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# Real-time frequency estimation of sinusoids with low-frequency disturbances

Shuonan Yang and Qing Zhao*

***Abstract*— This paper mainly concentrates on extracting dominant frequency estimates from sinusoidal signals corrupted with low frequency disturbances perturbation by means of a synthetic identifier in a real-time manner. It is assumed that the frequencies of such signals are unknown, and the magnitudes of the dominant signals are sufficiently large compared with those of disturbances. A state space based model for such sinusoidal signal is presented. The proposed frequency identifier consists of a high-gain observer and a gradient based estimator. The utilization of the observer is for effectively attenuating the low frequency disturbances so that the dominant frequency components can be isolated. The gradient estimators can then operate on the estimated states from the observer to generate frequency estimation.**

## I. INTRODUCTION

The online estimation of frequency and amplitude of sinusoidal signals is an important and classical problem, which has received much attention from the systems and controls community. In [1], a globally convergent frequency estimator was proposed based on adaptive notch filter design for a sinusoidal signal with single frequency. By using a state space realization of the sinusoidal signal, the frequency estimation problem can be converted to combined state and parameter estimation problem, a well-known yet challenging problem in controls. Adaptive observers based global frequency estimator was then developed, [2],[3]. By considering white noise and time-varying frequencies, a modified Kalman filter was designed for frequency estimation, [4]. Reference [5] provides a disturbance observer incorporating adaptive parameter dynamics based on the least-mean-square (LMS) and recursive least square (RLS) method and positive result on a discrete signal with white noise. Reference [6] and [7] uses robust adaptive modified Newton optimization to accomplish frequency estimation on discrete sinusoids with white noise, where instant changes on frequencies are tolerant. Recently, the approach was extended to simultaneous reconstruction of multiple frequencies and amplitudes for a signal containing $n$ unknown sinusoids, [8]. Although the convergence and boundedness of the estimator can be proved analytically, [9] tunes parameters analytically by analyzing the convergence of the frequency estimator with the nonlinear contraction theory, in order to guarantee satisfactory estimation results.

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S. Yang and Q. Zhao are with the Department of Electrical and Engineering, University of Alberta, Edmonton, AB T6G 2V4, Canada. qingz@ualberta.ca

*: the corresponding author.

It is noticeable that the current research results of frequency estimation generally deal with purely sinusoids or sinusoids corrupted with white noise. In contrast, this paper focuses on a type of signals with unknown but bounded low frequency disturbances, technically hardly to be removed by the methods above. A state space model of the signal is formed so that each frequency component is represented by two state. Due to the lack of information, the perturbations can only be attenuated rather than removed. Based on this model, a synthetic frequency identifier is proposed, which consists of a high-gain observer and a gradient based estimator. The utilization of the observer is for effectively attenuating the low frequency disturbances so that the dominant frequency components can be isolated. The gradient estimators can then operate on the estimated states from the observer to generate frequency estimation. An output transformation is applied that can help compress the range of the observation error. The synthetic identifier has provided more satisfactory simulation results on perturbed signals, compared with those from [8].

The remainder of this paper is organized as follows: Modeling of the signal and the problem formulation are given in section II; The real-time estimation of perturbed sinusoids are presented in section III, where a high-gain observer based on [10],[11] is introduced and an adaptive frequency estimator is developed; The section IV includes the simulation results that demonstrates the effectiveness of the proposed identifier. Finally section V concludes the paper.

## II. MODELING & FORMULATION

Consider a sinusoidal signal with $n$ nominal frequency components and the disturbance:

$$y(t) = \sum_{i=1}^{n}(A_i \sin(w_i t + \phi_i)) + d(t). \tag{1}$$

It is assumed that only $y(t)$ is measurable and the number of nominal (main) frequency components $n$ is known, whereas all the parameters are unknown, including $w_i$, $\phi_i$ and $A_i$, $i = 1,\ 2,\ \ldots,\ n$. $d(t)$ is the bounded disturbance containing perturbation components other $w_i$, whose magnitudes are minor ones compared with $\{A_i\}$.

Define sinusoidal states $\mathbf{x} = \begin{bmatrix} x_1, & x_2, & \cdots, & x_{2n} \end{bmatrix}^T$ with

$$\begin{cases} x_{2i-1}(t) = A_i \sin(w_i t + \phi_i) \\ x_{2i}(t) = A_i \cos(w_i t + \phi_i), \end{cases} \tag{2}$$

we have

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & w_1 & 0 & 0 & \cdots & 0 & 0 \\ -w_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & w_2 & \cdots & 0 & 0 \\ 0 & 0 & -w_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & w_n \\ 0 & 0 & 0 & 0 & \cdots & -w_n & 0 \end{bmatrix} \mathbf{x}$$
$$= \mathbf{A}\mathbf{x} = \mathbf{A}_0\mathbf{x} + \mathbf{f}. \quad (3)$$

Here we define $\mathbf{A}_0$ and $\mathbf{f}$ considering that the system matrix $\mathbf{A} \triangleq \mathbf{A}(\mathbf{w})$ contains the unknown frequency $w$. Give an initial guess of $\mathbf{w}$, denoted as $\mathbf{w}_0$, and define $\mathbf{A}_0 = \mathbf{A}(\mathbf{w}_0)$, then the estimation deviation $\Delta\mathbf{A}(\mathbf{w}, \mathbf{w}_0) = \mathbf{A} - \mathbf{A}_0$, and $\mathbf{f}(t) = \Delta\mathbf{A}\mathbf{x}(t)$.

