Explicit and systematic selection of controller parameters to quantitatively meet various performance requirements is also given. Comparative simulation studies have been performed on a third order integrator chain and the obtained results have verified the effectiveness of the

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S. Gayaka received his PhD from the School of Mechanical Engineering, Purdue University, West Lafayette, IN 47907, USA and is now with Western Digital Inc., 5863 Rue Ferrari, San Jose, CA 95122, USA; *gshreekant@gmail.com*

B. Yao is with the School of Mechanical Engineering, Purdue University, West Lafayette, IN 47907, USA; *byao@purdue.edu*

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proposed scheme.

II. Motivation and Problem formulation

Consider a chain of integrator with input disturbance

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= u + d(t) \\ y &= x_1 \end{aligned} \tag{1}$$

where $|u| \leq u_M$, $|d(t)| \leq d_M$ and d_M is a known constant. The objective of the present work is to design a globally stabilizing controller such that

1) x_1 tracks y_d with steady-state error $|e_{ss}| \leq \delta$
2) $e_{ss}(t)$ should have the desired transient performance in the unsaturated region.

We will make the following assumption regarding the extent of input disturbance and the reference trajectory

A1: The extent of disturbances and the desired trajectory is such that

$$d_M < u_M - \lambda_M \quad \text{and} \quad \left|y_d^{(n)}(t)\right| \leq \lambda_M, \ \forall t \tag{2}$$

Let us first investigate the effect of this disturbance on Teel's coordinate transformation based designs (e.g., [12]). All those designs use the following coordinate transformation [1]

$$y_i = \sum_{j=i}^{n} t_{ij}\tilde{x}_j, \quad j = 1, \ldots, n \tag{3}$$

where $\tilde{x}_j = x_j - y_d^{(j-1)}$ and $T = [t_{ij}]$ is the transformation matrix given by

$$t_{ij} = \begin{cases} 1, & 1 \leq i \leq n, \quad j = n \\ 0, & i = n, \quad 1 \leq j \leq n-1 \\ \sum_{m=i+1}^{n} k_m t_{m,j+1}, & i \leq n-1, \quad j \leq n-1 \end{cases}$$

With this, the dynamics of the transformed states can be expressed as

$$\begin{pmatrix} \dot{y}_1 \\ \vdots \\ \dot{y}_{n-1} \\ \dot{y}_n \end{pmatrix} = \begin{bmatrix} 0 & k_2 & \cdots & k_n \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & k_n \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} + \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \left[u(t) - y_d^{(n)} + d(t)\right] \tag{4}$$

The control law presented in [12] takes the following form

$$u = y_d^{(n)}(t) - \sigma_n\left(k_n y_n + \sigma_{n-1}\left(... + \sigma_1(k_1 y_1)\right)\right) \tag{5}$$

where σ_i are non-decreasing saturation functions satisfying

$$\begin{aligned} &(1) \quad s\sigma_i(s) > 0, \quad \forall s \neq 0 \\ &(2) \quad \sigma_i(s) = s, \quad \forall |s| < L_i \\ &(3) \quad |\sigma_i(s)| \leq M_i, \quad \forall s \in \mathbb{R} \end{aligned} \tag{6}$$

where M_i and L_i are some positive design parameters with $M_i > L_i$. To account for the disturbances, the inequalities proposed in [1] should be modified as follows to ensure that the tracking error can be steered into a neighborhood of zero globally:

$$\begin{aligned} |y_d^{(n)}| &\leq u_M - M_n \\ L_{i+1} &\geq 2M_i + d_M, \quad i = 0, ..., n-1 \end{aligned} \tag{7}$$

with $M_0 = 0$. From (4), we see that due to the coordinate transformation, the disturbance affects all the states, although it appears only in the input channel in the original coordinates. This makes the design procedure rather conservative in terms of the level of disturbances which can be handled. Specifically, for the conditions in (7) to be satisfied, it is necessary that

$$\begin{array}{lccc} u_M & > & M_n & > & L_n \\ & \geq & d_M + 2M_{n-1} & > & d_M + 2L_{n-1} \\ & \geq & d_M + 2(d_M + 2M_{n-2}) & > & d_M + 2(d + 2L_{n-2}) \\ & \vdots & & \vdots & \\ & \geq & d_M + \cdots + 2^{n-1}d_M & = & (2^n - 1)d_M \end{array}$$

