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Consistent Identification of Spatially Interconnected Systems

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Abstract— **This paper presents an identification technique to consistently identify multi-dimensional systems with additive output colored noise in transfer function representation. The method is an extension of a one-dimensional refined instrumental variable method to multi-dimensional systems. It can be used to identify models for multi-dimensional systems with Box-Jenkins structure; the method may give estimates with minimum bias and variance. In this paper we consider a general multi-dimensional system, which may be separable or non-separable, causal, semi-causal (spatially interconnected systems) or non-causal. Furthermore the algorithm can handle boundary conditions. The effectiveness of the method is shown with application to simulation examples.**

I. INTRODUCTION

The field of distributed interconnected systems has attracted the attention of many researchers for several decades. Such systems can be seen as a subclass of multidimensional systems and consist of similar subsystems interacting with their closest neighbours. Examples of such systems include vehicle platoons, satellites and unmanned aerial vehicles flying in formations, automated highway systems, spatially distributed flexible structures and fluid flow among others, as well as systems whose dynamics are governed by partial differential equations. A highly active research area, the control of such systems inevitably demands accurate models. An apparent choice is to use physical models, but such an approach would compromise the accuracy due to the presence of unmodelled dynamics.

Fewer results are available in the field of system identification of two-dimensional (2-D) or multi-dimensional (m-D) systems. Identification of transfer function models of 2-D causal systems is presented in [1]; the method is based on the 2-D Hankel theory. Methods to identify 2-D and m-D non-causal rational transfer functions are given in [2], [3] but are less practical due to their dependence on the impulse response of the system. Identification of 2-D state-space models for separable-in-denominator systems based on the impulse response of the system is discussed in [4], while identification from input-output data is presented in [5]. The latter has the advantage that it gives a state-space model in balanced form.

A state-space based identification method for spatially distributed interconnected systems is proposed in [6], which can be seen as decentralized subspace identification of spatially interconnected systems. In this method each subsystem is identified as multiple-input single-output (MISO) system. A similar method for spatially distributed systems was also proposed in [7] .

A method based on least squares estimation is proposed in [8] to identify transfer function models for m-D systems. It has been extended in [9] where measurement noise is also considered.

The work proposed here is an extension of [9] to systems with Box-Jenkins model structure. We apply the Refined Instrumental Variable method [10], [11] to identify m-D systems with noise model having non-common polynomials with the process model. Unlike the simple IV method which is sub-optimal, this identification scheme can yield estimates with minimum bias and variance. Furthermore, the method can be applied to both separable and non-separable systems. It is also equally applicable to causal, semi-causal (spatially interconnected) and non-causal systems with the additional benefit that it can handle arbitrary boundary conditions.

The paper is organized as follows. Section II describes preliminaries to the problem under study. Section III presents the main idea of the paper: the application of the Refined IV algorithm to identify Box-Jenkins models for linear invariant spatially interconnected systems. Section IV discusses examples, which include applications of the approach to the identification of semi-causal systems. Conclusions are drawn in section V.

II. PRELIMINARIES

For simplicity of notation we consider 2-D systems in this paper. The results are equally valid for m-D systems.

Let $u(n_1,n_2)$ be a two-dimensional discrete input signal applied to a linear invariant spatially interconnected 2-D discrete single-input single-output (SISO) system $G(q_1,q_2)$. Then its output $y(n_1,n_2)$ is given as

$$y(n_1,n_2) = G(q_1,q_2)u(n_1,n_2) + v_1(n_1,n_2) \tag{1}$$

where q_1, q_2 are the forward shift operators, n_1, n_2 are the independent variables, and

$$G(q_1,q_2) = \frac{B(q_1,q_2)}{A(q_1,q_2)} \tag{2}$$

where

$$\begin{aligned} B(q_1,q_2) &= \sum_{(i_1,i_2)\in M^u} b_{i_1,i_2} q_1^{-i_1} q_2^{-i_2} \\ A(q_1,q_2) &= 1 + \sum_{(i_1,i_2)\in M_0^y} a_{i_1,i_2} q_1^{-i_1} q_2^{-i_2} \end{aligned} \tag{3}$$

and $v(n_1,n_2)$ is a noise term. If equation (1) defines a system having AutoRegressive with eXogeneous input (ARX)