Having proposed a state equation with known coefficients and uncertainties, we start to focus on the establishment of an output equation. Define an augmented output vector $\bar{\mathbf{y}} = \begin{bmatrix} y, & \dot{y}, & \cdots, & y^{(2n-1)} \end{bmatrix}^T$ and similarly an augmented disturbance vector $\bar{\mathbf{d}} = \begin{bmatrix} d, & \dot{d}, & \cdots, & d^{(2n-1)} \end{bmatrix}^T$ by including the measurement of the signal and its derivatives up to $(2n-1)^{\text{th}}$ order. Similarly, the output matrix $\mathbf{C}$ can be written as $\mathbf{C} = \mathbf{C}_0 + \Delta\mathbf{C}$, where $\mathbf{C}_0 = \mathbf{C}(\mathbf{w}_0)$ as $\mathbf{C} = \mathbf{C}(\mathbf{w})$. Here we have formed the output equation (4) with $\mathbf{v}(t) = \Delta\mathbf{C}\mathbf{x}(t) + \bar{\mathbf{d}}(t)$.

A state space model is now established for the sinusoidal signal corrupted with the disturbance:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x} + \mathbf{f} \\ \bar{\mathbf{y}} = \mathbf{C}_0\mathbf{x} + \mathbf{v}. \end{cases} \quad (5)$$

***Remark** 1:* Note that the derivatives of the output up to $(2n-1)^{\text{th}}$ order are computed in this model for the proposed state estimator to work properly. It will be shown later that absence of the derivatives will lead to the infeasible solutions of the observer. The reasons will be explained in detail in Remark 2. In other words, the measurement of the signal itself alone cannot provide sufficient information about the unknown frequencies, which appear as the unknown parameters in the state space model.

The state space models for sinusoidal signals are not unique, and different models (linear or nonlinear) have been presented in previous work for frequency estimation using model based techniques, e.g. [3],[8]. The model presented in this paper is one of the linear models and it highlights the treatment of the disturbances.

## III. Real-time Estimation of Perturbed Sinusoids

Based on the model in (3-4), our goal is to design a frequency identifier that can identify and estimate in real time the dominant frequencies of the sinusoidal signals, despite the existence of frequency perturbations in the low or medium spectrum range. This design can find important applications in machine condition monitoring and diagnosis, for instance, trending and monitoring non-stationary vibration signals of the rotating machine. Reference [8] provided a complete coverage on frequency identification and amplitude estimation for pure sinusoidal signals without noise or disturbances, whereas it fails to converge to the desired values in the case of obvious disturbances. On the contrary, this paper has provided restriction on these disturbance frequency components, as long as their magnitudes are minor compared with those of the main frequency components.

The proposed frequency identifier contains an observer and a frequency estimator. The observer estimates every state as each one contains one main frequency, while the frequency estimator generates the real-time frequency estimates $\{\hat{w}_i\}$ from the state estimates. Fig. 1 shows the design scheme.

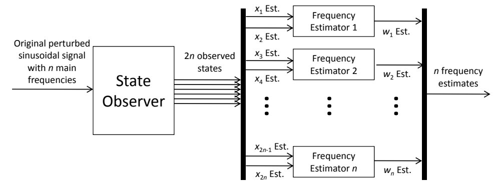


Fig. 1. The Scheme of the Synthetic Identifier

### A. High-gain observer based state estimation

Note that the unknown frequency information of the signal are treated as modeling uncertainties and the disturbance as unknown input terms, the objective for the observer is to reduce the effects of the uncertainties and disturbance as much as possible in the first step of the identification process. In this case, a typical solution is to adopt a robust observer. Hereby a high gain observer is selected for this purpose. A high gain observer may have great improvement on state estimation for systems with modeling errors or external disturbances, in which their effects can be effectively compressed within a bounded zone that can be made "small" by proper tuning of the design. Reference [10],[11] have provided such a high gain observer for a descriptor system, based on which our research has been developed. For the model given in (3-4), one can augment the states as: $\bar{\mathbf{x}} \triangleq \begin{bmatrix} \mathbf{x}^T & \mathbf{f}^T & \mathbf{v}^T \end{bmatrix}^T \in \mathcal{R}^{6n}$. As a result, an augmented $6n^{\text{th}}$ order descriptor state space expression can be obtained:

$$\begin{cases} \bar{\mathbf{E}}\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{H}}\dot{\mathbf{f}} + \bar{\mathbf{N}}\mathbf{v} \\ \bar{\mathbf{y}} = \bar{\mathbf{C}}\bar{\mathbf{x}}. \end{cases} \quad (6)$$

where $\bar{\mathbf{E}}$, $\bar{\mathbf{A}}$, $\bar{\mathbf{H}}$, $\bar{\mathbf{N}}$ are exactly in the same form as in [10], and $\bar{\mathbf{C}} \triangleq \begin{bmatrix} \mathbf{C}_0 & \mathbf{0} & \mathbf{I}_{2n} \end{bmatrix}$.