It is thus clear that if the input disturbance $d(t)$ is large enough such that $d_M \geq u_M/(2^n - 1)$, then this approach would not work as the robust stability conditions (7) can never be met. For example, consider the stabilization problem for a third order chain of integrator with $u_M = 10$, and $d_M = 4$. As $d_M = 4 > 10/7$, the above design cannot be applied.

The above analysis shows the conservativeness of approaches which rely on coordinate transformation. In this work, it will be shown that such conservativeness can be removed by using a backstepping like technique, which does not require coordinate transformation. In contrast to the condition $u_M > (2^n - 1)d_M$ a much less restrictive condition of $u_M > d_M$ as assumed in assumption **A1** is sufficient for a guaranteed global stability in the presence of disturbances.

III. Main Result

In this section, we present the proposed backstepping based controller design for robust global stabilization of an integrator chain and the main theoretical results. We begin the section by introducing the saturation functions used in the present work. Next, a set of inequalities are proposed such that when satisfied, the states of the tracking error dynamics will reach in a finite time an invariant region, where the controller is unsaturated independent of the initial conditions. Once within the unsaturated region, desired properties of a linear controller, e.g., exponential rate of convergence, desired transient response and arbitrarily good disturbance rejection performance are guaranteed. Necessary and sufficient conditions for the existence of such a control law, and a systematic way of choosing the controller parameters to achieve the desired closed-loop performance are also presented.

A. Saturation Functions

For each of $z_i, i = 1, \ldots, n$, to be defined later, the non-decreasing saturation function $\sigma_i(z_i)$ used in the proposed

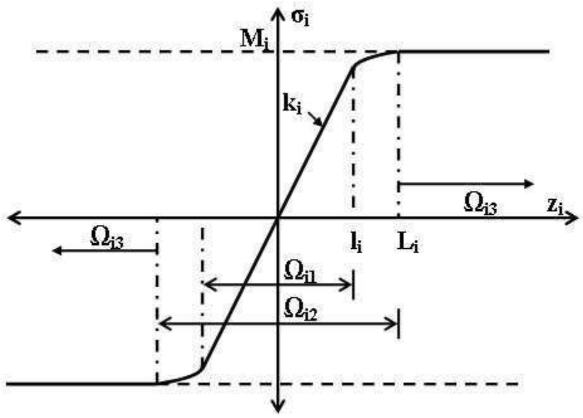

Fig. 1. Saturation function

design (see Fig.1) is required to satisfy a set of more stringent conditions than those used in Teel's work [1]:

$$(a) \qquad z_i\sigma_i(z_i) \geq 0,\ \forall z_i \tag{8}$$

$$(b) \qquad \sigma_i(z_i) = k_i z_i,\ \forall |z_i| \leq l_i \tag{9}$$

$$(c) \qquad \sigma_i(z_i) = M_i(\text{sign}(z_i)),\ \forall |z_i| \geq L_i \tag{10}$$

$$(d) \qquad 0 \leq \frac{\partial \sigma_i}{\partial z_i} \leq k_i,\ \forall z_i \tag{11}$$

where l_i, L_i, M_i and k_i are some positive design parameters to be specified later. These parameters satisfy $l_i = \beta_i L_i$ with $\beta_i \leq 1$, and $M_i = k_i l_i(1+\gamma_i)$ with $\gamma_i \geq 0$. Essentially, the interval for z_i is divided into three different regions - $\Omega_{i1} = \{z_i : |z_i| \leq l_i\}$, $\Omega_{i2} = \{z_i : |z_i| \leq L_i\}$ and $\Omega_{i3} = \{z_i : |z_i| > L_i\}$. For the last saturation function, we can choose $\gamma_n = 0$, $\beta_n = 1$ for σ_n, which implies $\Omega_{n1} = \Omega_{n2}$.

