2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Constrained stabilization of a two-input buck-boost DC/DC converter using a set-theoretic method

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Abstract— **This paper considers the problem of constrained stabilization of a two-input buck-boost DC/DC converter by linear state-feedback. It is demonstrated that, via an appropriate change of coordinates, a recent synthesis technique for constrained bilinear discrete-time systems can be applied to an averaged nonlinear model of the converter. Moreover, it is proven that the synthesis method yields a polyhedral constrained control invariant set for the converter model in the original coordinate system. The synthesis algorithm requires solving a single linear program off-line. An extensive simulation case study along with a preliminary, successful real-time experiment, demonstrate the effectiveness of the proposed methodology.**

Index Terms— **DC/DC converters, Bilinear systems, Invariance, Polyhedral Lyapunov functions, Constraints.**

I. Introduction

Buck-boost DC/DC converters are switching devices that have strong nonlinear dynamics and are subject to hard constraints on inputs and states. A very fast switching frequency and small sampling time (ranging from μs to ns) pose a serious challenge to controller synthesis and implementation. That is why simple control solutions, such as PID and Fuzzy controllers, are dominant in PWM controlled low-cost power converters, see, e.g., [1] and [2]. The main issues with this type of controllers are a lack of an a priori stability guarantee and inability to cope with constraints in a non-conservative way. As far as stability is concerned, a direct switching Lyapunov approach was proposed for stabilization of DC/DC converters, see, e.g., [3]. However, this approach can lead to arbitrarily fast switching and does not handle constraints. Recently, model predictive control was proposed as a viable alternative to deal with constraints in control of power converters, see, e.g., [4]–[8] and the references therein. However, due to the bilinear nature of the typical averaged model of a buck-boost converter, these algorithms are computationally intensive and not suitable for low-end converters. Tractable solutions can only be obtained for linear or piecewise affine approximations, which introduce errors and lack an a priori stability guarantee as well. For such classes of systems, an explicit piecewise affine predictive control law can be obtained and stability can be checked a posteriori, see, e.g., [9]. As such, it would be desirable to obtain a tractable synthesis method that is applicable to the full bilinear model of a buck-boost converter, results in a low complexity feedback law and which also offers an a priori guarantee of stability and constraint satisfaction.

This paper indicates that a recent synthesis technique [10] developed for constrained stabilization of general discrete-time bilinear systems with zero as equilibrium can be applied to DC/DC converters. The method of [10] employs invariance conditions [11] for a polyhedral set and yields a stabilizing linear static state-feedback control law that satisfies constraints. The method is computationally advantageous as it requires solving a single linear program off-line. Along with the controller synthesis it yields a *polyhedral* Lyapunov function for the closed-loop system. Notice that polyhedral Lyapunov functions are preferable to quadratic ones, as they induce polyhedral constrained control invariant (CCI) sets. Moreover, for bilinear systems, quadratic Lyapunov functions lead to fourth order matrix inequalities, which are hardly solvable. However, the results in [10] cannot be applied directly to DC/DC converters, as the corresponding averaged converter model, although bilinear, does not have zero as an equilibrium point. Notice that a simple shift of coordinates does not preserve invariance of a polyhedral set, when the system model is bilinear.

The main contribution of this paper consists of a set of sufficient conditions that render the results of [10] applicable to a standard two-input buck-boost DC/DC converter averaged model. It is shown that if these conditions hold for an auxiliary bilinear model obtained via an appropriate, specific coordinate change, then the resulting control law is stabilizing and satisfies constraints for the original converter model with a non-zero equilibrium. Moreover, it is indicated how a polyhedral CCI set can be obtained for the original model via a suitable Minkowski translation. Such a set, besides providing a region of attraction for the closed-loop system, is very useful for model predictive control algorithms, i.e., it can be employed as a terminal set, see [12] for more details on this topic. The polyhedral CCI set, obtained with the method proposed in this paper, is much larger than the region where the linear approximation of the bilinear model is reasonable and it virtually covers the entire desired range of operation.

II. Preliminaries

A. Mathematical notation and definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, $\mathbb{Z}_+$ denote the set of real numbers, the set of non-negative reals, the set of integer numbers and non-negative integers, respectively. Given two sets $\mathbb{P}$ and $\mathbb{S}$, $\mathbb{P}_{\mathbb{S}} := \mathbb{P}\cap\mathbb{S}$. For a $\lambda \in \mathbb{R}$ and a set $\mathbb{P} \subset \mathbb{R}^n$, let $\lambda\mathbb{P} := \{\lambda x | x \in \mathbb{P}\}$.