Define

$$\bar{\mathbf{S}} \triangleq \begin{bmatrix} \mathbf{I}_{2n} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{2n} & \mathbf{0} \\ \mathbf{M}\mathbf{C}_0 & \mathbf{0} & \mathbf{M} \end{bmatrix}, \bar{\mathbf{L}} = \begin{bmatrix} \mathbf{0}_{2n} & \mathbf{0}_{2n} & \mathbf{M}^T \end{bmatrix}^T; \quad (7)$$

$\mathbf{M} \in \mathcal{R}^{n \times n}$ is a nonsingular matrix to be selected. It provides

$$\bar{\mathbf{C}}\bar{\mathbf{S}}^{-1}\bar{\mathbf{L}} = \mathbf{I}_{2n}, \quad \bar{\mathbf{A}}\bar{\mathbf{S}}^{-1}\bar{\mathbf{L}} = -\bar{\mathbf{N}}.$$

By adopting the high-gain observer structure in [10], the observer based on the signal model given in (3-4) is given as follows with the estimate error dynamics:

$$\bar{\mathbf{S}}\dot{\hat{\bar{\mathbf{x}}}} = \bar{\mathbf{A}}\hat{\bar{\mathbf{x}}} + \bar{\mathbf{K}}(\bar{\mathbf{y}} - \bar{\mathbf{C}}\hat{\bar{\mathbf{x}}}) + \bar{\mathbf{L}}\dot{\bar{\mathbf{y}}}, \quad (8)$$
$$\dot{\bar{\mathbf{e}}} = \bar{\mathbf{S}}^{-1}(\bar{\mathbf{A}} - \bar{\mathbf{K}}\bar{\mathbf{C}})\bar{\mathbf{e}} + \bar{\mathbf{N}}\mathbf{M}^{-1}\mathbf{v} + \bar{\mathbf{H}}\dot{\mathbf{f}}, \quad (9)$$

$$
\begin{aligned}
\bar{\mathbf{y}} &= \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ 0 & w_1 & 0 & w_2 & \cdots & 0 & w_n \\ -w_1^2 & 0 & -w_2^2 & 0 & \cdots & -w_n^2 & 0 \\ 0 & -w_1^3 & 0 & -w_2^3 & \cdots & 0 & -w_n^3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-w_1^2)^{n-1} & 0 & (-w_2^2)^{n-1} & 0 & \cdots & (-w_n^2)^{n-1} & 0 \\ 0 & w_1(-w_1^2)^{n-1} & 0 & w_2(-w_2^2)^{n-1} & \cdots & 0 & w_n(-w_n^2)^{n-1} \end{bmatrix} \mathbf{x} + \begin{bmatrix} d \\ \dot{d} \\ \ddot{d} \\ d^{(3)} \\ \vdots \\ d^{(2n-2)} \\ d^{(2n-1)} \end{bmatrix} \\
&= \mathbf{C}\mathbf{x} + \bar{\mathbf{d}} = \mathbf{C}_0\mathbf{x} + \mathbf{v} \qquad (4)
\end{aligned}
$$

$\bar{\mathbf{K}}, \bar{\mathbf{L}}$ are the observer matrices. Reference [10] has shown a stabilizing $\bar{\mathbf{K}}$ can be obtained for the error dynamic equation, i.e.

$$\bar{\mathbf{K}} = \bar{\mathbf{S}}\bar{\mathbf{P}}^{-1}\bar{\mathbf{C}}^T, \qquad (10)$$

where $\bar{\mathbf{P}}$ is the solution of the Lyapunov equation

$$\tilde{\mathbf{A}}^T\bar{\mathbf{P}} + \bar{\mathbf{P}}\tilde{\mathbf{A}} = -\bar{\mathbf{C}}^T\bar{\mathbf{C}}. \qquad (11)$$

$\tilde{\mathbf{A}} \triangleq -(\mu\mathbf{I} + \bar{\mathbf{S}}^{-1}\bar{\mathbf{A}})$, and $\mu$ is a positive scalar satisfying $\Re[\lambda_i(\bar{\mathbf{S}}^{-1}\bar{\mathbf{A}})] > -\mu$ to make $\tilde{\mathbf{A}}$ Hurwitz. Lemma 1 provides a sufficient and necessary condition for the Lyapunov equation to have a unique positive definite solution $\bar{\mathbf{P}}$:

***Lemma** 1:* The Lyapunov equation in (11) has a unique positive definite solution $\bar{\mathbf{P}}$ if and only if $\mathrm{rank}(\mathbf{C}_0) = 2n$. The proof is included in the appendix.

The high-gain observer and the above lemma can guarantee the the estimation error $\bar{\mathbf{e}}$ converges to a range with finite bounds. Select the Lyapunov function as $V(t) = \bar{\mathbf{e}}^T\bar{\mathbf{P}}\bar{\mathbf{e}}$. Reference [10],[11] has shown that $\bar{\mathbf{e}}$ will be finally bounded in a range given as follows,

$$
\begin{aligned}
\|\bar{\mathbf{e}}(t)\| &\leq \frac{1}{\mu}\frac{\lambda_{\max}[\bar{\mathbf{P}}^{1/2}]}{\lambda_{\min}[\bar{\mathbf{P}}^{1/2}]}(\|\dot{\mathbf{f}}\| + \|\mathbf{M}^{-1}\|\|\mathbf{v}\|) \\
&\triangleq \alpha\|\dot{\mathbf{f}}\| + \beta\|\mathbf{v}\| \qquad (12) \\
&\leq \alpha\|\Delta\mathbf{A}\|\|\dot{\mathbf{x}}\| + \beta(\|\Delta\mathbf{C}\|\|\mathbf{x}\| + \|\bar{\mathbf{d}}\|) \qquad (13)
\end{aligned}
$$

where

$$\alpha \triangleq \frac{\lambda_{\max}[\bar{\mathbf{P}}^{1/2}]}{\mu\lambda_{\min}[\bar{\mathbf{P}}^{1/2}]}, \qquad \beta \triangleq \frac{\lambda_{\max}[\bar{\mathbf{P}}^{1/2}]}{\mu\lambda_{\min}[\bar{\mathbf{P}}^{1/2}]}\|\mathbf{M}^{-1}\| \qquad (14)$$