B. Finite-time Convergence to Unsaturated Region

We will use the states of the tracking error dynamics given by $\tilde{x}_i = x_i - y_d^{(i-1)}$ to simplify the analysis. System (1) can be rewritten as follows

$$\begin{aligned}
\dot{\tilde{x}}_1 &= \tilde{x}_2 \\
\dot{\tilde{x}}_2 &= \tilde{x}_3 \\
&\vdots \\
\dot{\tilde{x}}_n &= u - y_d^{(n)} + d(t)
\end{aligned} \tag{12}$$

Step 1: A modified backstepping based approach will be presented here to stabilize the tracking error dynamics (12). Define $z_1 = \tilde{x}_1 - \alpha_0$, where α_0 is used for uniformity of notation and is given by $\alpha_0 = 0$. Let the virtual control law for the first step be α_1, and the virtual control law discrepancy be given by $z_2 = \tilde{x}_2 - \alpha_1$. Then, we have

$$\dot{z}_1 = \dot{\tilde{x}}_1 = \tilde{x}_2 = z_2 + \alpha_1 \tag{13}$$

In order to stabilize the z_1-dynamics, we choose

$$\alpha_1(z_1) = -\sigma_1(z_1) \tag{14}$$

where $\sigma_1(z_1)$ is as defined in the preceding section. Substituting (14) in (13), we get

$$\dot{z}_1 = z_2 - \sigma_1(z_1) \tag{15}$$

Step 2: As in step 1, we choose $\alpha_2(z_2) = -\sigma_2(z_2)$ and denote the virtual control input discrepancy by $z_3 = \tilde{x}_3 - \alpha_2$. Then, taking derivative of $z_2 = \tilde{x}_2 + \sigma_1(z_1)$, we obtain

$$\begin{aligned}
\dot{z}_2 &= \dot{\tilde{x}}_2 + \frac{\partial \sigma_1}{\partial z_1}\dot{z}_1 = \tilde{x}_3 + \frac{\partial \sigma_1}{\partial z_1}(z_2 - \sigma_1(z_1)) \\
&= z_3 + \alpha_2(z_2) + \frac{\partial \sigma_1}{\partial z_1}(z_2 - \sigma_1(z_1)) \\
&= (z_3 - \sigma_2(z_2)) + \frac{\partial \sigma_1}{\partial z_1}(z_2 - \sigma_1(z_1))
\end{aligned} \tag{16}$$

The difference in the standard "cancellation" based backstepping design and the proposed approach should be clear from (16). As opposed to typical backstepping based approach where the effect of $\dot{\alpha}_1$ is completely canceled by incorporating appropriate terms in α_2, we *do not* cancel the terms resulting from $\dot{\alpha}_1$. Furthermore, the α_2 is a function of z_2 only, whereas in standard backstepping designs, the virtual control depends on z_1 and z_2. *This results in a much simpler control law for the proposed approach. Naturally, the resulting closed-loop error dynamics of the proposed approach will be significantly different from those in the standard backstepping based designs and novel global stability and performance analysis will be needed as detailed later.*

Step i: At the ith step of the proposed design, let the desired virtual control law α_i and the virtual control input error z_i be defined as follows

$$\begin{aligned}
z_i &= \tilde{x}_i - \alpha_{i-1}(z_{i-1}) \\
\alpha_i &= -\sigma_i(z_i)
\end{aligned} \tag{17}$$

Then, the following lemma can be proved.