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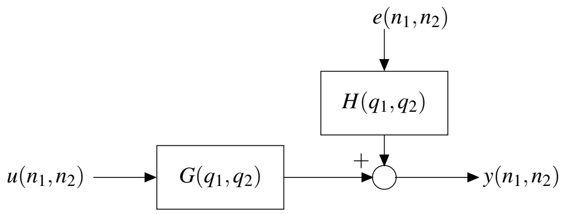

Fig. 1. Two-dimensional data-generating system

structure i.e if $v_1(n_1,n_2) = \frac{1}{A(q_1,q_2)}$, then it can be written in difference equation form as

$$\begin{aligned} y(n_1,n_2) = &- \sum_{(i_1,i_2)\in M_0^y} a_{i_1,i_2} y(n_1-i_1,n_2-i_2) \\ &+ \sum_{(i_1,i_2)\in M^u} b_{i_1,i_2} u(n_1-i_1,n_2-i_2) + e(n_1,n_2) \end{aligned} \tag{4}$$

where $e(n_1,n_2)$ is 2-D zero-mean white-noise with normal distribution, M^y and M^u denote the support regions (mask) for output and input terms, respectively, where M_0^y stands for $M^y \backslash (0,0)$. The support region is defined as a subset of the two-dimensional space where the indices of the coefficients of output and input terms in the difference equation lie. For further details see [8].

A. Identification of two-dimensional systems with noisy data

A least squares approach was presented in [8] for the identification of systems as in (4). The method is based on a least squares estimate and does not give consistent estimates if there is additive colored output noise in the data. Now consider a data-generating SISO 2-D system with output noise as in Fig. 1. Its output is given as

$$y(n_1,n_2) \;\; = \;\; G(q_1,q_2)u(n_1,n_2) \;\; + \;\; H(q_1,q_2)e(n_1,n_2)$$

where $H(q_1,q_2)$ is a 2-D linear filter.
Thus we have

$$\begin{aligned} A(q_1,q_2)y(n_1,n_2) = &B(q_1,q_2)u(n_1,n_2) \\ &+ A(q_1,q_2)H(q_1,q_2)e(n_1,n_2) \end{aligned} \tag{5}$$

With $v(n_1,n_2) = A(q_1,q_2)H(q_1,q_2)e(n_1,n_2)$, equation (5) can be written in difference equation form as

$$\begin{aligned} y(n_1,n_2) = &- \sum_{(i_1,i_2)\in M_0^y} a_{i_1,i_2} y(n_1-i_1,n_2-i_2) \\ &+ \sum_{(i_1,i_2)\in M^u} b_{i_1,i_2} u(n_1-i_1,n_2-i_2) + v(n_1,n_2) \end{aligned} \tag{6}$$

where $u(n_1,n_2)$, $y(n_1,n_2)$ and $v(n_1,n_2)$ denote the input, output and colored noise signal, respectively, and (6) can be written in linear regression form as

$$y(n_1,n_2) = \varphi^\top(n_1,n_2)\theta_0 + v(n_1,n_2). \tag{7}$$

The following notation is used for the filtered data,

$$\varphi_f(n_1,n_2) = L(q_1,q_2)\varphi(n_1,n_2) \tag{8}$$

where $L(q_1,q_2)$ is a 2-D stable filter.