***Remark** 2:* In the right hand of the inequality above, the state vector $\mathbf{x}$ and the disturbance $\mathbf{d}$ are bounded; $\|\Delta\mathbf{A}\|$ and $\|\Delta\mathbf{C}\|$ depend on the initial frequency estimation errors, $w_i - w_{0i}$. Hence, it is clear that the coefficients $\alpha$ and $\beta$ play an important role in determining the size of the estimation error bounds. One way to reduce $\alpha$ and $\beta$ as suggested in [10] is to select a large $\mu$ value. However, unlike what was initially claimed in [10], larger $\mu$ value may not always guarantee smaller $\alpha$ or $\beta$ because of the relationship between $\mu$ and the eigenvalues of the matrix $\mathbf{P}$; one way to reduce $\alpha$ and $\beta$ is demonstrated in Part III-B. Furthermore, according to Lemma 1, (13), and (14), it is necessary to include the derivatives of $y$ up to the $(2n-1)^{\mathrm{th}}$ order in the extended output vector so that a finite estimation error bound exists.

*B. Output Transformation*

In this section, an output transformation is proposed in order to improve the observation performance of the high-gain observer. Define $\mathbf{T}$ as the transformation matrix operated on the output matrix, i.e., $\mathbf{C_T} \triangleq \mathbf{T}\mathbf{C}_0$. The modified output equation becomes:

$$\bar{\mathbf{y}}_{\mathbf{T}} = \mathbf{T}\mathbf{C}_0\mathbf{x} + \mathbf{T}(\Delta\mathbf{C}\mathbf{x} + \bar{\mathbf{d}}) = \mathbf{C_T}\mathbf{x} + \mathbf{T}\mathbf{v}. \qquad (15)$$

With the redefinition $\bar{\mathbf{C}} = \begin{bmatrix}\mathbf{C_T} & \mathbf{0} & \mathbf{I}\end{bmatrix}$, the expression of $\alpha$ remains unchanged, whereas the expression of $\beta$ becomes $\beta \triangleq \lambda_{\max}[\bar{\mathbf{P}}^{1/2}]\|\mathbf{M}\|^{-1}\|\mathbf{T}\|/(\mu\lambda_{\min}[\bar{\mathbf{P}}^{1/2}])$ so that $\|\bar{\mathbf{e}}\| \leq \alpha\|\dot{\mathbf{f}}\| + \beta\|\mathbf{v}\|$ still holds.

One intuition on pursuing a convenient $\mathbf{T}$ appears as the reduction of the ratio $\lambda_{\max}[\bar{\mathbf{P}}]/\lambda_{\min}[\bar{\mathbf{P}}]$, whose square root is a factor in $\alpha$ and $\beta$; now we start to figure out the relation between this ratio and $\mathbf{T}$. As $\bar{\mathbf{P}}$ is strictly positive definite in the case that $(\tilde{\mathbf{A}}, \bar{\mathbf{C}})$ is observable, a positive lower bound of $\lambda_{\min}[\bar{\mathbf{P}}]$ exists, so does a finite upper bound of the ratio. Reference [13] takes advantage of the observability of and separates a positive definite matrix $\mathbf{G}$, which is only decided by $\tilde{\mathbf{A}}$:

$$
\begin{aligned}
\lambda_{\min}[\mathbf{G}]\lambda_{\max}[\mathcal{O}^T\mathcal{O}] \leq \lambda_{\max}[\bar{\mathbf{P}}] \leq \lambda_{\max}[\mathbf{G}]\lambda_{\max}[\mathcal{O}^T\mathcal{O}], \\
\lambda_{\min}[\mathbf{G}]\lambda_{\min}[\mathcal{O}^T\mathcal{O}] \leq \lambda_{\min}[\bar{\mathbf{P}}] \leq \lambda_{\max}[\mathbf{G}]\lambda_{\min}[\mathcal{O}^T\mathcal{O}].
\end{aligned}
$$

We then achieve

$$\frac{\lambda_{\min}[\mathbf{G}]\lambda_{\max}[\mathcal{O}^T\mathcal{O}]}{\lambda_{\max}[\mathbf{G}]\lambda_{\min}[\mathcal{O}^T\mathcal{O}]} \leq \frac{\lambda_{\max}[\bar{\mathbf{P}}]}{\lambda_{\min}[\bar{\mathbf{P}}]} \leq \frac{\lambda_{\max}[\mathbf{G}]\lambda_{\max}[\mathcal{O}^T\mathcal{O}]}{\lambda_{\min}[\mathbf{G}]\lambda_{\min}[\mathcal{O}^T\mathcal{O}]}, \qquad (16)$$

where $\mathcal{O}$ is the observability matrix of $(\tilde{\mathbf{A}}, \bar{\mathbf{C}})$, i.e., $\mathcal{O} \triangleq \begin{bmatrix}\bar{\mathbf{C}}^T, & \tilde{\mathbf{A}}^T\bar{\mathbf{C}}^T, & \cdots, & (\tilde{\mathbf{A}}^{6n-1})^T\bar{\mathbf{C}}^T\end{bmatrix}^T$.