Lemma 1. The derivative of the virtual control input error z_i is given by

$$\dot{z}_i = z_{i+1} - \sigma_i(z_i) + \sum_{j=1}^{i-1}\left\{\left[\prod_{r=1}^{j}\frac{\partial \sigma_{i-r}}{\partial z_{i-r}}\right](z_{i-j+1} - \sigma_{i-j}(z_{i-j}))\right\} \tag{18}$$

Proof. This lemma can be proved by induction and has been omitted due to space restrictions. □

Thus, the virtual control law error dynamics $\dot{z}_i$ can be written as

$$\begin{aligned}
\dot{z}_1 &= z_2 - \sigma_1(z_1) \\
&\cdots \\
\dot{z}_i &= z_{i+1} - \sigma_i(z_i) \\
&\quad + \sum_{j=1}^{i-1}\left\{\left[\prod_{r=1}^{j}\frac{\partial \sigma_{i-r}}{\partial z_{i-r}}\right](z_{i-j+1} - \sigma_{i-j}(z_{i-j}))\right\} \\
&\cdots \\
\dot{z}_n &= u + d(t) - y_d^{(n)} \\
&\quad + \sum_{j=1}^{n-1}\left\{\left[\prod_{r=1}^{j}\frac{\partial \sigma_{n-r}}{\partial z_{n-r}}\right](z_{n-j+1} - \sigma_{n-j}(z_{n-j}))\right\}
\end{aligned}$$

Theorem 1. *Consider system (19). Let the control input be*

$$u = y_d^{(n)} - \sigma_n(z_n) \quad (19)$$

For simplicity, choose the design parameters used in $\sigma_n(z_n)$ as $l_n = L_n$ and $M_n = k_n l_n = u'_M \triangleq u_M - \lambda_M$. Choose the design parameters of other saturation functions such that

$$\begin{aligned} k_i l_i &> l_{i+1} + k_{i-1} N_i, \quad i = 1, 2, ..., n-1 \quad (20) \\ k_n l_n &> k_{n-1} N_n + d_M \quad (21) \end{aligned}$$

where

$$N_i \triangleq L_i + M_{i-1} + \sum_{j=1}^{i-2} \left[(\prod_{r=1}^{j} k_{i-1-r})(L_{i-j} + M_{i-1-j}) \right]$$

and $k_0 \triangleq 0$, $l_{n+1} \triangleq 0$. Then, for any *set of initial conditions, all states reach the linear unsaturated region (i.e., $\bigcap_{i=1}^{n} \Omega_{i1}$) in a finite time.*

Proof. We first show that the following claims are true.

Claim1. For any initial conditions

(a) if $|z_n(0)| > L_n$, then $z_n(t)$ reaches $\Omega_{n1} = \Omega_{n2}$ in a finite time

(b) if $|z_n(0)| \leq L_n$, i.e., $z_n(0) \in \Omega_{n1}$, then $z_n(t) \in \Omega_{n1}$, $\forall$ $t > 0$.

Claim 2. Assume $|z_{i+1}| \leq l_{i+1}, \forall t > t_0$, i.e., $z_{i+1} \in \Omega_{(i+1)1}$. Then

(a) if $|z_i(t_0)| > L_i$, $z_i(t)$ reaches Ω_{i2} in a finite time

(b) if $z_i(t_0) \in \Omega_{i2} \backslash \Omega_{i1}$, $z_i(t)$ reaches Ω_{i1} in a finite time

(c) if $z_i(t_0) \in \Omega_{i1}$, $z_i(t) \in \Omega_{i1}$, $\forall t > t_0$

If both claims are true. Then, from claim 1, we have $|z_n| < L_n = l_n$ after a finite time. Theorem 1 then follows from a recursive application of claim 2 to the intermediate states z_j, $j = n-1, \cdots, 1$. Thus, we will be done if we can show that both the claims are true. Due to space restrictions, only proof of claim 1(a) and 1(b) will be given.