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978-1-4577-0079-8/11/$26.00 ©2011 AACC

$\mathbb{R}^{n\times m}$ denotes the set of real $n \times m$ matrices. For a matrix $Z \in \mathbb{R}^{n\times m}$, $[Z]_{ij} \in \mathbb{R}$ denotes the element on the i-th row and the j-th column of Z, $[Z]_{i\bullet} \in \mathbb{R}^{1\times m}$ denotes the i-th row of Z and $[Z]_{\bullet j} \in \mathbb{R}^{n\times 1}$ denotes the j-th column of Z. Given a vector $x \in \mathbb{R}^n$, $\|x\|_\infty := \max_{i=1,\ldots,n} |[x]_i|$ denotes the infinity norm, where $[x]_i$ is the i-th element of x and $|\cdot|$ denotes the absolute value. $I_n \in \mathbb{R}^{n\times n}$ denotes n-th dimensional identity matrix. Given a matrix $A \in \mathbb{R}^{n\times m}$, $A^+ \in \mathbb{R}^{n\times m}$ denotes a matrix where $[A^+]_{ij} = \max\{0, [A]_{ij}\}$ and $A^- \in \mathbb{R}^{n\times m}$ denotes a matrix where $[A^-]_{ij} = \max\{0, -[A]_{ij}\}$. Thus, $A^+ - A^- = A$. Given a matrix $D \in \mathbb{R}^{n\times n}$, $D^\delta \in \mathbb{R}^{n\times n}$ is a matrix with $[D^\delta]_{ii} = [D]_{ii}$ and $[D^\delta]_{ij} = 0$ for $i \neq j$. Given a matrix $D \in \mathbb{R}^{n\times n}$ the matrix $D^\mu \in \mathbb{R}^{n\times n}$ is defined by $D^\mu := D - D^\delta$. For two matrices $A, B \in \mathbb{R}^{n\times m}$, $A \odot B = \sum_{i=1}^{n}\sum_{j=1}^{m}[A]_{ij}[B]_{ij}$ denotes the Frobenius inner product. Given two arbitrary sets $\mathbb{P}, \mathbb{X} \subset \mathbb{R}^n$, $\mathbb{P}\oplus\mathbb{X} = \{p+x | p \in \mathbb{P}, x \in \mathbb{X}\}$ denotes their Minkowski sum. Define the set valued map $\Psi : \mathbb{R}^{r\times n} \times \mathbb{R}^r \times \mathbb{R}^r \rightrightarrows \mathbb{R}^n$, with $\Psi(G, w_1, w_2) := \{x \in \mathbb{R} | -w_2 \leq Gx \leq w_1\}$. Given matrices $C_i \in \mathbb{R}^{n\times m}$ with $i \in \mathbb{Z}_{[1,n]}$, we define $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ with $\mathcal{C}(a) = \begin{bmatrix} a^\top C_1 \\ \vdots \\ a^\top C_n \end{bmatrix}$ and $\mathcal{C}^\top : \mathbb{R}^m \to \mathbb{R}^{n\times n}$ with $\mathcal{C}^\top(b) := \begin{bmatrix} (C_1 b)^\top \\ \vdots \\ (C_n b)^\top \end{bmatrix}$. Notice that $\mathcal{C}(a)b = \mathcal{C}^\top(b)a$.

Consider the closed-loop nonlinear discrete time system

$$x(k+1) = f(x(k), u(k)) \tag{1a}$$
$$u(k) = g(x(k)) \tag{1b}$$

where $x(k) \in \mathbb{X} \subset \mathbb{R}^n$ is the state and $u(k) \in \mathbb{U} \subset \mathbb{R}^m$ is the input at time instant $k \in \mathbb{Z}_+$, and $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ and $g : \mathbb{X} \to \mathbb{U}$ are arbitrary maps. It is assumed that the sets $\mathbb{X}$ and $\mathbb{U}$ are bounded.

Definition II.1 A state $x^s \in \mathbb{X}$ with $u^s := g(x^s) \in \mathbb{U}$ is called an equilibrium state for system (1) if $f(x^s, u^s) = x^s$.

Definition II.2 Let $\varepsilon \in \mathbb{R}_{[0,1)}$. A subset $\mathbb{P}$ of $\mathbb{X}$ is said to be *ε-constrained control contractive* (or shortly, ε-contractive) for an equilibrium state x^s of system (1), if for all $x \in \mathbb{P}$ it holds that $g(x) \in \mathbb{U}$ and $f(x, g(x)) - x^s \in \varepsilon(\mathbb{P} \oplus \{-x^s\})$. A set $\mathbb{P} \subseteq \mathbb{X}$ is said to be *constrained control invariant* (or shortly, invariant) for the system (1) if for all $x \in \mathbb{P}$ it holds that $g(x) \in \mathbb{U}$ and $f(x, g(x)) \in \mathbb{P}$.

