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Coordinated Decentralized Estimation Over Random Networks

Marsh Nabi-Abdolyousefi and Mehran Mesbahi

***Abstract*— In this paper, we consider three representative problems on observability and estimation over networks in presence of randomness. The aim of this work is to highlight that these problems, among many others, can be approached via a unified formulation, that can be subsequently be utilized for proving almost sure stability and convergence of filtering algorithms over distinct classes of random networks. More specifically, we show stability and convergence properties of random variations on the coordinated decentralized estimation using this approach.**

***Index Terms*— Decentralized information filter; contractive maps; Random Riccati equation; random networks**

I. Introduction

Consider a network of N spatially distributed autonomous sensors, in which each sensor collects measurements in some modality of interest, e.g., temperature, sound, vibration, pressure, motion, or pollutant. Each sensor in the network is equipped with small storage, a radio transceiver, a micro-controller, and a battery power source on a single chip. Some engineering sensor networks consist of large numbers of sensors, from hundreds to even hundred thousand nodes [1]–[3]. Sending large amounts of raw measurements to the fusion center requires large bandwidth for transferring data, which in turn has adverse effect on power usage for sensor networks. Specially in some cases, for instance habitat monitoring, non-rechargeable batteries are employed to observe the evolution of a particular phenomenon in nature for a few consecutive months [4]. In some other applications, "reactive" sample rates for the sensing application is important. For example in soil moisture sensor networks [5], sensors observe a dynamics which changes rapidly during certain intervals and slowly during others. Moreover, considering a large wireless multi-hop sensor network, there is generally a large amount of data and insufficient bandwidth for transferring the data in its entirety between the sensors and a centralized fusion center. Moreover, in such networks, data travels along unreliable communication channels and the effect of communication delays and loss of information cannot be neglected.

The subject of decentralized estimation is studied in [10]–[13]. Estimation over stochastic systems has also been explored in [6], [8], [15], [21]–[23]

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The authors are with the Department of Aeronautics and Astronautics, University of Washington, Seattle, WA 98195-2400 USA (emails: Manzi+mesbahi@uw.edu)

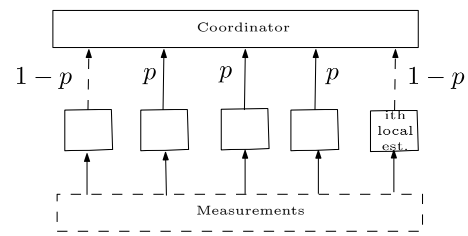

Fig. 1. The communication link between each sensor and the coordinator switches on with a probability $0 < p < 1$.

II. Problems considered

In this work, we examine coordinated decentralized estimator design over networks in three different scenarios involving randomness:

(a) A sensor network is observing a dynamic process. A coordinated decentralized filter is considered that uses the local computational capability of each sensor while also utilizing the presence of a coordinator. Each sensor calculates, based on the partial information available to it, an estimate of the state of the dynamic process. A partitioned information filter in a simple hierarchical estimation architecture, shown on Fig. 1, provides a decentralized estimation architecture for this problem [9]. Meanwhile, the proposed architecture implicitly requires that the sensors communicate with the coordinator at every time step. However, this assumption might be unrealistic within the operational constraints. For more economical energy usage and/or in presence of unreliable communication links and packet drop-outs, a random communication scheme between the sensors and the coordinator is thus assumed. Hence at every time step, the link between each sensor and the coordinator becomes active with probability $p \in (0, 1)$. A natural question is thereby whether this randomized coordinated decentralized estimation still has convergence properties that parallel the original deterministic decentralized estimation scheme.

(b) A group of dynamic agents are running a formation task based on the consensus protocol over a static network. The consensus algorithm is corrupted by Gaussian noise on the interaction/communication links between these agents. Estimating the states of agents in this setup is facilitated by interfacing with a group of nodes called "output ports" and observing their states over time; see Fig. 2. The observability condition requires us to find a set of nodes that led to an "observable" process.

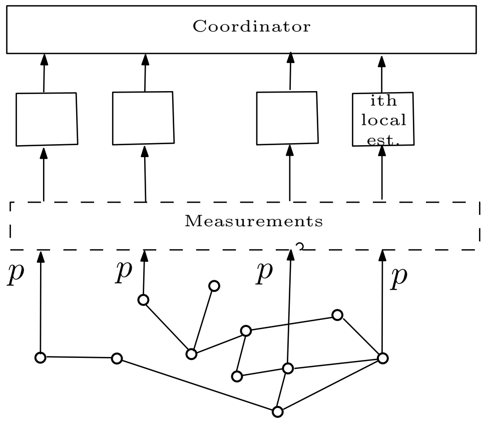

Fig. 2. The communication architecture; each link switches on with a probability $0 < p < 1$.