Equation (16) implies that an output transformation matrix $\mathbf{T}$ tends to reduce the ratio $\lambda_{\max}[\bar{\mathbf{P}}]/\lambda_{\min}[\bar{\mathbf{P}}]$ if it is able to bring down $\lambda_{\max}[\mathcal{O}^T\mathcal{O}]/\lambda_{\min}[\mathcal{O}^T\mathcal{O}]$, which is proportional to both the upper and the lower bounds of that ratio.

*C. Gradient Based Frequency Estimator*

The model above has shown the $2n$ real states $\mathbf{x}$ are respectively corresponding to $n$ frequencies. As the high gain observer generates $2n$ observed states $\hat{\mathbf{x}}$ are the counterpart of $\mathbf{x}$, we may define $n$ gradient-based identifiers following the general concept of gradient algorithms for parameter identification [14], in order to provide convergent or bounded estimation of $n$ main frequencies. Each estimator will use the $i^{\mathrm{th}}$ pair of the estimated states $\begin{bmatrix}\hat{x}_{2i-1}, & \hat{x}_{2i}\end{bmatrix}^T$ as its input.

Define the identifier dynamics

$$\dot{\hat{w}}_i = g_i \hat{x}_{2i}(\dot{\hat{x}}_{2i-1} - \hat{w}_i \hat{x}_{2i}), \quad (17)$$

where $g_i$ is a selected positive gain.

As (13) has shown that the state observation error $\bar{\mathbf{e}}$ is bounded, $e_{2i}$, $e_{2i-1}$ are bounded, also resulting in the boundedness of $\dot{e}_{2i-1}$ (see (9)). Since $x_{2i-1}$ has the property that $\dot{x}_{2i-1} = w_i x_{2i}$, the estimate error dynamics is:

$$\begin{aligned} \dot{\tilde{w}}_i = -\dot{\hat{w}}_i &= -g_i \hat{x}_{2i}(\dot{\hat{x}}_{2i-1} - \hat{w}_i \hat{x}_{2i}) \\ &= -g_i \hat{x}_{2i}(w_i x_{2i} - \dot{e}_{2i-1} - w_i \hat{x}_{2i} + \tilde{w}_i \hat{x}_{2i}) \\ &= -g_i \hat{x}_{2i}^2 \tilde{w}_i + g_i \hat{x}_{2i}(\dot{e}_{2i-1} - w_i e_{2i}). \quad (18) \end{aligned}$$

which is actually a linear time-varying (LTV) system with a BIBS dynamics and a bounded input. The following lemma provides the feasibility of such a gradient frequency-based estimator, for which the proof is in Appendix 2.

***Lemma** 2:* The gradient frequency-based estimator as in (30) generates bounded $\tilde{w}_i$, $\forall i = 1, 2, \cdots, n$, when $\begin{bmatrix}\hat{x}_{2i-1}, & \hat{x}_{2i}\end{bmatrix}^T$ is the corresponding estimated state of the high gain observer discussed in Subsection III-A and $\hat{x}_{2i}$ satisfies the persistency of excitation condition, i.e. $\exists$ constant $T_i > 0$, $k_i > k_{0_i} > 0$, $\forall t$, s.t. $\int_t^{t+T_i} \hat{x}_{2i}^2(\tau) d\tau \geq k_i T_i > k_{0_i} T_i > 0$.

Lemma 2 can be proved in the following theoretic structure: find positive constants $b_i < b_i'$, while proving the following steps in order:

1) $\forall t, \exists \varepsilon > 0$, s.t. $|\tilde{w}_i(t+T_i)| - |\tilde{w}_i(t)| < -2g_i\varepsilon < 0$, if $|\tilde{w}_i(\tau)| \geq b_i, \forall \tau \in [t, t+T_i]$.
2) Assume at arbitrary $t_0$, $|\tilde{w}_i(t_0)| > b_i$, then $\exists n \in \mathbb{N}$, $t_1 \in [t_0, t_0 + (n+1)T_i]$, s.t. $|\tilde{w}_i(t_1)| = b_i$.
3) Assume at arbitrary $t_1$, $|\tilde{w}_i(t_1)| = b_i$, then $\max_{t \geq t_1} |\tilde{w}_i(t)| \leq b_i' < \infty$.

*Proof:*

*1) Proof of Step 1:* Define the Lyapunov function $V_i(t) \triangleq \tilde{w}_i^2/(2g_i)$. We have

$$\dot{V}_i(t) = -\hat{x}_{2i}^2 \tilde{w}_i^2 + \hat{x}_{2i}\tilde{w}_i(\dot{e}_{2i-1} - w_i e_{2i}). \quad (19)$$

As $e_{2i}$, $e_{2i-1}$ and $\dot{e}_{2i-1}$ are bounded, $\hat{x}_{2i} = x_{2i} - e_{2i}$ is also bounded. Hence, we may define

$$b_i = \|\hat{x}_{2i}(\dot{e}_{2i-1} - w_i e_{2i})\| / k_{0_i}, \quad (20)$$

where $k_{0_i}$ is the positive constant defined in the assumption of PE condition. As a result, (19) can be transformed into $\dot{V} \leq -\hat{x}_{2i}^2 \tilde{w}_i^2 + |\tilde{w}_i(t)| b_i k_{0_i}$.