Proof of Claim 1, Part (a). Without the loss of generality (w.l.o.g), assume $z_n(0) \geq L_n$. To prove case (a), the contradiction method will be used. Namely, assuming that $z_n(t)$ does not reach $\Omega_{n1} = \Omega_{n2}$ in a finite time, then, $z_n(t) \geq L_n, \forall t$. Thus, $u = -M_n + y_d^{(n)}$ and from (17) we get

$$\begin{aligned} & \dot{\tilde{x}}_n = -M_n + d(t) \\ \Rightarrow \quad & \tilde{x}_n(t) \leq \tilde{x}_n(0) - (M_n - d_M)t \\ \Rightarrow \quad & z_n(t) + \alpha_{n-1}(t) \leq z_n(0) + \alpha_{n-1}(0) - (M_n - d_M)t \\ \Rightarrow \quad & z_n(t) \leq z_n(0) + \alpha_{n-1}(0) - \alpha_{n-1}(t) - (M_n - d_M)t \\ & \quad \leq z_n(0) + 2M_{n-1} - (M_n - d_M)t \quad (22) \end{aligned}$$

in which the fact that $|\alpha_{n-1}(t)| \leq M_{n-1}$ due to the use of the saturation function is used. Since the right hand side of (22) goes to negative infinity as $t \to \infty$, a contradiction results. Thus $z_n(t)$ will reach $\Omega_{n1} = \Omega_{n2}$ in a finite time t_{32}^n. In fact, an upper-bound for t_{32}^n can be obtained by letting the right hand side of (22) equal to L_n:

$$t_{32}^n \leq \frac{z_n(0) - L_n + 2M_{n-1}}{M_n - d_M} \quad (23)$$

This completes the proof of case (a).

Part (b). To prove (b), all one has to do is to show that the tangent vector points inward at the boundaries of Ω_{n1}, i.e., to show $z_n \dot{z}_n \leq 0$ whenever $z_n = \pm L_n$. For this purpose, first note that when (21) is true, there exists $\delta_n > 0$ such that

$$k_n l_n - \delta_n = \sum_{j=1}^{n-1} \left[(\prod_{r=1}^{j} k_{n-r})(L_{n-j+1} + M_{n-j}) \right] + d_M \quad (24)$$

When $z_n = \pm L_n$, from (19), $u = y_d^{(n)} - M_n sign(z_n)$. Thus, from (19),

$$\begin{aligned} z_n \dot{z}_n &= z_n \Bigg(-M_n sign(z_n) + d(t) + \\ & \sum_{j=1}^{n-1} \left\{ \left[\prod_{r=1}^{j} \frac{\partial \sigma_{n-r}}{\partial z_{n-r}} \right] (z_{n-j+1} - \sigma_{n-j}(z_{n-j})) \right\} \Bigg) \\ &\leq |z_n| \Bigg(-k_n l_n + d_M + \\ & \sum_{j=1}^{n-1} \left[(\prod_{r=1}^{j} k_{n-r})(L_{n-j+1} + M_{n-j}) \right] \Bigg) \quad (25) \end{aligned}$$

in which the fact that $\frac{\partial \sigma_{n-j+1}}{\partial z_{n-j+1}} = 0$ whenever $|z_{n-j+1}| > L_{n-j+1}$ and $|\frac{\partial \sigma_{n-r}}{\partial z_{n-r}}| \leq k_{n-r}$ are used in deriving the upper bound of the terms inside the summation operation. Combining (24) and (25), when $z_n = \pm L_n$,

$$z_n \dot{z}_n \leq -|z_n| \delta_n < 0 \quad (26)$$

This completes the proof of the claim (b). □

C. Controller Parameter Selection

There are two important questions which need to be answered at this stage: (i) the existence of a solution to inequalities (20)-(21) assumed in the controller design, and (ii) how to select the controller gains such that the desired closed-loop performance can be achieved when such a solution exists. In the following subsection, we first state and prove the main result regarding the necessary and sufficient condition for the existence of a solution to the inequalities. Then, a systematic way of choosing the controller gains in accordance with this theorem is proposed such that the desired closed-loop performance can be achieved.