The predicted model output $\hat{y}(n_1,n_2)$ and prediction error $\varepsilon(n_1,n_2)$ are given as

$$\hat{y}(n_1,n_2) = \varphi^\top(n_1,n_2)\theta \tag{9}$$

and

$$\varepsilon(n_1,n_2,\theta) = y(n_1,n_2) - \hat{y}(n_1,n_2) = y(n_1,n_2) - \varphi^\top(n_1,n_2)\theta \tag{10}$$

Let $Z \in R^{N_1 \times N_2}$ represent the input-output data set for the system, then the problem is to identify the model by minimizing the quadratic objective function

$$V(Z,\theta) = \frac{1}{2N_1N_2} \sum_{n_2=1}^{N_2} \sum_{n_1=1}^{N_1} \varepsilon^2(n_1,n_2,\theta) \tag{11}$$

In order to obtain consistent parameter estimates, the instrumental variable (IV) technique can be used [12]. The IV parameter estimates for 2-D systems are given by [9]

$$\begin{aligned} \theta^{IV} = &\left[\tfrac{1}{N_1N_2} \textstyle\sum_{n_2=1}^{N_2} \sum_{n_1=1}^{N_1} \zeta(n_1,n_2)\varphi^\top(n_1,n_2) \right]^{-1} \\ &\left[\tfrac{1}{N_1N_2} \textstyle\sum_{n_2=1}^{N_2} \sum_{n_1=1}^{N_1} \zeta(n_1,n_2)y(n_1,n_2) \right] \end{aligned} \tag{12}$$

where $\zeta(n_1,n_2)$ is the instrumental variable vector, which must be correlated with the regressors $\varphi(n_1,n_2)$ but uncorrelated with the noise $v(n_1,n_2)$ in order to have a consistent estimate. The instrumental variable $\zeta(n_1,n_2)$ here is assumed to satisfy the following two conditions:

$$\sum_{n_2=1}^{N_2} \sum_{n_1=1}^{N_1} \zeta(n_1,n_2)\varphi^\top(n_1,n_2) \text{ non-singular} \tag{13}$$

and

$$\sum_{n_2=1}^{N_2} \sum_{n_1=1}^{N_1} \zeta(n_1,n_2)v(n_1,n_2) = 0 \tag{14}$$

III. Refined Instrumental Variable for Identification of Box-Jenkins Models

The IV method referred to in the previous section gives consistent estimates when applied to a multidimensional system [9]. However this method is sub-optimal and it is required to run the algorithm several times (Monte Carlo simulations) in order to get a reasonable estimate. To deal with the problem of non-optimality (in terms of minimum variance), adaptive IV and multi-step approaches are suggested in [12]. It is a known fact that obtaining an optimal IV estimator is dependent upon finding the true noise model $H(q_1,q_2)$, see for example [10] for the closed loop case.

A. Model Considered

In this paper we are interested in identifying models having Box-Jenkins (BJ) structure for linear invariant spatially interconnected systems. Since it does not constrain the process and noise models to have common polynomials, the BJ model structure is a natural choice for many practical purposes. A BJ model for the 2-D system (5) can be given in the form

$$\begin{aligned} y(n_1,n_2) &= \frac{B(q_1,q_2,\theta)}{A(q_1,q_2,\theta)}u(n_1,n_2)+v(n_1,n_1) \\ v(n_1,n_2) &= H(q_1,q_2,\eta)e(n_1,n_2) \end{aligned} \tag{15}$$

Where

$$H(q_1,q_2,\eta)=\frac{C(q_1,q_2)}{D(q_1,q_2)} \tag{16}$$

is a 2-D filter where

$$\begin{aligned} C(q_1,q_2) &= 1+\sum_{(i_1,i_2)\in M_0^c} c_{i_1,i_2}q_1^{-i_1}q_2^{-i_2} \\ D(q_1,q_2) &= 1+\sum_{(i_1,i_2)\in M_0^d} d_{i_1,i_2}q_1^{-i_1}q_2^{-i_2} \end{aligned} \tag{17}$$

with $C(q_1,q_2)$ and $D(q_1,q_2)$ being stable polynomials where M_0^c and M_0^d stand for $M^c\backslash(0,0)$ and $M^d\backslash(0,0)$ respectively. M^c and M^d here correspond to the support region for numerator and denominator of the 2-D filter respectively. The associated parameters of $H(q_1,q_2)$ i.e. $c_{i_1,i_2},\forall i_1,i_2\in M_0^c$ and $d_{i_1,i_2},\forall i_1,i_2\in M_0^d$ are stacked columnwise in the parameter vector $\eta\in\mathbb{R}^{n_\eta}$.