Define $V_{s_i}(t) \triangleq \sqrt{V_i/2g_i} = |\tilde{w}_i(t)|/(2g_i)$. Then

$$\dot{V}_{s_i}(t) = \frac{V^{-1/2}}{\sqrt{2g_i}} \dot{V} = \frac{\dot{V}}{|\tilde{w}_i(t)|} \leq -\hat{x}_{2i}^2(t)|\tilde{w}_i(t)| + b_i k_{0_i}. \quad (21)$$

When $|\tilde{w}_i(\tau)| \geq b_i, \forall \tau \in [t, t+T_i]$, it is deduced that

$$V_{s_i}(t+T_i) - V_{s_i}(t) \leq \int_t^{t+T_i} (-\hat{x}_{2i}^2(\tau)|\tilde{w}_i(\tau)| + b_i k_{0_i}) d\tau$$

$$\leq -b_i \int_t^{t+T_i} \hat{x}_{2i}^2(\tau) d\tau + b_i k_{0_i} T_i \leq b_i(-k_i + k_{0_i})T_i < 0.$$

Define $\varepsilon \triangleq (k_i - k_{0_i}) b_i T_i$, then

$$V_{s_i}(t+T_i) - V_{s_i}(t) \leq -\varepsilon < 0 \quad (22)$$

$$\Rightarrow \quad |\tilde{w}_i(t+T_i)| - |\tilde{w}_i(t)| \leq -2g_i\varepsilon < 0 \quad (23)$$

Step 1 is thus proved.

*2) Proof of Step 2:* As $V_{s_i}(t_0) = |\tilde{w}_i(t_0)|/(2g_i) > b_i/(2g_i)$, we define

$$n_i = \left\lfloor \frac{V_{s_i}(t_0) - \frac{b_i}{2g_i}}{\varepsilon} \right\rfloor + 1 > \frac{V_{s_i}(t_0) - \frac{b_i}{2g_i}}{\varepsilon}. \quad (24)$$

Here we begin to prove that $\exists t_1 \in [t_0, t_0 + (n_i+1)T_i]$, s.t. $|\tilde{w}_i(t)| = b_i$, by showing that its negative proposition, i.e. $\forall t \in [t_0, t_0 + (n_i+1)T_i]$, s.t. $|\tilde{w}_i(t)| > b_i$, is false. The following analysis is carried out in a cycle-by-cycle manner, and only the worst cases (as defined in the following paragraphs) are considered, since any other case automatically provides the reality of Step 2.

Assume that the worst case happens in the first cycle $[t_0, t_0 + T_i]$, i.e. $\forall t \in [t_0, t_0 + T_i]$, $|\tilde{w}_i(t)| > b_i$. Then according to Step 1, we have $V_{s_i}(t_0 + T_i) - V_{s_i}(t_0) \leq -\varepsilon < 0$. Repeat this deduction for $(n_i + 1)$ cycles in total under the assumption of the accumulation of the worst cases, i.e. the occurrence of the negative proposition of Step 2, then we have $V_{s_i}(t_0 + (n_i+1)T_i) - V_{s_i}(t_0) \leq -n_i\varepsilon < 0$. However,

$$\begin{aligned} (24) \quad &\Rightarrow \quad V_{s_i}(t_0 + (n_i+1)T_i) \leq V_{s_i}(t_0) - n_i\varepsilon < b_i/(2g_i) \\ &\Rightarrow \quad |\tilde{w}_i(t_0 + (n_i+1)T_i)| < b_i, \end{aligned}$$

which is contradict to the negative proposition of Step 2. In other words, $\exists t_1 \in [t_0, t_0 + (n_i+1)T_i]$, s.t. $|\tilde{w}_i(t)| = b_i$.

*3) Proof of Step 3:* Define

$$b_i' = (1 + 2g_i k_{0_i} T_i) b_i. \quad (25)$$

Here we begin to prove that $\max_{t \geq t_1} |\tilde{w}_i(t)| \leq b_i'$, by showing that its negative proposition, i.e. $\exists t_2 > t_1$, s.t. $|\tilde{w}_i(t_2)| > b_i'$, is false.

Assume the negative proposition holds. Without loss of generality, assume $t_1$ as the last time instant before $t_2$ that $\tilde{w}_i$ escapes from the bounded zone, i.e., $|\tilde{w}_i(t_1)| = b_i$ and $\forall t \in [t_1, t_2]$, $|\tilde{w}_i(t)| \geq b_i$ (equality only applies at the two ends). Obviously, $t_2 < t_1 + T_i$, otherwise $V_{s_i}(t_1 + T_i) \geq V_{s_i}(t_1)$, which is contradict to Step 1. Then it can be deduced that

$$\begin{aligned} V_{s_i}(t_2) &= V_{s_i}(t_1) + \int_{t_1}^{t_2} \dot{V}_{s_i}(\tau) d\tau \\ &< V_{s_i}(t_1) + \int_{t_1}^{t_1+T_i} b_i k_{0_i} d\tau \text{ (from (21))} \\ &= \frac{b_i}{2g_i} + b_i k_{0_i} T_i = \frac{b_i'}{2g_i} \\ \Rightarrow |\tilde{w}_i(t_2)| &< b_i', \end{aligned}$$

which is contradict to the negative proposition under the condition $\forall t \in [t_1, t_2]$, $|\tilde{w}_i(t)| > b_i$, i.e., such $t_2$ does not exist after the nearest hitting time $t_1$.