1) Necessary and sufficient conditions for the existence of controller parameters: After a series of derivations, (20)-(21) can be rewritten in a matrix form as

$$AL < D, \quad (27)$$

where $L = [l_1, l_2, \cdots, l_{n-1}, l_n]^T$, $D = [0, 0, \cdots, 0, -d_M]^T$. And A is a matrix whose elements are functions of k_i given by

$$A = \begin{bmatrix} -k_1 & 1 & 0 & \cdots & 0 \\ k_1^2(1+\gamma_1) & \left(\frac{k_1}{\beta_2} - k_2\right) & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \left(\frac{k_{n-1}}{\beta_n} - k_n\right) \end{bmatrix}, \quad (28)$$

where

$$
\begin{aligned}
a_{ij} &= k_{i-1}k_j \prod_{r=1}^{i-j-1} k_{i-r-1}(1+\gamma_j) \\
&\quad + k_{i-1}\prod_{r=1}^{i-j} k_{i-r-1}\tfrac{1}{\beta_j},\ \forall i > j. \\
a_{ij} &= -k_i + \tfrac{k_{i-1}}{\beta_i},\ \forall i = j \\
a_{ij} &= 1,\ \forall j = i+1 \\
a_{ij} &= 0,\ \forall j > i+1
\end{aligned} \tag{29}
$$

For any $\gamma_i > 0$ and $0 < \beta_i \leq 1$, if we fix a set of positive k_i, then the control law is **feasible** if and only if there exist l_1, l_2,..., $l_n > 0$ such that (27) is satisfied. In other words, at least one solution to (27) should lie in the region $\{(l_1, \cdots, l_n):\ l_i > 0\}$. The following theorem gives the necessary and sufficient condition for k_is such that the control law is feasible.

Theorem 2. *For any $\gamma_i > 0$ and $0 < \beta_i \leq 1$, with a set of positive k_i, at least one solution to (27) lie in the region $L \in \{(l_1, \cdots, l_n):\ l_i > 0\}$* **iff** *the k_is satisfy the following set of inequalities:*

$$
\begin{array}{lcl}
k_1 & > & 0, \\
k_2 & > & \frac{a_{21}p_1 + \frac{k_1}{\beta_2}p_2}{p_2} = (1+\gamma_1+\frac{1}{\beta_2})k_1, \\
 & \cdots & \\
k_i & > & \frac{\sum_{j=1}^{i-1} a_{ij}p_j + \frac{k_{i-1}}{\beta_i}p_i}{p_i},\ \forall i < n \\
 & \cdots & \\
k_n & > & \frac{u_M - \lambda_M}{u_M - (\lambda_M + d_M)} \cdot \frac{\sum_{j=1}^{n-1} a_{nj}p_j + \frac{k_{n-1}}{\beta_n}p_n}{p_n},
\end{array} \tag{30}
$$

where the coefficients p_is are computed recursively using the formula

$$
\begin{array}{lcl}
p_1 & = & 1 \\
p_i & = & -\sum_{j=1}^{i-1} a_{i-1j}p_j
\end{array} \tag{31}
$$

Proof 2. Due to space restrictions, the proof has been omitted and can be obtained from the authors upon request.