Also let $\mathscr{H}=\{\,\mathsf{H}_\eta\mid\eta\in\mathbb{R}^{n_\eta}\,\}$ denote the collection of all noise models in the following form

$$\mathsf{H}_\eta:(H(q_1,q_2,\eta)) \tag{18}$$

$A(q_1,q_2)$ and $B(q_1,q_2)$ are polynomials in q_1 and in q_2 as in (3), whose coefficients $a_{i_1,i_2},\forall i_1,i_2\in M_0^y$ and $b_{i_1,i_2},\forall i_1,i_2\in M^u$ are stacked columnwise to form the parameter vector θ of the process model. Now let the process model be $\mathscr{G}_\theta$, and $\mathscr{G}=\{\mathscr{G}_\theta|\theta\in\mathbb{R}^{n_\theta}\}$ be the collection of all process models. The parameters corresponding to a given process and noise model $(\mathscr{G}_\theta,\mathscr{H}_\eta)$ are represented as

$$\rho=[\,\theta^T\ \eta^T\,]^T \tag{19}$$

Based on the model structure in (15) if $\mathscr{M}_\rho$ denote the model set, parametrized independently with process $(\mathscr{G}_\theta)$ and noise $(\mathscr{H}_\eta)$ model, can be represented as

$$\mathscr{M}_\rho=\{(\mathscr{G}_\theta,\mathscr{H}_\eta)\mid col(\theta,\eta)=\rho\in\mathbb{R}^{n_\theta+n_\eta}\} \tag{20}$$

This set corresponds to the set of candidate models in which we seek our model corresponding to the data generating system in (5). The model structure in (15) is nonlinear in parameters and as a consequence the simple IV estimation can not be applied directly. $y(n_1,n_2)$ can be written in the linear regression form as:

$$y(n_1,n_2)=\varphi^\top(n_1,n_2)\theta+\tilde{v}(n_1,n_2) \tag{21}$$

where

$$\tilde{v}(n_1,n_2)=A(q_1,q_2)v(n_1,n_2) \tag{22}$$

The prediction error of (21) w.r.t (15) is given by

$$\varepsilon(n_1,n_2)=\frac{D(q_1,q_2)}{C(q_1,q_2)}[y(n_1,n_2)-\frac{B(q_1,q_2)}{A(q_1,q_2)}u(n_1,n_2)] \tag{23}$$

Since the error is nonlinear in the parameters of unknown polynomials, an alternative expression is,

$$\varepsilon(n_1,n_2)=\frac{D(q_1,q_2)}{C(q_1,q_2)A(q_1,q_2)} [A(q_1,q_2)y(n_1,n_2)-B(q_1,q_2)u(n_1,n_2)]$$

or

$$\varepsilon(n_1,n_2)=A(q_1,q_2)y_f(n_1,n_2)-B(q_1,q_2)u_f(n_1,n_2) \tag{24}$$

where $y_f(n_1,n_2)=Q(q_1,q_2)y(n_1,n_2)$ and $u_f(n_1,n_2)=Q(q_1,q_2)u(n_1,n_2)$ with

$$Q(q_1,q_2)=\frac{D(q_1,q_2)}{C(q_1,q_2)A(q_1,q_2)} \tag{25}$$

Therefore, (21) is equivalent to

$$y_f(n_1,n_2)=\varphi_f^\top(n_1,n_2)\theta+\tilde{v}_f(n_1,n_2) \tag{26}$$

where $\varphi_f(n_1,n_2)$ is the filtered regressor constructed from $y_f(n_1,n_2)$ and $u_f(n_1,n_2)$.

$$\tilde{v}_f(n_1,n_2)=A(q_1,q_2,\theta)v_f(n_1,n_2)=e(n_1,n_2) \tag{27}$$

This means that if the optimal filter (25) is known *a priori*, it is possible to filter the data such that a simple LS algorithm applied to the data pre-filtered with (25) leads to the statistically optimal estimate. But as is the case in practice, filter coefficients are not known *a priori*. One solution lies in the so called iterative IV methods. The idea of using such algorithms can be traced back at least to [13]. A refined version of that approach has been used for closed-loop identification in [10] and recently for open-loop identification of linear parameter-varying temporal systems for Box-Jenkin models in [11] referred to as Refined Instrumental Variable (RIV) method.