While finding the set of output nodes for guaranteeing an observable network is challenging, in Rahmani *et al.* [20] it is shown that, for example, the symmetric structure of network with respect to the observation ports has to be broken to avoid an unobservable dynamics. In our setup, we assume that at every time step, with probability $p \in (0, 1]$, each node is selected as the observation port; see Fig. 2. It is of interest to determine whether the subsequent coordinated decentralized estimation on such a process has the desired (albeit, probabilistic) stability and convergence properties.

(c) In this case, the static formation network described in the previous scenario also randomly changes at every time step, according to the Erdös-Renyi model, as is demonstrated in Fig. 3. Thus both the underlying dynamics and the measurement scheme evolve randomly over time. The main focus of this research is designing stable decentralized estimator in the stochastic setup.

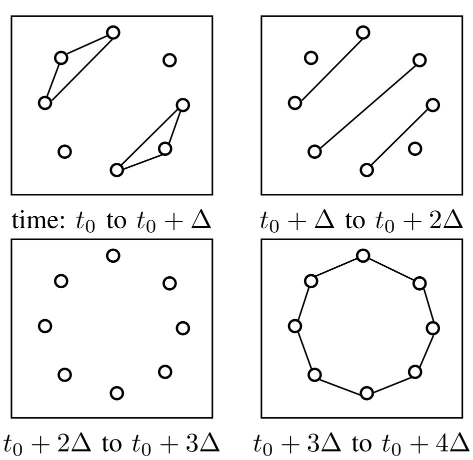

Fig. 3. Behavior of a random network in some time intervals

A. Deterministic Setup

Consider a sensor network that is employed to observe the evolution of the dynamical process described

$$x(k+1) = Ax(k) + w(k), \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector with initial condition $x(0)$ distributed as zero mean Gaussian with covariance Σ_0 and $w(k)$ is an uncorrelated zero mean Gaussian sequence with covariance $Q(k)$. The N sensors have a measurement map of the form $z_i(k) = H_i x(k) + v_i(k), \quad i = 1, 2, \ldots, N$, which when combined together, can be represented as

$$z(k) = H(k)x(k) + v(k), \tag{2}$$

$$H = \begin{bmatrix} H_1 \\ \vdots \\ H_N \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} v_1(k) \\ \vdots \\ v_N(k) \end{bmatrix}. \tag{3}$$

Under the hypothesis of stabilizability of the pair (A, Q) and detectability of the pair (A, H), the estimation error covariance, $\Sigma(k|k)$, using a Kalman filter converges to a unique value from any initial condition [15]. The Kalman filter updates the state according to

$$\hat{x}(k|k) = \hat{x}(k|k-1) + K(z(k) - H\hat{x}(k|k-1)), \tag{4}$$

with Kalman gain $K = \Sigma(k|k)H$, where $\Sigma(k|k)$ is the covariance matrix of the error vector $\hat{x}(k|k) - x(k)$. Thus

$$\begin{aligned} \Sigma(k|k) &= \mathbb{E}\{(x(k) - \hat{x}(k|k))(x(k) - \hat{x}(k|k))^T\} \\ &= (\Sigma(k|k-1)^{-1} + H^T H)^{-1}, \end{aligned} \tag{5}$$

where $\Sigma(k|k-1) = A\Sigma(k-1|k-1)A^T + Q$. The information filter, which proves to be advantages in distributed formulation of Kalman filtering, is an alternative representation of the filter in terms of the information-state vector and information matrix [15] defined via $y(k) = H^T z(k)$ and $Y = H^T H$. In this venue,

$$\hat{y}(k|k) = I(k|k)\hat{x}(k|k), \hat{y}(k|k-1) = I(k|k-1)\hat{x}(k|k-1),$$

where the information matrix $I(k|k) = \Sigma(k|k)^{-1}$ and $I(k|k-1) = \Sigma(k|k-1)^{-1}$. The linear Kalman filter may now be written in terms of the information state vector and the information matrix as

$$\begin{aligned} I(k|k) &= I(k|k-1) + Y(k) \\ \hat{y}(k|k) &= \hat{y}(k|k-1) + y(k) \end{aligned} \tag{6}$$

or in additive form,

$$\begin{aligned} I(k|k) &= I(k|k-1) + \sum_{i=1}^{N} Y_i(k) \\ \hat{y}(k|k) &= \hat{y}(k|k-1) + \sum_{i=1}^{N} y_i(k) \end{aligned} \tag{7}$$