2) Controller gain selection: a recursive root-locus design: Suppose, the steady-state error is required to be less than δ and the slowest closed-loop pole needs to be to the left of p_0 for fast enough transient responses, then it is sufficient to place all the closed-loop poles to the left of $p_{cl} = \min\{-\sqrt[n]{\frac{d_M}{\delta}}, -p_0\}$. We make the following observation from the general principles of root-locus design: if a system is such that the open-loop transfer function (OLTF) is given by $OLTF = K\frac{P_m(s)}{s^{m+1}}$, where K is the open-loop gain, P_m is a polynomial of degree m, such that all the roots of $P_m(s)$ lie to the left of $-p_{cl}$. Then, as the difference in number of poles and zeros is one, for sufficiently large gain K there always exists an asymptote along the negative real axis. We shall use this fact for the recursive root-locus design. In the first step of the design, select $k_1 > -p_{cl}$ such that the root of the equation $s + k_1 = 0$ lie to the left of $s = p_{cl}$. In the second step, let the virtual open-loop system be $k_2\frac{s+k_1}{s^2}$ such that the virtual closed-loop characteristic equation is $s^2 + k_2 s + k_1 k_2 = 0$. To determine k_2, draw the root locus of $k_2\frac{s+k_1}{s^2}$. Then, from the observation above, there exists a k_2 large enough such that: (a) the first inequality of (30) is satisfied and, (b) all the roots of $s^2 + k_2 s + k_1 k_2 = 0$ lie to the left of $s = p_{cl}$, on the real axis. Continuing in this fashion, for the last we let the virtual open-loop system be $k_n\frac{s^{n-1} + k_{n-1}s^{n-2} + \cdots + \prod_{j=1}^{n-1} k_j}{s^n}$, such that the virtual closed-loop characteristic equation is exactly the same as that of the actual system, i.e., $k_n s^{n-1} + k_n k_{n-1} s^{n-2} + \cdots + \prod_{j=1}^{n} k_j$. Then, from the observation mentioned above, we can always find a large enough k_n such that (a) the last two inequalities of (30) are satisfied and (b) all the closed-loop poles lie to the left of $s = p_{cl}$ on the real axis. Thus, we can choose the controller gains such that the desired closed-loop performance is achieved, as well as the conditions imposed for the existence of a feasible control law given by (30) are also satisfied simultaneously.

IV. Simulation Example

A 3^{rd} order chain of integrators is used to demonstrate the effectiveness of the proposed scheme. In figures (2) and (3), the following correspondence is noted between the various control laws and the legend - *Marchand and Hably* refers to Thm.3 of [11]; *Zhou and Duan 07* refers to control law 3 of [15]; *Zhou and Duan 08* refers to Thm. 2 of [16]; *Zhou and Duan 09* refers to Thm. 13 of [17]; and *Kaliora and Astolfi* refers to proposition 4 of [8]. For all except last control law, please refer to the corresponding article for parameter values. For Kaliora and Astolfi, we use $\epsilon_1 = 0.24, \epsilon_2 = 0.25, \epsilon_3 = 0.51$ and $\lambda_1 = .008, \lambda_2 = 0.12, \lambda_3 = 0.6$. Note that we cover a wide variety of designs for comparative studies - two designs which use state-dependent saturation functions ([11], [17]) to improve performance, one approach which uses $[(n+1)/2]$ saturation functions to improve convergence ([15]) and, two designs which do not require coordinate transformation and need only assumption A1 ([8], [16]).

Simulation 1: The goal of this simulation study is to investigate the convergence rate of the proposed scheme against a transformation based approach for large initial conditions. The third order chain of integrators $\dot{x}_1 = x_2, \dot{x}_2 = x_3, \dot{x}_3 = u$ has been used in most of the articles used for comparative studies and thus, provides a level platform to compare the various approaches. Same initial conditions $[-3, -3, 3]$ as used in [16], the other non-transformation based approach, has been used for ease of comparison. The controller parameters for the proposed scheme are: $[k_1, k_2, k_3] = [0.2, 0.9, 100]$, $[l_1, l_2, l_3] = [6.5, 0.7, 0.01]$, $[\beta_1, \beta_2, \beta_3] = [0.99, 0.99, 1]$, $[\gamma_1, \gamma_2, \gamma_3] = [0.001, 0.001, 0]$. The parameter selection scheme proposed in the previous section was used as a starting point, and then it was tuned to improve the performance. As seen from Fig. (2), we achieve slightly faster convergence with the proposed scheme. This shows that the performance of the proposed technique in terms of achievable convergence rate is at least as good as that of a transformation based approach for large initial conditions. However, the true strength of the proposed controller and its performance robustness against input disturbances is demonstrated in the next set of simulations when large disturbances are considered.