In the same spirit, our approach here is to use iterative RIV method for identifying BJ models for multidimensional systems.

B. The Iterative RIV Algorithm

Step 1 *ARX model estimation*

Select the input and output masks for the model of the system as

$M^u=\{(i_1,i_2)\mid\ 0\le i_1\le k_{u_1},\ -k_{u_2}\le i_2\le k_{u_2}\}$

$M^y=\{(i_1,i_2)\mid\ 0\le i_1\le k_{y_1},\ -k_{y_2}\le i_2\le k_{y_2}\}$.

Compute an initial ARX estimate using the LS approach. This gives $\hat{B}^{(0)}(q_1,q_2)$ and $\hat{A}^{(0)}(q_1,q_2)$. Set $\hat{D}^{(0)}(q_1,q_2)=1,\hat{C}^{(0)}(q_1,q_2)=1$ and $i=0$

Step 2 *Generate data*
Compute an estimate of the noise free output $\hat{x}(n_1,n_2)$ by simulating the auxiliary model

$$\hat{x}(n_1,n_2) = \frac{\hat{B}^{(i)}(q_1,q_2)}{\hat{A}^{(i)}(q_1,q_2)}u(n_1,n_2)$$

based on the estimated parameters $\hat{\theta}^{(i)}$ of the previous iteration.
Step 3 *Estimated filter*
Compute

$$\hat{Q}(q_1,q_2) = \frac{\hat{D}^{(i)}(q_1,q_2)}{\hat{C}^{(i)}(q_1,q_2)\hat{A}^{(i)}(q_1,q_2)}$$

and obtain $u_f(n_1,n_2)$,$y_f(n_1,n_2)$ and $\hat{x}_f(n_1,n_2)$ by filtering $u(n_1,n_2)$,$y(n_1,n_2)$ and $\hat{x}(n_1,n_2)$ respectively with the obtained filter.
Step 4 *Estimated regressor*
Form the filtered estimated regressor as

$$\begin{aligned}\hat{\varphi}_f(n_1,n_2) = [&-\hat{y}_f(n_1,n_2+k_{y_2}),\ldots,\\ &-\hat{y}_f(n_1,n_2-k_{y_2}),\ldots\ldots\ldots,\\ &-\hat{y}_f(n_1-k_{y_1},n_2+k_{y_2}),\ldots,-\hat{y}_f(n_1-k_{y_1},n_2-k_{y_2}),\\ &\hat{u}_f(n_1,n_2+k_{u_2}),\ldots,\hat{u}_f(n_1,n_2-k_{u_2}),\\ &\ldots\ldots\ldots,\hat{u}_f(n_1-k_{u_1},n_2+k_{u_2}),\ldots,\hat{u}_f(n_1-k_{u_1},n_2-k_{u_2})]^\top\end{aligned}$$

with the filtered instrument

$$\begin{aligned}\hat{\zeta}_f(n_1,n_2) = [&-\hat{x}_f(n_1,n_2+k_{y_2}),\ldots,\\ &-\hat{x}_f(n_1,n_2-k_{y_2}),\ldots\ldots\ldots,\\ &-\hat{x}_f(n_1-k_{y_1},n_2+k_{y_2}),\ldots,-\hat{x}_f(n_1-k_{y_1},n_2-k_{y_2}),\\ &\hat{u}_f(n_1,n_2+k_{u_2}),\ldots,\hat{u}_f(n_1,n_2-k_{u_2}),\\ &\ldots\ldots\ldots,\hat{u}_f(n_1-k_{u_1},n_2+k_{u_2}),\ldots,\hat{u}_f(n_1-k_{u_1},n_2-k_{u_2})]^\top\end{aligned}$$