where $Y_i = H_i^T H_i$ and $y_i(k) = H_i^T z_i(k)$. The interpretation of this additive form of the information filter is as follows: each sensor performs a local Kalman filter based on local sensor measurements; in this case we have $H_i^T H_i = I_i(k|k) - I_i(k|k-1)$. The information matrix can then be updated at each sensor node by receiving the difference $I_i(k|k) - I_i(k|k-1)$, summing them up across all sensors, and then adding them to obtain $I(k|k)$. Similarly, the information vector can be updated by summing up the received $y_i(k)$ from each sensor. The above scheme can then be considered in terms of the state and covariance update by including a coordinator. The global update assumes the form $\hat{x}(k|k) = (I - K(k)H)\hat{x}(k|k-1) + K(k)z(k)$, with $K(k)z(k) = \Sigma(k|k)H^T z(k) = \Sigma(k|k)\sum_i H_i^T z_i$, and $I - K(k)H = \Sigma(k|k)\Sigma(k|k-1)^{-1}$. In the meantime, $H_i^T z_i = I_i(k|k)\hat{x}_i(k|k) - I_i(k|k)(I - K_i H_i)\hat{x}_i(k|k-1)$ and therefore

$$\begin{aligned}\hat{x}(k|k) = &\Sigma(k|k)(I(k|k-1)\hat{x}(k|k-1) \\ &+\sum_i I_i(k|k)\hat{x}_i(k|k-1) - I_i(k|k-1)\hat{x}_i(k|k-1)).\end{aligned} \tag{8}$$

More details on this section are presented in [9], [16].

We note that the error covariance matrix update in (5) can be written as

$$\begin{aligned}\Sigma(k|k) \quad =&(A\Sigma(k-1|k-1)A^T + Q) \\ &(I + R(k)Q + R(k)A\Sigma(k-1|k-1)A^T)^{-1},\end{aligned} \tag{9}$$

where $R(k) = H^T(k)H(k)$.

III. Coordinated Decentralized estimation over random networks

Now consider the sensor network and the coordinator displayed in Figs. 1, 2, and 3. Note that in reference to the deterministic setup, in all these cases the measurement matrix H is a random matrix. That is when the i^{th} sensor sends its estimate to the coordinator, H_i becomes "active" in the i^{th} row, which happens with probability $0 < p < 1$ at every time step. Note that $\{H(k), k \geq 1\}$ is a strictly stationary ergodic process and that there is a σ-algebra $\mathfrak{F}$ generated by all measurements $z_1, z_2, \ldots, z_k$ for all $k \geq 1$.

This sensor network can be modeled by a star graph with $N+1$ nodes that has the coordinator at its center. Let us denote the coordinator as node with label 1. Hence, at every time step, sensor i communicates with the coordinator with probability p, i.e., the edge set $\mathcal{E}$ contains the edge $(i, 1)$ with probability p. The graph, $\mathcal{G}_k(N+1, \mathcal{E}, p)$, at the k^{th} time step can then be represented in terms of its $(N+1)\times(N+1)$ symmetric adjacency matrix $\mathcal{A}$ as

$$\mathcal{A}_{i1} = \begin{cases} 1 & \text{if } (i,1) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Define the set $\mathcal{M}$ as the resulting set of all possible adjacency matrices as well as the symmetric stochastic

$$\bar{\mathcal{A}} = \mathbb{E}[\mathcal{A}(k)] = \int_{\mathcal{M}} \mathcal{A}\, d\mathcal{D}(\mathcal{A}), \tag{11}$$

where $\mathcal{D}$ is the probability distribution on the space $\mathcal{M}$. It follows that $\bar{\mathcal{A}}$ for the considered random graph model is irreducible.

We will subsequently assume that the following conditions hold:

Hypothesis 3.1: . (1) The matrix A in the dynamic process (1) is invertible, and (2) $\log^+ ||A||$, $\log^+ ||A^{-1}||$, and $\log^+ ||Q||$ are integrable, where $\log^+ x = \max\{0, \log x\}$.

We now to explore conditions on the pair (A, H) in the scenarios (a), (b), and (c) discussed in §II that would allow a randomized estimation scheme with guaranteed (probabilistic) stability and convergence. A key technical construct that proves to be instrumental in this direction is that of weak detectability [7], [18], [19]. To get a general model for cases (a), (b), and (c), assume the matrix in (1) is time varying and is denoted by $A(k)$.

Definition 3.2: Let $R(k) = H^T(k)H(k)$. The sequence $\{(A(k), H(k)), k \in \mathbb{N}\}$ is said to be weakly observable if, for some $k \geq 1$, $\mathbb{P}\{\mathbf{det}[\Omega(k)] \neq 0\} \neq 0$, where

$$\begin{aligned}\Omega(k) \;&=\; R(1) + A^T(1)R(2)A(1) + \ldots \\ &+\; A^T(k-1)\ldots A^T(1)R(k)A(k-1)\ldots A(1) \\ &=\; \sum_{i=1}^{k} A^T(i-1)\ldots A^T(1)R(i)A(i-1)\ldots A(1).\end{aligned} \tag{12}$$

The weak observability condition holds if the matrices $A(k)$ and $\sum_{i=1}^{k} R(i)$ are invertible. In the same manner, the stabilizability assumption of the pair $(A(k), Q(k))$ can be explored.