B. Constrained stabilization of bilinear systems

This subsection reproduces the main result on constrained stabilization of bilinear systems with a static linear state-feedback presented in [10]. The synthesis method developed in [10] is applicable to a special case of system (1), i.e.,

$$\begin{aligned} x(k+1) &= f(x(k), u(k)) \\ &:= Ax(k) + Bu(k) + \mathcal{C}(x(k))u(k), \end{aligned} \tag{2a}$$
$$u(k) = g(x(k)) := Kx(k),\ k \in \mathbb{Z}_+, \tag{2b}$$

where $C_i \in \mathbb{R}^{n\times m}$ for all $i \in \mathbb{Z}_{[1,n]}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $K \in \mathbb{R}^{m\times n}$. Let a subset of initial conditions be given, i.e., $Q := \Psi(G, w_1, w_2) \subseteq \mathbb{X}$, for some G, w_1, w_2 and let the input constraints set be defined as $\mathbb{U} := \Psi(I_m, u^M, u^m)$ for some $u^m, u^M \in \mathbb{R}^m$.

Theorem II.3 [10] *Suppose there exist matrices $D_j, H \in \mathbb{R}^{p\times p}$, $j \in \mathbb{Z}_{[1,p]}$, $K \in \mathbb{R}^{m\times n}$, a non-negative matrix $L \in \mathbb{R}^{2m\times 2p}$ and $\varepsilon \in \mathbb{R}_{[0,1]}$ that satisfy*

$$G(A + BK) = HG, \tag{3}$$
$$\sum_{i=1}^{n}[G]_{ji}C_iK = G^\top D_j G,\ j \in \mathbb{Z}_{[1,p]} \tag{4}$$

$$\begin{bmatrix} H^+ & H^- \\ H^- & H^+ \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} D_1^{\delta+} \odot W^M + D_1^{\mu+} \odot W^M + D_1^{\mu-} \odot W^m \\ \vdots \\ D_p^{\delta+} \odot W^M + D_p^{\mu+} \odot W^M + D_p^{\mu-} \odot W^m \\ D_1^{\delta-} \odot W^M + D_1^{\mu-} \odot W^M + D_1^{\mu+} \odot W^m \\ \vdots \\ D_p^{\delta-} \odot W^M + D_p^{\mu-} \odot W^M + D_p^{\mu+} \odot W^m \end{bmatrix} \leq \varepsilon \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \tag{5}$$

$$L\begin{bmatrix} G \\ -G \end{bmatrix} = \begin{bmatrix} K \\ -K \end{bmatrix},\ L\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \leq \begin{bmatrix} u^M \\ u^m \end{bmatrix}, \tag{6}$$

where,

$$[W^M]_{ij} = \max\{[w_1]_i[w_1]_j, [w_2]_i[w_2]_j\},$$
$$[W^m]_{ij} = \max\{[w_1]_i[w_2]_j, [w_2]_i[w_1]_j\}.$$

Then the set Q is invariant for the system (2a) *in closed-loop with the control law* (2b).

For the proof of Theorem II.3, we refer to [10]. If the sign of each element of the matrices H and D_j, respectively, is fixed a priori, then the matrix K can be determined by solving a single *linear program* (LP), as it is summarized in the following problem.

Problem II.4 For a given system (2), matrix G, vectors w_1, w_2, u^m, u^M, and a fixed sign of each element of the matrices H and D_j, $j \in \mathbb{Z}_{[1,p]}$, respectively, solve

$$\min_{\varepsilon,\, H,\, L,\, D_j,\, K} \varepsilon \tag{7}$$

subject to the linear constraints (3)-(6).

Corollary II.5 [10] *Suppose that Problem II.4 has a feasible solution with $\varepsilon \in \mathbb{R}_{[0,1)}$. Then the set-induced function*

$$\hat{V}(x) := \max_{j\in\mathbb{Z}_{[1,p]}} \left\{ \frac{[G]_{j\bullet}x}{[w_1]_j}, \frac{[-G]_{j\bullet}x}{[w_2]_j} \right\} \tag{8}$$

is a Lyapunov function for the closed-loop system (2).

For a formal definition of a Lyapunov function for discrete-time systems, the interested reader is referred to [11].

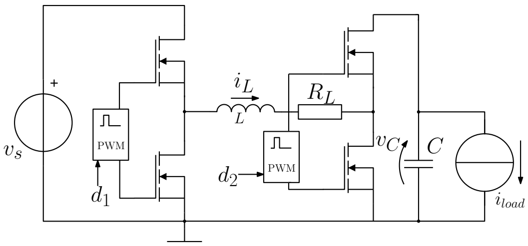

Fig. 1. A schematic representation of the non-inverting buck-boost converter.