Step 5 *Compute the IV estimate*
Compute the IV estimate as

$$\begin{aligned}\hat{\theta}^{i+1}(N) = [&\frac{1}{N_1N_2}\sum_{n_2=1}^{N_2}\sum_{n_1=1}^{N_1}\hat{\zeta}_f(n_1,n_2)\hat{\varphi}_f^\top(n_1,n_2)]^{-1}\\ &[\frac{1}{N_1N_2}\sum_{n_2=1}^{N_2}\sum_{k_1=1}^{N_1}\hat{\zeta}_f(n_1,n_2)y_f(n_1,n_2)] \quad (28)\end{aligned}$$

where $\hat{\theta}^{i+1}$is the IV esimate of the process model parameter vector at iteration $i+1$ based on prefiltered data.
Step 6 *Noise model estimate*
Estimate the noise signal as

$$\hat{v}(n_1,n_2) = y(n_1,n_2) - \hat{x}(n_1,n_2) \quad (29)$$

Based on this, the noise model parameter vector $\hat{\eta}^{i+1}$ is estimated using the ARMA estimation algoritm of the MATLAB identification toolbox (if we take $H(q_1,q_2) = 1$ at this step and avoid the noise model estimation, the method is referred to as simplified RIV (SRIV)).

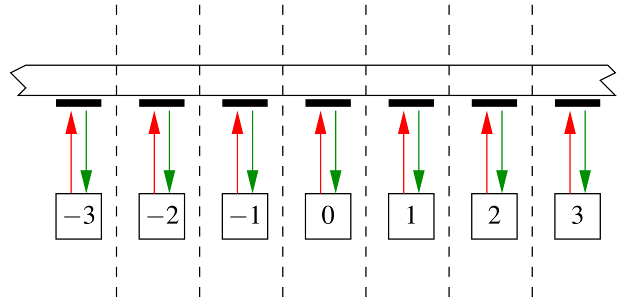

Fig. 2. Part of the rod; green shows sensor and red shows linear actuator signals from each segment

TABLE II
ESTIMATOR BIAS AND VARIANCE NORM AT DIFFERENT SNR

Method		15dB	10dB	5dB
LS	Bias Norm	0.0623	0.1274	0.1694
	Variance Norm	3.4×10^{-6}	5.4×10^{-6}	9.3×10^{-6}
SRIV	Bias Norm	0.0002	0.0010	0.0026
	Variance Norm	3.9×10^{-5}	1.1×10^{-4}	3.7×10^{-4}
RIV	Bias Norm	2.8×10^{-5}	5.0×10^{-5}	0.00012
	Variance Norm	5.7×10^{-8}	2.1×10^{-7}	3.4×10^{-7}

Step 7 *Stopping criterion*
If convergence has occured or the maximum number of iterations reached then stop, else set $i = i+1$ and go to Step 2.

Remarks: The above algorithm gives optimal estimates if the noise filter is known and if the algorithm converges. As the filter is unknown in practice, optimality can not be guaranteed. However, our experience show that the algorithm conveges most of the time.

IV. Illustrative Examples

In this section the performance of the approach is illustrated by two simulation examples.

A. Example 1: Heat transfer through a rod

As a practical application of the approach discussed in the previous section, a distributed system is considered: a heat conduction in a rod of length one meter as in [14]. It is an example of a 2-D semi-causal system. Fig. 2 shows a schematic diagram of a rod with an array of temperature sensors and linear actuators. The system is described by the equation

$$\frac{\partial y(t,x)}{\partial t} = \kappa\frac{\partial^2 y(t,x)}{\partial x^2} + u(t,x) \quad (30)$$

where $y(t,x)$ denotes temperature (°C), t and x denote time (seconds s) and spatial (meter m) co-ordinates, respectively, κ is a constant (m^2s^{-1}) and u is the linear heat source.
Equation (30) is discretized using the central difference method to approximate partial derivatives as

$$\left(\frac{\partial y(t,x)}{\partial t}\right)_{n_1,n_2} = \frac{y(n_1+1,n_2) - y(n_1,n_2)}{T} \quad (31)$$