Simulation 2: TThe purpose of this simulation study was to compare the convergence rate and steady-state error for a third order integrator chain in presence of input disturbances. The input disturbance and initial conditions were chosen to be $d(t) = 0.1\sin(\frac{\pi t}{2})$ and $x(0) = [0.8, -0.6, 0.5]$ respectively. The initial conditions were chosen to be smaller than the previous case in order to highlight the effect of disturbance on steady-state error. Controller parameters are same as used in the previous simulation. The disturbance attenuation capability of the proposed scheme over a coordinate transformation based approach is evident from Fig. (3). One of the main observations from the simulation studies was that as compared to other techniques, higher gains could be chosen in the proposed technique to improve disturbance rejection without significantly sacrificing convergence properties. On the other hand, increasing controller gains in other designs result in improved disturbance rejection, but at the expense of drastic degradation in convergence performance. Thus, the proposed scheme has better convergence and disturbance rejection properties as compared to other designs.

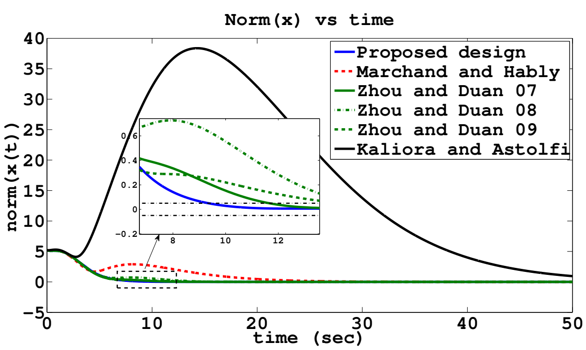

Fig. 2. Comparative results for stabilization in absence of disturbances

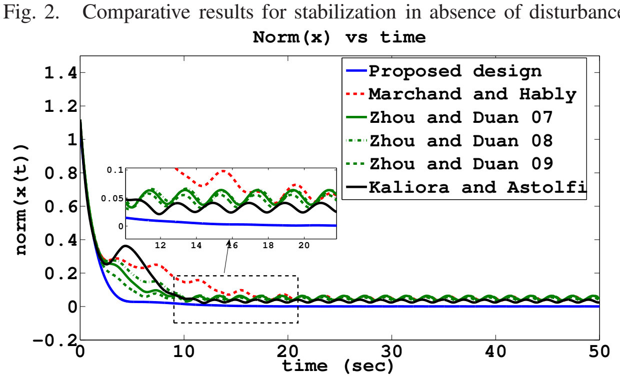

Fig. 3. Comparative results for stabilization in presence of disturbance

V. CONCLUSIONS

The main contribution of the paper lies in proposing a conceptually different approach to solve the problem of global stabilization of a chain of integrators in presence of input disturbance with desired performance in the unsaturated region. Based on Teel's work, many modifications have been proposed in the literature to improve the performance of the controller. In our analysis, it was clearly shown that all such schemes exhibit poor robustness properties with respect to input disturbance and leads to conservative design. These limitations cannot be overcome by any modification based on Teel's work, as coordinate transformation is an essential step in all such designs. In order to remove these limitations, we take a fundamentally different viewpoint and propose a scheme which does not rely on coordinate transformation, and is directly based on the actual tracking error dynamics. The resulting controller is easy to implement and tune, as we only deal with the original coordinates. Comparative studies have been performed on a third order chain of integrator to show the superior performance of the proposed technique over other designs.

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