To summarize, in order to synthesize a stabilizing state-feedback law, one has to impose a candidate ε-contractive set Q, which satisfies the state constraints and has the origin in its interior, and solve the Problem II.4. If $\varepsilon < 1$ then the resulting state-feedback is stabilizing and the set Q is indeed ε-contractive, otherwise one has to chose another candidate set and repeat the procedure.

III. Plant description and problem formulation

The non-inverting buck-boost converter is essentially one buck and one boost converter connected in series. This type of converter can produce lower as well as higher output voltages than the supplied one. For more information on the subject of power conversion, see [1], [13], [14]. The converter topology employed in this paper has a separate control input for each stage. The control signal is a PWM waveform with a constant frequency and controlled duty-cycle. The schematic representation of such a converter is shown in Fig. 1.

A. Nonlinear averaged model

No dead-time nor other nonlinear behavior of circuitry components were considered during the mathematical modeling of the converter. The only parasitic elements taken into account are the resistances of power transistors, output capacitor and inductor, which are lumped into R_L and R_C. Thus, the resulting average discrete-time model of the system is bilinear in input and states, i.e.,

$$\begin{aligned} x(k+1) &= \phi(x(k), u(k)) \\ &:= Ax(k) + Bu(k) + \begin{bmatrix} x(k)^\top C_1 \\ x(k)^\top C_2 \end{bmatrix} u(k) + w \end{aligned} \tag{9}$$

where $x(k) := \begin{bmatrix} v_C(k) & i_L(k) \end{bmatrix}^\top \in \mathbb{R}^2$ is the system state (i.e., the capacitor voltage and the inductor current) and $u(k) := \begin{bmatrix} d_1(k) & d_2(k) \end{bmatrix}^\top$ is the system input (i.e., the duty cycles) at the time instant $k \in \mathbb{Z}_+$, respectively. In this paper, a constant current source is considered as load. Notice that other load types can be accommodated in a similar fashion. The matrix coefficients from (9) are specific to the circuitry implementation and they are described in terms of system parameters such as inductance, capacitance and resistance, i.e.,

$$A = I_2 + T_s \begin{bmatrix} 0 & 0 \\ 0 & -\frac{R_L}{L} \end{bmatrix}, B = T_s \begin{bmatrix} 0 & 0 \\ \frac{v_s}{L} & -\frac{i_{load}}{C} \end{bmatrix}, \tag{10}$$
$$w = T_s \begin{bmatrix} -\frac{i_{load}}{C} \\ 0 \end{bmatrix}, C_1 = T_s \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{C} \end{bmatrix}, \; C_2 = T_s \begin{bmatrix} 0 & -\frac{1}{L} \\ 0 & \frac{R_C}{L} \end{bmatrix},$$

where v_s is the supply voltage, i_{load} is the load current and T_s is the sampling time. The sampling time and the PWM period are assumed to be equal throughout this paper. The system is subject to hard constraints, i.e., $u(k) \in \mathbb{U} := \Psi(I_2, u^M, -u^m)$ and $x(k) \in \mathbb{X} := \Psi(I_2, x^M, -x^m)$, for all $k \in \mathbb{Z}_+$, where $u^m, u^M, x^m, x^M \in \mathbb{R}^2$ are suitable vectors. The constraints on the states can be softened within certain limits in most situations.

Remark III.1 In (10), it can be observed that some of the system matrices are functions of the supply voltage v_s and the load current i_{load}. In this paper their values are considered constant and known a priori. In practice, they are either measured or estimated. Further work deals with parametrization of the control law with respect to v_s and i_{load}. $\square$

B. Control problem formulation

The goal of the controller is to maintain the output voltage of the converter at a prescribed value while maintaining the system state and input within specified limits. The stationary value of the inductor current can vary in certain limits without affecting the value of the output voltage. Generally, a low value of the stationary inductor current is preferred to minimize the power dissipation of the converter.

Throughout the paper it is assumed that a specific reference is provided for both the output voltage V_{ref} and the inductor current I_{ref}. The notation $x^s := \begin{bmatrix} V_{ref} & I_{ref} \end{bmatrix}^\top$ will be employed throughout the rest of the paper.

In conclusion, the controller synthesization problem has the following formulation.