TABLE I

MEAN AND STANDARD DEVIATION OF THE ESTIMATED PARAMETERS AT SNR = 5DB

		$a_{1,1}$	$a_{1,0}$	$a_{1,-1}$	$b_{1,0}$	d_1	d_2	d_3
Method	True Value	-0.1	-0.8	-0.1	1	-2.42	2.1	-0.663
LS(MCS)	mean	-0.0255	-0.9327	-0.0255	1.0002	x	x	x
	std	0.0015	0.0014	0.0014	0.0030	x	x	x
SRIV(MCS)	mean	-0.0997	-0.8008	-0.0995	0.9976	x	x	x
	std	0.0037	0.0063	0.0035	0.0192	x	x	x
RIV(MCS)	mean	-0.0999	-0.8000	-0.1000	1.0000	-2.4206	2.1013	-0.6637
	std	0.0004	0.0004	0.0004	0.0004	0.0084	0.0166	0.0089

$$\left(\frac{\partial^2 y(t,x)}{\partial x^2}\right)_{n_1,n_2} = \frac{y(n_1,n_2-1)-2y(n_1,n_2)+y(n_1,n_2+1)}{h^2} \tag{32}$$

where T is the sampling time, h is the spatial sampling distance between two nodes along the rod and n_1 and n_2 are the time and spatial indices respectively. Then at instance (n_1,n_2) equation (30) can be approximated as

$$\begin{aligned} y(n_1+1,n_2) = &\frac{T}{h^2}y(n_1,n_2-1)+(1-2\frac{T}{h^2})y(n_1,n_2) \\ &+\frac{T}{h^2}y(n_1,n_2+1)+Tu(n_1,n_2) \end{aligned} \tag{33}$$

Here we are assuming that $\kappa = 1$ and let $\hat{u}_{n_1,n_2} = Tu_{n_1,n_2}$. The difference equation (33) can be represented in transfer function form as

$$G(q_1,q_2) = \frac{b_{1,0}q_1^{-1}}{1+a_{1,1}q_1^{-1}q_2^{-1}+a_{1,0}q_1^{-1}+a_{1,-1}q_1^{-1}q_2} \tag{34}$$

where q_1 is the temporal forward shift operator, q_2 is the spatial shift operator, and

$$b_{1,0} = 1, \;\; a_{1,0} = -1+2\frac{T}{h^2}$$

$$a_{1,1} = -\frac{T}{h^2}, \;\; a_{1,-1} = -\frac{T}{h^2}$$

Let the rod be divided spatially into 10 segments, i.e $h = 1/10$, and let $T = 0.001s$. Noisy data is generated from this system by considering a structure as in Fig. 1 with $H(q_1,q_2) = \frac{1}{1+d_1q_1^{-1}+d_2q_1^{-2}+d_3q_1^{-3}}$. The input $u(n_1,n_2)$ and $e(n_1,n_2)$ are taken as 2-D zero-mean normally distributed white-noise. The data used for identification is of size 10000×10. Monte-Carlo simulations of 100 runs are carried out at different signal-to-noise ratios SNR. Table I and II show a comparison of the estimated parameters, bias norm ($\|\theta_0 - E[\hat{\theta}]\|_2$) and variance norm ($\|E[(\hat{\theta}-E[\hat{\theta}])^2]\|_2$), where θ_0 is the true and $\hat{\theta}$ the estimated parameter vector, respectively.