The proof above actually covers all the time pieces during which $|\tilde{w}_i(t)| \geq b_i$ (equality only applies at the two ends), as well as the period after the last hitting time in which

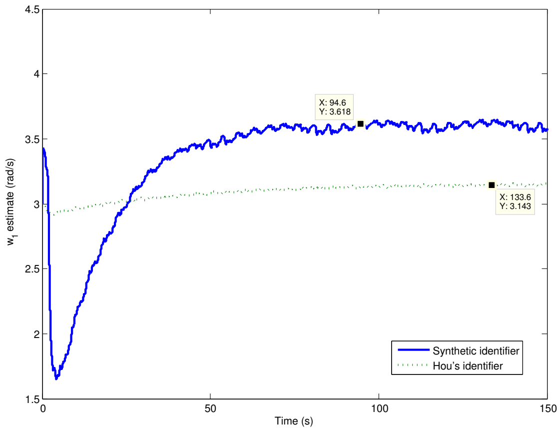


Fig. 2. Curves of frequency estimate by synthetic identifier and Hou's identifier. Nominal value of the main frequency: $w$ = 3.6.

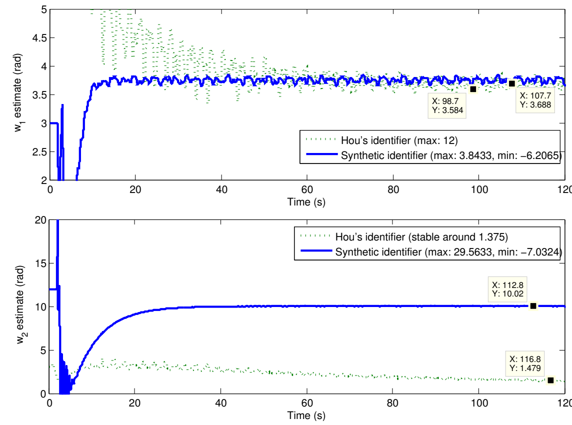


Fig. 3. Curves of frequency estimate by synthetic identifier and Hou's identifier. Nominal Values of the main frequencies: $w_1 = 3.6$, $w_2 = 10$.

$|\tilde{w}_i(t)| > b_i$. For all uncovered ones, during which $|\tilde{w}_i(t)| \leq b_i$, $|\tilde{w}_i(t)|$ is naturally in the zone bounded by $\pm b_i'$. Hence $|\tilde{w}_i(t)| \leq b_i'$ from $t_1$, i.e., Step 3 is proved. ■

## IV. Simulation

Two different simulations[1], respectively regarding one and more main frequencies, have been carried out. Our synthetic identifier has presented more convincing performance than Hou's real-time identifier [8].

### A. One Useful Frequency

Consider the case with one useful frequency and two sinusoidal perturbations. Select the useful frequency $w = 3.6$, two perturbation frequencies $w_{d1} = 1$, $w_{d2} = 5$, corresponding magnitudes $A_1 = 1$, $A_{d1} = 0.18$, $A_{d2} = 0.25$ and phases $\phi_1 = 0$, $\phi_{d1} = \pi/6$, $\phi_{d2} = -\pi/5$.

Take an initial guess of useful frequency: $w_0 = 3$, which determines the system model for high gain observer and some coefficient matrices. Now we start to evaluate the two important parameters—$\mu$ and $\mathbf{M}$: for simplicity, $\mathbf{M}$ is selected as a diagonal matrix with identical elements valued $m$. Select $m = 20$ and $\mu = 0.4$ to keep both $\alpha$, $\beta$, and thus $\|\bar{\mathrm{e}}(t)\|$ relatively small. Select the gradient coefficient of our gradient estimator $g = 0.4$ and that of Hou's identifier $\gamma = 10$. Use the input given above respectively on the synthetic identifier (integrating the high gain observer and the gradient identifier) and Hou's identifier [8], then both curves are generated as in Fig. 2. The synthetic identifier has generally reached the estimate objective, reflecting the obvious static error in Hou's result.

### B. Two Useful Frequencies: Output Transformation

Consider the case with two useful frequencies and two sinusoidal perturbations. Select the useful frequency $w_1 = 3.6$, $w_2 = 10$, two perturbation frequencies $w_{d1} = 1$, $w_{d2} = 5$, corresponding magnitudes $A_1 = 1$, $A_2 = 1$, $A_{d1} = 0.18$, $A_{d2} = 0.25$, and corresponding phases $\phi_1 = 0$, $\phi_2 = 4\pi/7$, $\phi_{d1} = \pi/6$, $\phi_{d2} = -\pi/5$.

Take an initial guess of useful frequency: $w_{1_0} = 3$ and $w_{2_0} = 12$. For simplicity, $\mathbf{M}$ is formed as a diagonal matrix with identical elements valued $m$. Computational results have shown that $\alpha$ and $\beta$ are much larger so that the boundary of observation errors are rather loose, and thus the observed states may be relatively inappropriate as inputs of gradient identifiers. For instance, select $\mu = 0.7$ and $\mathbf{M} = 8\mathbf{I}$, then the ratio $\lambda_{\max}[\bar{\mathbf{P}}^{1/2}]/(\mu\lambda_{\min}[\bar{\mathbf{P}}^{1/2}])$, a factor in both $\alpha$ and $\beta$, is $2.9977 \times 10^5$, which is even a small value among those generated by various selections of $\mu$ and $\mathbf{M}$. It reflects the necessity of output transformation: when the output transformation gain $\mathbf{T} = \mathbf{C}_0^{-1}$ is exerted, the ratio is reduced to 229.0673, only weighting about 0.1 percent of that without output transformation.