Problem III.2 Given the system (9), sets $\mathbb{U}$, $\mathbb{X}$ and $\mathbb{P} \subseteq \mathbb{X}$, and desired equilibrium state x^s (along with corresponding control input u^s), construct an affine state-feedback control law such that $\mathbb{P}$ is ε-contractive for the resulting closed-loop system. $\square$

IV. Main results

As mentioned in Section II-B, the algorithm described in [10] is applicable only to systems with zero as equilibrium. Thus, system (9) must be transformed in order to render the results from [10] applicable. Moreover, this transformation must be such that its corresponding reverse transformation preserves stability and invariance and thus, constraint satisfaction.

To this end, let us begin with the analysis of a coordinate change problem for affine systems. Consider an affine system

$$x(k+1) = Ax(k) + Bu(k) + w \tag{11}$$

and the coordinate transformation

$$z(k) = x(k) - x^s, \tag{12a}$$
$$s(k) = u(k) - u^s, \tag{12b}$$

where x^s is the desired equilibrium point for the closed-loop system and u^s is selected such that $Bu^s = x^s - w - Ax^s$. Then, one obtains

$$z(k+1) = Az(k) + Bs(k), \tag{13}$$

which has exactly the same form as the linear part of (11).

Next, a separate linear feedback law is considered for each of the systems (11) and (13), i.e.,

$$u(k) = K(x(k) - x^s) + u^s, \tag{14a}$$
$$s(k) = Kz(k), \tag{14b}$$

for all $k \in \mathbb{Z}_+$. For some sets $\mathbb{P} \subset \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^m$ let $\hat{\mathbb{P}} := \mathbb{P} \oplus \{-x^s\}$ and $\hat{\mathbb{U}} := \mathbb{U} \oplus \{-u^s\}$. Let $\varepsilon \in \mathbb{Z}_{[0,1]}$.

Theorem IV.1 *The set $\mathbb{P}$ is ε-contractive for the equilibrium state x^s of the system* (11) *in closed-loop with the state-feedback* (14a) *and input constraints set $\mathbb{U}$ if and only if the set $\hat{\mathbb{P}}$ is ε-contractive for the zero equilibrium of the system* (13) *in closed-loop with the state-feedback* (14b) *and input constraints set $\hat{\mathbb{U}}$.*

Proof: The proof of this theorem follows straightforward from the equivalence of the two closed-loop systems. Let $k \in \mathbb{Z}_+$ be arbitrary. More precisely, by construction, for any $x(k) \in \mathbb{P}$, $z(k) = x(k) - x^s \in \hat{\mathbb{P}}$ it holds that $z(k+1) \in \varepsilon\hat{\mathbb{P}} = \varepsilon(\mathbb{P} \oplus \{-x^s\})$ and thus, $x(k+1) - x^s = z(k+1) \in \varepsilon(\mathbb{P} \oplus \{-x^s\})$ with $s(k) \in \hat{\mathbb{U}}$ and $u(k) \in \mathbb{U}$, respectively. The same reasoning applies when starting with any $z(k) \in \hat{\mathbb{P}}$, which yields $z(k+1) \in \varepsilon\hat{\mathbb{P}}$ with $s(k) \in \hat{\mathbb{U}}$. ■

Theorem IV.1 shows that if a set is ε-contractive for the zero equilibrium of the dynamics (A, B), i.e., the linear part of system (11), with translated input constraints set $\hat{\mathbb{U}}$, then the same set translated back to the original coordinates is ε-contractive for the non-zero equilibrium of the original affine system. Unfortunately, this result does not apply to the considered buck-boost DC/DC converter model, due to the bilinear nature of (9). In what follows, an auxiliary bilinear system that enables a result similar to Theorem IV.1 is constructed.

Consider a bilinear system with a non-zero equilibrium point

$$x(k+1) = Ax(k) + Bu(k) + \mathcal{C}(x(k))u(k) + w, \tag{15}$$

where $w \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ and B, $C_i^\top \in \mathbb{R}^{n\times m}$ for all $i \in \mathbb{Z}_{[1,n]}$. By applying the coordinate transformation (12) with u^s obtained as a solution of

$$(B + x^s\mathcal{C})u^s = x^s - w - Ax^s, \tag{16}$$

one obtains the following auxiliary bilinear system with zero as equilibrium:

$$z(k+1) = \hat{A}z(k) + \hat{B}s(k) + \mathcal{C}(z(k))s(k), \tag{17}$$

where

$$\hat{A} := A + \mathcal{C}^\top(u^s),\ \hat{B} := B + \mathcal{C}(x^s). \tag{18}$$

Next, the main result is stated.