The key observations is now that *weak detectability is a generic property for all three scenarios discussed in §II.* As such, the main results of [18], [19], in the context of estimation over random networks become directly applicable. We now briefly review relevant results that are instrumental for proving stability and convergence of the corresponding coordinated distributed estimators over random networks pertaining to these three scenarios.

Let us start by first describing the asymptotic behavior of the error covariance matrix $\Sigma(k|k)$ in (9). In this venue, let $\mathcal{P}$ (respectively, $\mathcal{P}_0$) denote the set of $N \times N$ nonnegative (respectively, positive) symmetric matrices. It can be shown that almost surely, for any solution $\Sigma(k|k)$ of (5), there is a constant covariance matrix $\bar{\Sigma}$ such that $||\Sigma(k|k) - \bar{\Sigma}||$ converges to zero as $k \to \infty$. The main result in the next section is that the error covariance matrices $\Sigma(k|k)$ are contractions on $\mathcal{P}_0$ with respect to the Riemannian metric to be defined shortly.

Define the symplectic group $Sp(N, \mathbb{R})$ as the set of all the matrices M of order $2N$ such that $M^T J M = J$ where $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. If we now write

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(N, \mathbb{R}), \tag{13}$$

where the entries are $N \times N$, then BA^T and $A^T C$ are symmetric and $A^T D - C^T B = I$. We associate to the system (1) and (2) the so-called Hamiltonian matrices $M(k)$ of order $2N$ written in block form as

$$M(k) = \begin{pmatrix} A(k) & Q(k)A(k)^{-T} \\ R(k)A(k) & (I + R(k)Q(k))A(k)^{-T}) \end{pmatrix}. \tag{14}$$

Therefore, the set of all Hamiltonian matrices can be represented as

$$\begin{aligned}\mathfrak{H} = \{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(d, \mathbb{R}); A \text{ is invertible}; \\ BA^T \in \mathcal{P},\ A^T C \in \mathcal{P}\ \}.\end{aligned} \tag{15}$$

We define three subsets $\mathfrak{H}_1$, $\mathfrak{H}_2$, and $\mathfrak{H}_0$ of $\mathfrak{H}$ by

$$\mathfrak{H}_1 = \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(d,\mathbb{R});\ BA^T \in \mathcal{P},\ A^T C \in \mathcal{P}_0 \right\},$$
$$\mathfrak{H}_2 = \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(d,\mathbb{R});\ BA^T \in \mathcal{P}_0,\ A^T C \in \mathcal{P} \right\},$$
$$\mathfrak{H}_0 = \mathfrak{H}_1 \cap \mathfrak{H}_2. \tag{16}$$

The Riemannian metric δ, and the Euclidean norm $||.||$, are used in this paper with the following definitions.

Definition 3.3: The Riemannian metric δ on $\mathcal{P}_0$ is defined, for any $P,\ Q \in \mathcal{P}_0$, as

$$\delta(P,Q) = \left\{ \sum_{i=1}^{N} \log^2 \lambda_i \right\}^{1/2}, \tag{17}$$

where $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of the matrix PQ^{-1}. On the other hand, the Euclidean norm on $\mathbb{R}^{N\times N}$ is defined as $||M|| = \sup\{||Mx||; x \in \mathbb{R}^N, ||x|| = 1\}$.

For any matrix $M \in \mathfrak{H}$ (13), we associate the map $\Phi_M : \mathcal{P}_0 \to \mathcal{P}_0$ by

$$\Phi_M(T) = (AT + B)(CT + D)^{-1}, \quad T \in \mathcal{P}_0. \tag{18}$$

Moreover, let us define $\Phi(T)$ for the Hamiltonian matrices (14) as

$$\begin{aligned}\Phi(T(k)) =& (A(k)T(k) + Q(k)A(k)^{-T}) \\ & (R(k)A(k)T(k) + (I + R(k)Q(k))A(k)^{-T})^{-1}.\end{aligned}$$

Therefore, by a straightforward modification of (9), the error covariance matrix $\Sigma(k|k)$ satisfies

$$\Sigma(k+1|k+1) = \Phi_{M(k)}(\Sigma(k|k)), \tag{19}$$

which is the *discrete Riccati equation*.