B. Example 2: Cantilever beam

In this section we consider the beam model shown in Fig. 3. The output $y(t,x)$ is a transverse deflection and the input is a force $u(t,x)$ applied at point x. The beam

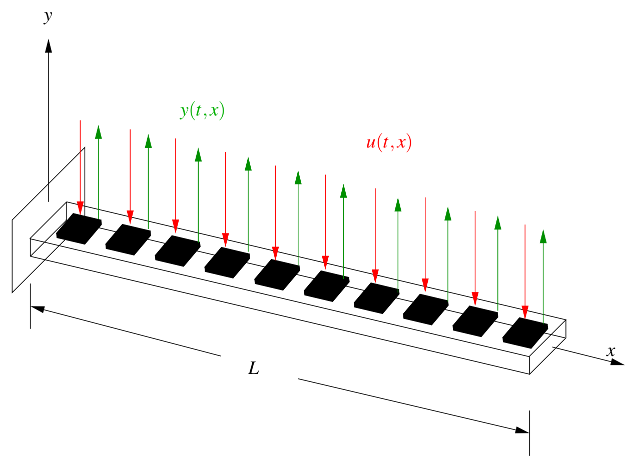

Fig. 3. Uniform Beam divided into N_2 segments; green shows sensor and red shows actuator signals from each segment

has Young's modulus E, I is the moment of inertia, ρ the density and L the length. This model is an example of a semi-causal system, being causal in time and non-causal in space. The model is discretized spatially into N_2 subsystems with a spatial sampling interval of $h = L/N_2$. In discrete domain, the transverse deflection of the beam is represented by $y(n_1,n_2)$ and the distributed force on the beam is represented as $u(n_1,n_2)$, where n_1 and n_2 are indices for temporal and spatial dimensions, respectively. The system has N_2 subsystems and each has a sensor and an actuator. If we take N_1 temporal data at each spatial segment, then we have $N_1 \times N_2$ data points for output $y(n_1,n_2)$ and input $u(n_1,n_2)$.

For data generation we simulate a beam model given in Fig. 3 with the following properties. Length $L = 1m$, density $\rho = 7800Kg/m^3$, Young's modulus $E = 20\times 10^{10}N/m^2$, maximum thickness $t_b = 5mm$, cross-sectional area $A = 1\times 10^{-4}m^2$, moment of inertia $I = 2.08\times 10^{-10}m^4$ and viscous damping $\gamma = 1kg/ms$. The given beam is divided into nine subsystems with sensors and actuators. The model is excited with 2-D zero-mean white-noise with normal distribution, the sampling time is taken as $1\times 10^{-4}\,sec$. The input and output data has size 10000×9 and is generated from the data generating system as in Fig. 1 with $H(q_1,q_2) = \frac{1}{1+d_1q_1^{-1}+d_2q_1^{-2}}$ where $d_1 = -1$ and $d_2 = 0.2$. The model structure selected is as in (15) with input and output masks given as

$$M^u = \{(1,0),(2,0)\}$$
$$M^y = \{(2,-2),(1,-2),(2,-1),(1,-1),(0,0),(2,0),\\ (1,0),(2,1),(1,1),(2,2),(1,2)\}$$

The parameters of this model are very sensitive to noise (as some parameters are very small), so it gives reasonable results above 40 dB SNR. Monte-Carlo simulations of 100 runs are carried out at SNR 40 dB. The least squares method gives parameters having bias norm 0.01 and variance norm 2.2×10^{-6} while the proposed RIV method identifies the parameters for the above model having bias norm 4.4×10^{-6} and variance norm 1×10^{-9}. The identified noise filter of the RIV method has mean values $d_1 = -1.0155$ and $d_2 = 0.187$.

V. CONCLUSIONS

In this paper a method has been presented to identify transfer function models for m-D systems with consistent estimates if there is additive colored noise in the output. The proposed method is an extension of the refined instrumental variable technique for one-dimensional systems to multi-dimensional systems. The method can give estimates with minimum bias and variance. As illustration the method is applied to two semi-causal (spatially interconnected) systems, but is equally applicable to causal and non-causal systems, and can be used for separable as well as non-separable systems. The method has also the advantage that boundary conditions can be easily included, as is discussed in [8].

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