Theorem IV.2 *The set $\mathbb{P}$ is ε-contractive for the equilibrium state x^s of the system* (15) *in closed-loop with the state-feedback* (14a) *and input constraints set $\mathbb{U}$ if and only if the set $\hat{\mathbb{P}}$ is ε-contractive for the zero equilibrium of the auxiliary system* (17) *in closed-loop with the state-feedback* (14b) *and input constraints set $\hat{\mathbb{U}}$.*

Proof: By substituting (14a) in (15), adding and subtracting several terms, and subtracting from both parts x^s one obtains

$$\begin{aligned} x(k+1) - x^s = {} & A(x(k) - x^s) + Ax^s + BK(x(k) - x^s) \\ & + Bu^s + \mathcal{C}(x(k) - x^s)K(x(k) - x^s) \\ & + \mathcal{C}x^sK(x(k) - x^s) + \mathcal{C}u^s \\ & + \mathcal{C}(x^s)u^s + w - x^s. \end{aligned} \tag{19}$$

Using the fact that $\mathcal{C}(x)u = \mathcal{C}^\top(u)x$ and substituting (12) in (19) with u^s a solution of (16), the equation (19) can be rewritten as follows:

$$\begin{aligned} z(k+1) = {} & \left(A + \mathcal{C}^\top(u^s)\right) z(k) + \\ & (B + \mathcal{C}(x^s))\, Kz(k) + \mathcal{C}(z(k))Kz(k). \end{aligned} \tag{20}$$

Observing that system (20) is system (17) in closed-loop with the state-feedback (14b) yields that for any $x(k) \in \mathbb{P}$, $z(k) = x(k) - x^s \in \hat{\mathbb{P}}$ and $s(k) = Kz(k) \in \hat{\mathbb{U}}$. As such, it holds that $z(k+1) = x(k+1) - x^s \in \varepsilon\hat{\mathbb{P}} = \varepsilon(\mathbb{P} \oplus \{-x^s\})$ and $u(k) = s(k) + u^s \in \mathbb{U}$. Then, the proof readily follows via the same reasoning used to prove Theorem IV.1. ■

Using Theorem IV.2 and [10], the existence of a polyhedral Lyapunov function can be established for a bilinear system with a non-zero equilibrium, as stated next.

Corollary IV.3 *Suppose that the set $\hat{\mathbb{P}}$ is ε-contractive with $\varepsilon \in \mathbb{R}_{[0,1)}$ for the zero equilibrium of the auxiliary system* (17) *in closed-loop with the state-feedback* (14b) *and input constraints set $\hat{\mathbb{U}}$. Then the set-induced function*

$$V(x) := \max_{j \in \mathbb{Z}_{[1,p]}} \left\{ \frac{[G]_{j\bullet}(x - x^s)}{[w_1]_j}, \frac{[-G]_{j\bullet}(x - x^s)}{[w_2]_j} \right\} \tag{21}$$

is a Lyapunov function for the system (15) *in closed-loop with the state-feedback* (14a) *and input constraints set $\mathbb{U}$.*

The proof follows directly from the equivalence of the two closed-loop systems established in Theorem IV.2 and Corollary II.5.

Using the methodology described above, an auxiliary bilinear system can be constructed for the converter model (9). The auxiliary system has zero as equilibrium. Thus, a solution to the constrained stabilization problem can be obtained by solving Problem II.4. Based on the resulting control law and ε-contractive set, a suitable, stabilizing control law and ε-contractive set can be calculated for the original converter model (9) with a non-zero equilibrium.

TABLE I

PARAMETERS OF THE SIMULATED PLANT

Name	Value	Name	Value
R_C	0.05Ω	i_{load}	$0.2A$
R_L	0.3Ω	v_s	$10V$
C	0.05μF	V_{ref}	$20V$
L	220μH	T_s	$10\mu s$

V. ILLUSTRATIVE CASE STUDY

In this section, we consider the buck-boost converter model (9)-(10) with the parameter values as summarized in Table I. The constraint sets on states and inputs are $\mathbb{X} := \Psi\left(I_2, \left[\begin{smallmatrix}22\\3\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\right)$ and $\mathbb{U} := \Psi\left(I_2, \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\right)$. Two different linear state-feedback controllers were constructed for a pre-specified, candidate ε-contractive set using Problem II.4. One of the controllers was synthesized using the average nonlinear model (9) of the system and the coordinate transformation proposed in Theorem IV.2. The other controller was obtained using a linearized model around the desired equilibrium state x^s and the coordinate transformation proposed in Theorem IV.1, i.e.,

$$\begin{aligned} z_l(k+1) &:= A_l z_l(k) + B_l s_l(k), \\ A_l &= \left[\begin{smallmatrix} 1.0000 & 0.1818 \\ -0.0182 & 0.9855 \end{smallmatrix}\right], \; B_l = \left[\begin{smallmatrix} 0 & 0.2273 \\ 0.4545 & -0.9098 \end{smallmatrix}\right], \end{aligned} \tag{22}$$

where the index l denotes the fact that the matrices A_l, B_l and vectors z_l, s_l correspond to the linearization of (9). Notice that matrices A_l and B_l in (22) are different from A and B in (9).