The following properties hold for deterministic system matrices in (1) and (2):

(a) For any M in $\mathfrak{H}$, and $T,\ S$ in $\mathcal{P}_0$,
$$\delta(\Phi_M(T), \Phi_M(S)) \le \delta(T,S).$$
(b) For any M in $\mathfrak{H}_1$ or in $\mathfrak{H}_2$, and $T,\ S$ in $\mathcal{P}_0$,
$$\delta(\Phi_M(T), \Phi_M(S)) < \delta(T,S).$$
(c) For any M in $\mathfrak{H}_0$, there exists $\rho(M),\ 0 < \rho(M) < 1$, such that, for all $T,\ S$ in $\mathcal{P}_0$,
$$\delta(\Phi_M(T), \Phi_M(S)) \le \rho(M)\delta(T,S).$$

In the random setup, $M(k), k \in \mathbb{N}$, as defined in (14) is the sequence of Hamiltonian matrices associated to the linear system (1) and (2). If the system is weakly detectable (respectively, weakly controllable), then, almost surely, $M(k)\ldots M(1)$ is in $\mathfrak{H}_1$ (respectively, $\mathfrak{H}_2$) for large enough values $k \in \mathbb{N}$. The main result of this section implies that the conditional error covariance matrix $\Sigma(k|k)$ does not diverge. In fact, the error is asymptotically stationary and is independent of the initial conditions. To prove this statement, let (E,δ) be a complete separable metric space. A Lipschitz map $\Psi : E \to E$ is one for which

$$\rho(\Psi) := \sup\{\frac{\delta(\Psi(x),\Psi(y))}{\delta(x,y)}; x, y \in E, x \ne y\} \tag{20}$$

is finite. Now consider the process $\{X(k), k \in \mathbb{N}\}$ generated by the following difference equation

$$X(k) = \Psi(X(k-1)). \tag{21}$$

The following theorem explores the ergodic stationary solution of (21) and its almost sure convergence properties.

Theorem 3.4: [19] Let $\{\Psi(k), k \in \mathbb{N}\}$ be a stationary ergodic sequence of Lipschitz maps from E into E. Suppose the following conditions hold:

(a) For some x in E, $\mathbb{E}[\log^+ \delta(\Psi(x), x)]$ is finite.
(b) The random variable $\log^+ \rho(\Psi(1))$ is integrable, and for some integer $\bar{k} > 0$, the real number
$$\alpha = \frac{1}{\bar{k}}\mathbb{E}[\log \rho(\Psi(\bar{k})\ o \ldots\ o\Psi(1))]$$
is strictly negative.

Then there exists an ergodic stationary process $\{\bar{X}(k), k \in \mathbb{N}\}$ with values in E, generated by (21), such that almost surely,
$$\lim_{k\to\infty} \frac{1}{k} \log \delta(X(k), \bar{X}(k)) \le \alpha < 0.$$
The next theorem proves the almost surely contraction property of the discrete Riccati equation defined in (19).

Theorem 3.5: Consider the linear system (1) and (2) with stochastic measurement matrix $H(k)$, that is weakly detectable and controllable. Then there exists an ergodic stationary $\mathcal{P}_0$-valued process $\{\bar{\Sigma}(k|k), k \in \mathbb{N}\}$ that is the solution of (19). Furthermore, there is a negative real number $\alpha < 0$ such that, almost surely, for any solution of (19) for which the initial covariance error is in $\mathcal{P}_0$,

$$\lim_{k\to\infty} \frac{1}{k} \log \delta(\Sigma(k|k), \bar{\Sigma}(k|k)) \le \alpha < 0. \tag{22}$$

Proof: Consider the random contractions $\{\Phi(k), k \in \mathbb{N}\}$ on the metric space $(\mathcal{P}_0, \delta)$ defined in (19). To apply Theorem 3.4, we first need to check the conditions of this theorem. Ccondition (a) is that for some $P \in \mathcal{P}_0$, $\mathbb{E}[\log \delta(\Phi(P), P)]$ is finite. Let us choose P to be the identity matrix and define $T = AA^T + Q$. Therefore, we obtain $\Phi(I) = T(I + R(1)T)^{-1} = (T^{-1} + R(1))^{-1}$. Consider the definition of the Riemannian metric δ and the smallest and the largest eigenvalues of the positive definite matrix $\Phi(I)$ as $\lambda_1 = 1/||\Phi(I)^{-1}||$ and $\lambda_N = ||\Phi(I)||$. Hence,

$$\delta(\Phi(I), I)^2 \le N \max(\log^2 ||\Phi(I)||, \log^2 ||\Phi^{-1}(I)||) \tag{23}$$

where

$$||\Phi^{-1}(I)|| = ||(T^{-1} + R(1))^{-1}|| \le ||T|| \le ||A||^2 + ||Q||. \tag{24}$$

The first inequality in (24) comes from the fact that $T - (T^{-1}+R(1))^{-1}$ is positive definite and the second inequality is the conseqence of the definition of T. We also have

$$\begin{aligned}||\Phi^{-1}(I)|| &\le ||(T^{-1} + R(1))|| \le ||T^{-1}|| + ||R(1)|| \\ &\le ||A^{-1}||^2 + ||H(1)||^2.\end{aligned} \tag{25}$$

From the assumptions of the Theorem 3.5 and inequalities in (24) and (25), we see that $\mathbb{E}[\log \delta(\Phi(I), I)]$ is finite. Regarding condition (b) in Theorem 3.4, we know that $\Phi(k)$ is a contraction.Thus, it suffices to show that $\rho(\Phi(k)\ o\ \ldots\ o\ \Phi(1))$ is smaller than one for some $k > 0$ with positive probability.

Let $M(k)$ be the sequence of Hamiltonian matrices associated to the system (1) and (2). It now follows that for all $k \in \mathbb{N}$ large enough $\mathbb{P}\{M(k)\ldots M(1) \in \mathfrak{H}_0\} \neq 0$ since $\mathfrak{H}_0 = \mathfrak{H}_1 \cap \mathfrak{H}_2$. And therefore, $\mathbb{P}\{\rho(\Phi(k)\ o\ \ldots\ o\ \Phi(1)) < 1\} \neq 0$. Since both conditions of Theorem 3.4 satisfy, the theorem implies the result in (22). ■

We now show that the evolution of the estimation error $x(k) - \hat{x}(k)$ is exponentially stable by the stochastic Lyapunov theory. In order to prove this, we need some preliminary results and introducing of a few new variables. First, define $B(k) = A(k) - K(k)H(k)$, $T(k) = K(k)K^T(k) + Q(k)$ and let $\Sigma(k|k) = B(k)\Sigma(k-1|k-1)B^T(k) + T(k)$.

It is proved in [19] that if $G(k) = T(k) + B(k)T(k-1)B^T(k) + \ldots + B(k)\ldots B(2)T(1)B^T(2)\ldots B^T(k)$, under the main Hypothesis 3.1, there exists $k \in \mathbb{N}$ such that $\mathbb{P}\{\mathbf{det}(G(k)) \neq 0\} > 0$.

Theorem 3.6: Consider the system (1) and (2) with the stochastic parameters for which the system is weakly detectable and controllable while Hypothesis (3.1) holds. Then the estimation error $x(k) - \hat{x}(k)$ is almost surely asymptotically stable.

Proof: For notational simplicity suppose that $\mathbb{P}\{\mathbf{det}(G(1)) \neq 0\} > 0$. To prove the exponential stability of the error $x(k) - \hat{x}(k)$, it suffices to show that there is a real number $\gamma > 0$, such that almost surely

$$\lim_{k\to\infty} \frac{1}{k} \log ||(A(k) - K(k)H(k)), \ldots, (A(k) - K(1)H(1))|| \leq -\gamma$$

for any solution $\{\Sigma(k|k),\ k \in \mathbb{N}\}$ of (9) such that the initial error covariance is in $\mathcal{P}_0$.

Assume that $\lambda(k) = ||T(k)^{-1}||^{-1}$, $\sigma(k) = ||\Sigma(k|k)||$, and $\alpha = ||\Sigma_0^{-1}||$. For positive integer $\bar{k}$, set $x_{\bar{k}} \in \mathbb{R}^N$ by the backward recursion to be $x(k) = B^T(k+1)x(k+1)$. Consider the candidate Lyapunov function $V(k) = x^T(k)\Sigma(k|k)x(k)$. We note now that

$$\begin{aligned} V(k+1) - V(k) &= x^T(k+1)\Sigma(k+1|k+1)x(k+1) \\ &\quad - x(k)^T\Sigma(k|k)x(k) \\ &= x^T(k+1)(\Sigma(k+1|k+1) \\ &\quad - B(k+1)\Sigma(k|k)B^T(k+1))x(k+1) \\ &= x^T(k+1)T(k+1)x(k+1) \\ &\geq \lambda(k+1)||x(k+1)||^2 \geq \frac{\lambda(k+1)}{\sigma(k+1)}V(k+1). \end{aligned}$$

Consequently, $V(k+1)(1 - \frac{\lambda(k+1)}{\sigma(k+1)}) \geq V(k)$. Define $\tau(k+1) = 1 - \frac{\lambda(k+1)}{\sigma(k+1)}$. We also have

$$\begin{aligned} ||x(0)||^2 &\leq ||\Sigma_0^{-1}|| \leq ||\Sigma_0^{-1}||\tau(1)\tau(2)\ldots\tau(\bar{k})V(\bar{k}) \\ &\leq ||\Sigma_0^{-1}||\tau(1)\tau(2)\ldots\tau(\bar{k})||\Sigma(\bar{k}|\bar{k})||||x(\bar{k})||^2. \end{aligned}$$