The first step in controller design is to define the coordinate transformations. Note that the u^s for both the linear and bilinear system is not uniquely defined in Section IV. For the particular case of the considered buck-boost converter, the matrices B_l and $\hat{B}$ are invertible. Thus, a unique u^s can be calculated for the chosen x^s. Next, the candidate ε-contractive set $\hat{\mathbb{P}}$ for the zero equilibrium auxiliary bilinear system is imposed and Problem II.4 is solved. For this particular case study, the elements of the matrices H and D_j were restricted to take only positive values. The corresponding Problem II.4 has 65 optimization variables, 26 equality and 70 inequality constraints, and yields the solution $K = \left[\begin{smallmatrix} 0.0037 & -0.2965 \\ 0 & 0 \end{smallmatrix}\right]$ and $\varepsilon = 0.9875$. This solution was obtained for the candidate ε-contractive set $\hat{\mathbb{P}} = \Psi(G, w_1, w_2)$ with

$$G = \left[\begin{smallmatrix} 0 & -1 \\ 0.8 & 1.16 \\ 1 & 0 \end{smallmatrix}\right], w_1 = \left[\begin{smallmatrix} 0.5 \\ 1.8 \\ 2.5 \end{smallmatrix}\right], w_2 = \left[\begin{smallmatrix} 2.5 \\ 14 \\ 20 \end{smallmatrix}\right]. \tag{23a}$$

The steady state input $u^s = \left[\begin{smallmatrix} 0.8157 \\ 0.4 \end{smallmatrix}\right]$ was computed using (16). The control input is computed at each time instant using (14a). The trajectories of closed-loop system starting from the vertices of the ε-contractive set $\mathbb{P}$ are shown in Fig. 2.

A similar technique, as described in Section IV, was applied to obtain a linear state-feedback controller for the linearized system model around x^s. The same candidate ε-contractive set was imposed as for the auxiliary bilinear system. Note that the uniqueness of the steady-state input for a specified x^s requires the same u^s for both linearized

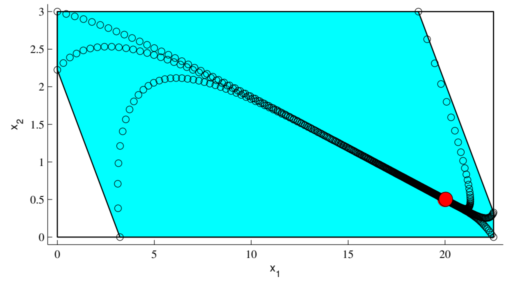

Fig. 2. Closed loop trajectories with linear state-feedback synthesized using average bilinear model of the converter - black circle; range of operation - white box with black borders; $\mathbb{P}$ - cyan; equilibrium state - red disk.

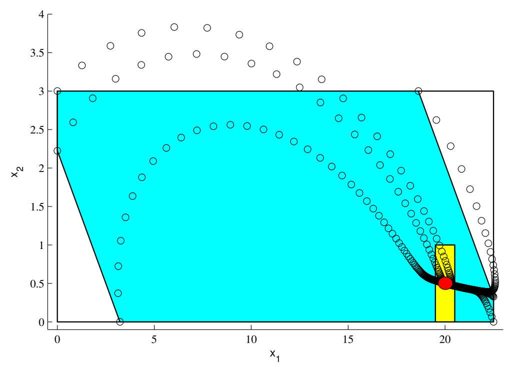

Fig. 3. Closed loop trajectories with linear state-feedback synthesized using linearized model of the converter - black circle; range of operation - white box with black borders; $\mathbb{P}$ - cyan; $\mathbb{S}$ - yellow box; equilibrium state - red disk.

and bilinear models. Problem II.4 in this setup has 38 optimization variables, 14 equality and 43 inequality constraints, and yields the solution $K_l = \left[\begin{smallmatrix} -0.0091 & -0.0635 \\ -0.0135 & 0.1324 \end{smallmatrix}\right]$ and $\varepsilon_l = 0.9823$. Note that in this case the feasibility of the controller synthesis methodology does not guarantee the stability of the closed-loop system due to differences between linearized and bilinear models. For example, to illustrate the significance of these differences, consider the set $\mathbb{S} := \Psi\left(I_2, \left[\begin{smallmatrix}0.5\\0.5\end{smallmatrix}\right], \left[\begin{smallmatrix}0.5\\0.5\end{smallmatrix}\right]\right)$. The average one-step prediction error of a linearized system model, in comparison with the bilinear model, turned out to be higher than 12% within $\mathbb{S}$. In Fig. 3, the yellow rectangle represents the set $\mathbb{S} \oplus \{x^s\}$.