Since $x(0) = B^T(1)\ldots B^T(\bar{k})x(\bar{k})$, this implies that

$$||x(0)|| = ||B^T(1)\ldots B^T(p)||^2||x(\bar{k})||^2 \leq \alpha\tau(1)\ldots\tau(\bar{k})\sigma(\bar{k}).$$

Therefore,

$$\begin{aligned} \frac{1}{\bar{k}} \log ||B^T(1)\ldots B^T(\bar{k})||^2 &\leq \frac{1}{\bar{k}}\log\alpha + \frac{1}{\bar{k}}\log\sigma(k) \\ &\quad + \frac{1}{\bar{k}}\sum_{i=1}^{\bar{k}}\log\tau(i). \end{aligned}$$

From the contraction property of $\Sigma(k|k)$ we know that $\lim_{\bar{k}\to\infty}\frac{1}{\bar{k}}\log\sigma(\bar{k}) \leq 0$. From Birkhoff's ergodic theorem it thus follows that

$$\lim_{\bar{k}\to\infty}\frac{1}{\bar{k}}\log||B^T(\bar{k})\ldots B^T(1)||^2 \leq \mathbb{E}(\log\tau(1)).$$

As by assumption $\mathbb{P}\{\mathbf{det}(G(1)) \neq 0\} > 0$, $\mathbb{E}(\log\tau(1)) < 0$, thus proving the statement of the theorem. ■

We conclude this section with a direct consequence of the above framework in the context of the three scenarios introduced in §II.

Corollary 3.7: The coordinated decentralized estimators, following the stochastic version of (8), for the models described by (a)-(b) §II converge almost surely.

IV. An Example

In this example, consider a network of four sensors in the coordinated decentralized estimator setup. The communication network between the sensors and the coordinator is shown in Fig. 4. The dynamics of the system, and the

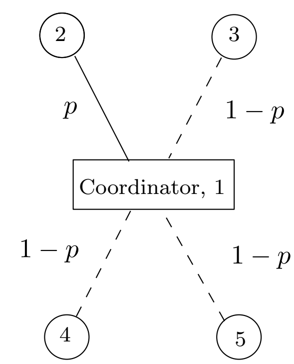

Fig. 4. The star communication network. The dashed line represent lack of communication between the sensor and the coordinator which happens with probability $1 - p$ at every time step.

corresponding observation network are as follows:

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & -0.5 & 0.1 & 0 & 0 & 0 \\ 0.5 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & -0.5 & 0.1 \\ 0 & 0 & 0 & 0 & 0.5 & 0.6 \end{bmatrix} x(k) \\ &\quad + [1\ 1\ 1\ 1\ 1\ 1]^T w(k) \\ z(k) &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} v(k). \end{aligned} \tag{26}$$

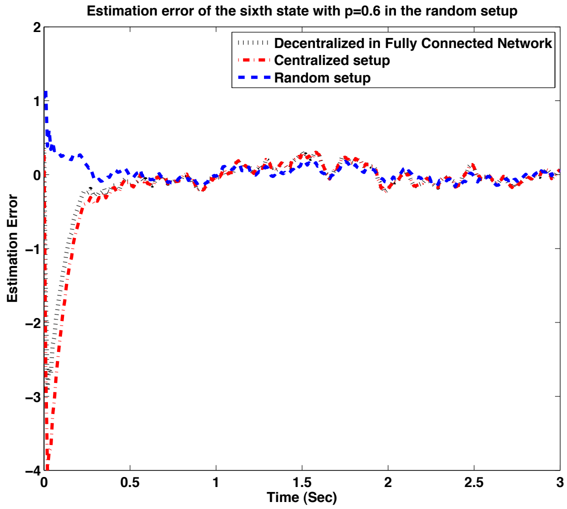

Fig. 5. Estimation of the sixth state in different setups with $p = 0.6$.

It is assumed that the system and the measurement noise signals are independent zero-mean Gaussian with covariances $Q = 10^{-2}$ and the identity matrix, respectively.

In the proposed estimation setup, the first sensor estimates the states of the system (26) as it observes the signal

$$z^1(k) = [0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0]x(k) + v(k).$$

Analogously, for example, the second sensor estimates the states of the system (26) as it observes the vector

$$z^2(k) = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0]x(k) + v(k).$$

In the random setup, at each time step, each sensor sends its estimate to the coordinator with probability $p = 0.6$. Fig. 5 demonstrates the estimation error of the sixth state in the centralized, decentralized, and the random estimation setup. Note that the system matrix in (26) is not observable by considering each measurement separately, while the weak observability condition holds in the random setup.