The trajectories of the closed-loop system with the controller designed using a linear model, for all vertices of $\mathbb{P}$, are plotted in Fig. 3. These trajectories clearly violate the state constraints. The results shown in Fig. 2 and Fig. 3 clearly illustrate the advantages of a synthesis method that is applicable to the full bilinear averaged model.

As it can be seen from Fig. 2, the ε-contractive set $\mathbb{P}$ does not contain the origin, which is a regular starting point for the converter. One possible solution is to apply a constant input $u(k) = u_{ct}$ until $x(k) \in \mathbb{P}$, $k \in \mathbb{Z}_+$. After the system state reaches the ε-contractive set the affine state-feedback control law (14a) can be applied. In this specific case it is also sufficient to clamp the control input, which is illustrated by the simulation result in Fig. 4. In any case, the satisfaction

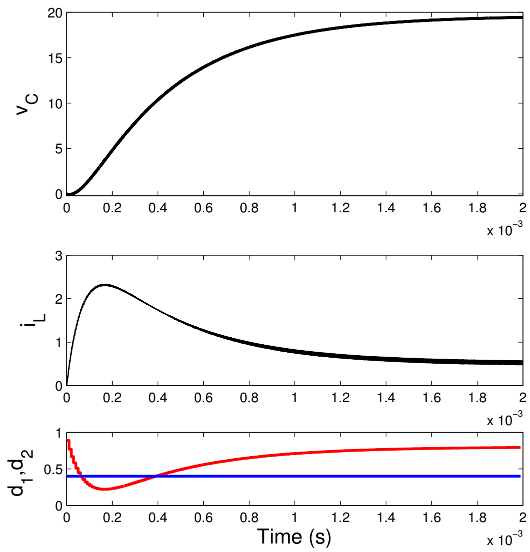

Fig. 4. Closed-loop trajectories for startup in $x(0) = [0,\ 0]^\top$ *(d_1-red, d_2-blue)*. Simulation performed using the continuous time switched model of the converter *(PWM frequency - 200kHz)*.

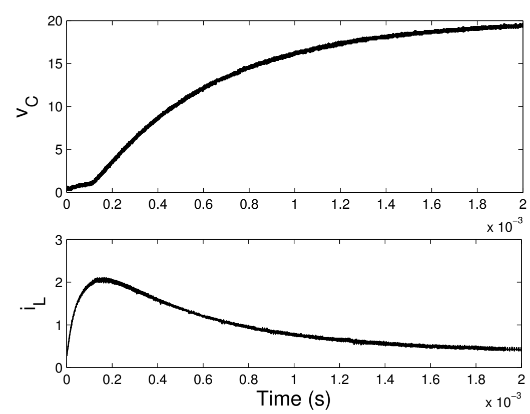

Fig. 5. Waveforms of output voltage and inductor current as measured by an oscilloscope.

of the constraint $v_C \geq 0$ is not guaranteed for the startup from the origin and under a constant load current.

The proposed design method of an affine state-feedback control law was tested in real-time on a real-life hardware platform. A circuit with the same parameters as the ones employed in the previous simulations was used for the experiments. A hardware implementation of the controller was obtained and executed on the Virtex 5 FPGA device on board of the NI PXI-7852R multifunction DAQ from National Instruments. The evolution of output voltage of the converter at startup is shown in Fig. 5.

VI. Conclusions

This paper proposed a novel, set-theoretic method for constrained stabilization of DC/DC power converters. The developed method makes use of a recent result [10] on the stabilization of bilinear discrete-time systems with zero as equilibrium. The main contribution was to design a coordinate transformation that renders this result applicable to buck-boost DC/DC power converters, which typically have a non-zero equilibrium. The resulting synthesis has several advantages, which include low computational complexity, an a priori guarantee of stability and constraints satisfaction and applicability to the full bilinear averaged model of the converter.

Future research deals with further enlarging the region of attraction via alternative synthesis methods for constrained discrete-time bilinear systems and optimizing the speed of convergence.

VII. Acknowledgements

The research presented in this paper is supported by the Veni grant “Flexible Lyapunov Functions for Real-time Control”, grant number 10230, awarded by STW and NWO as well as by the IOP EMVT-II project “Ultra-High Precision Power Amplifier”, number 08201.

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