V. CONCLUSION

In this paper a coordinated decentralized information filter over certain classes of random networks has been examined. The proposed distributed filter uses the local computational capability of each sensor and uploads the processed measurements to the coordinator in a random fashion. In order to account for energy efficiency, as well as the presence of potentially unreliable communication links and time delays, a random communication scheme between the sensors and the coordinator has been considered. We have shown that distinct variations of this problem satisfies a generic property of being weakly detectable, thus making it suitable for the application of stability and convergence results on stochastic Kalman filtering.

VI. ACKNOWLEDGMENTS

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REFERENCES

[1] Smart Dust Project Home Page. Univ. California, Berkeley. [On- line] http://robotics.eecs.berkeley.edu/ pister/SmartDust/

[2] NEST Project at Berkeley Home Page. Univ. California, Berkeley. [On- line] http://webs.cs.berkeley.edu/nest-index.html

[3] Dust Network Inc. Home Page, [On- line] http://www.dustnetworks.com/about/careers

[4] A. Mainwaring, D. Culler, J. Polstre, R. Szewczyk, and J. Anderson, Wireless sensor networks for habitat monitoring, *Proceedings of the 1st ACM international workshop on wireless sensor networks and applications-WSNA '02*, 2002.

[5] R. Cardell-Oliver, M. Kranz, K. Mayer, A reactive soil moisture sensor network: design and field evaluation, *International Journal of Distributed Sensor Networks*. Vol. 1, 2005, pp. 149-162.

[6] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. Jordan, and S. Sastry, Kalman filtering with intermittent observations, *IEEE Transaction on Automatic Control*, Vol. 49, 2004, pp. 1453-1464.

[7] S. Kar, and J. M. F. Moura, Gossip and distributed Kalman filtering: weak consensus under weak detectability, *http://arxiv.org/abs/1004.0381v1*

[8] S. Kar, B. Sinopoli, and J. M. F. Moura, A random dynamical systems approach to filtering in large-scale networks, *http://arxiv.org/abs/0910.0918v1*

[9] S. Grime, and H.F. Durraat-Whyte, Data fusion in decentralized sensor networks, *Control Engineering Practice*,Vol. 2, No.5, 1994, pp. 849-863.

[10] A. Simonetto, T. Keviczky, and R. Babuska, Distributed nonlinear estimation for robot localization using weighted consensus, *IEEE International Conference on Robotics and Automation*, Anchorage, Alaska, USA, 2010.

[11] P. Alriksson and A. Rantzer, Model based information fusion in sensor networks, *Proceedings of the 17th IFAC World Congress*, Seoul, Korea, July 2008.

[12] R. Carli, A. Chiuso, L. Schenato, and S. Zampieri, Distributed Kalman filtering using consensus strategies, *Proceedings of the 46th Conference on Decision and Control*, New Orleans, LA, USA, December, 2007.

[13] R. Olfati-Saber, Kalman-consensus filter: optimality, stability, and performance, *Proceedings of the 46th IEEE Conference on Decision and Control*, 2009.

[14] A. Speranzon, C. Fischione, and K. H. Johansson, Distributed and collaborative estimation over wireless sensor networks, *Proceedings of the 45th IEEE Conference on Decision and Control*, San Diego, CA, USA, 2006.

[15] P.S. Maybeck, *Stochastic Models, Estimation and Control, Vol. I.* Academic Press, 1979.

[16] M. Mesbahi and M. Egerstedt. *Graph Theoretic Methods for Multiagent Networks*. Princeton University Press, 2010.

[17] M. Nabi, M. Mesbahi, N. Fathpour, F. Y. Hadaegh. Local Estimators for Multiple Spacecraft Formation Flying. *GNC*, 2008.

[18] P. Bougerol, Almost sure stabilizability and Riccati equation of linear systems with random parameters, *SIAM Journal on Control and Optimization*, Vol. 33, No. 3, 1995, pp. 702-717.

[19] P. Bougerol, Kalman filtering with random coefficients and contractions, *SIAM Journal on Control and Optimization*, Vol. 31, No. 4, 1993, pp. 942-959.

[20] A. Rahmani, M. Ji, M. Egerstedt, and M. Mesbahi, Controllability of Multi-agent Systems from a Graph-theoretic Perspective, SIAM Journal on Control and Optimization, 48 (1): 162-186, 2009.

[21] A.S. Willsky, M.G. Bello, D.A. Castanon, B.C. Levy, and G.C. Verghese. "Combining and updating of local estimates and regional maps along sets of one-dimensional tracks. " *IEEE Transactions on Automatic Control*, 27(4), 1982.

[22] V. Gupta, N. C. Martins, and J. S. Baras. "Stabilization over erasure channels using multiple sensors." *IEEE Transactions on Automatic Automatic Control,* 57(7), July 2009.

[23] A. Chiuso, L. Schenato. "Information fusion strategies and performance bounds in packet-drop networks." submitted to Automatica.