Select the two gradient coefficients of our gradient estimators $g_1 = 0.3$, $g_2 = 5$, and the gradient coefficients of Hou's identifier $\gamma_1 = 2000$, $\gamma_2 = 50$. When $\mu = 0.7$ and $\mathbf{M} = 8\mathbf{I}$, the curves in Fig. 3 are generated by the synthetic identifier (with $\mathbf{T} = \mathbf{C}_0^{-1}$ applied) and Hou's estimator. Fig. 3 has shown both identifiers have performed satisfactorily with a static error around 0.1 regarding the estimate of $w_1$. However, Hou's identifier cannot converge $\hat{w}_2$ to the nominal value 10 or anywhere nearby, whereas the synthetic identifier has provided a convergent result. In summary, the estimation result of the synthetic estimator has obvious advantage on that from Hou's identifier.

## V. Conclusion

In this paper, we have investigated the problem of estimating unknown frequencies of a given sinusoidal signal with sinusoid perturbation(s). The error convergence property of high gain observer and the Lyapunov stability are selected as the basic guiding theory. Accordingly, a synthetic identification framework integrating high gain observer and gradient estimation components has been formed as the specific tool for solution. The high gain observer provides observed states corresponding to individual frequencies, and the observation

[1]ODE 45 is adopted as the solver through the simulation. Each first-order derivative is replaced by a differential filter $D(s) = 500s/(s + 500)$ to prevent non-causality. The input in the two-main-frequency test is screened in the first 2 seconds in order to dodge the huge spike due to the differential filter based computation.

error is limited within certain bounds, which can also be tightened with output transformation. Gradient estimation components extract frequency estimates from the high gain observer states. Compared with previous research results, the synthetic identifier presents improved performance in numerical simulations, demonstrating its advantages.

## APPENDIX: PROOF OF LEMMA 1

### A. Proof of "⇐"

The following part has demonstrated that such $\mathbf{C}_0$ with rank $< 2n$ will lead to a $\bar{\mathbf{P}}$ which is not strictly positive definite. Firstly we deduce the infeasibility caused by any $q \times 2n$ ($q < 2n$) dimensional $\mathbf{C}_0$. For $n$ frequencies to be estimated, when $q \times 2n$ dimensional $\mathbf{C}$ and $\mathbf{C}_0$ apply, the dimension of output equation becomes

$$\begin{aligned} & \bar{\mathbf{y}}_{q\times 1} = \begin{bmatrix} y & \dot{y} & \ddot{y} & \cdots & y^{(q-1)} \end{bmatrix}^T \\ = \; & \mathbf{C}_{q\times 2n}\mathbf{x}_{2n\times 1} + \bar{\mathbf{d}}_{q\times 1} = (\mathbf{C}_0)_{q\times 2n}\mathbf{x} + \mathbf{v}_{q\times 1}. \end{aligned}$$

It also leads to the variation on the dimension of the augmented states $\hat{\mathbf{x}}_{(4n+q)\times 1} \triangleq \begin{bmatrix} \mathbf{x}^T & \mathbf{f}^T & \mathbf{v}^T \end{bmatrix}^T$, as well as most high gain observer matrices, including $\bar{\mathbf{E}}_{(4n+q)\times(4n+q)}$, $\bar{\mathbf{A}}_{(4n+q)\times(4n+q)}$, $\bar{\mathbf{N}}_{(4n+q)\times q}$, $\bar{\mathbf{C}}_{q\times(4n+q)}$, $\bar{\mathbf{S}}_{(4n+q)\times(4n+q)}$, $\mathbf{M}_{q\times q}$, and $\bar{\mathbf{L}}_{(4n+q)\times q}$. Note the modification is limited to the sense of dimension; all the matrix block expressions concerning the observer, including the Lyapunov equation, remain unchanged.

At this stage the observability of the coefficient pair consisting the Lyapunov equation is used for proving $\bar{\mathbf{P}} \not> \mathbf{0}$. Now we start to briefly show the observability pair $(\tilde{\mathbf{A}}, \bar{\mathbf{C}})$ is unobservable when $q < 2n$. $\bar{\mathbf{S}}^{-1}\bar{\mathbf{A}}$ can be rewritten into

$$\bar{\mathbf{S}}^{-1}\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_0 & \mathbf{I}_{2n} & \mathbf{0}_{2n\times q} \\ \mathbf{0}_{2n\times 2n} & \mathbf{0}_{2n\times 2n} & \mathbf{0}_{2n\times q} \\ -(\mathbf{C}_0\mathbf{A}_0)_{q\times 2n} & -(\mathbf{C}_0)_{q\times 2n} & -\mathbf{M}_{q\times q}^{-1} \end{bmatrix},$$

which is not of full rank; its rank is at most $2n+q$. Besides, it has been figured out that $-\mu$ is one eigenvalue of $\tilde{\mathbf{A}}$, as $\mathrm{rank}(\bar{\mathbf{S}}^{-1}\bar{\mathbf{A}}) < 4n+q \Rightarrow \exists \mathbf{z} \in \mathcal{R}^{4n+q}, \mathbf{z} \neq \mathbf{0}, \bar{\mathbf{S}}^{-1}\bar{\mathbf{A}}\mathbf{z} = \mathbf{0}$ $\Rightarrow (-\mu\mathbf{I} - \tilde{\mathbf{A}})\mathbf{z} = (-\mu\mathbf{I} + \mu\mathbf{I} + \bar{\mathbf{S}}^{-1}\bar{\mathbf{A}})\mathbf{z} = \bar{\mathbf{S}}^{-1}\bar{\mathbf{A}}\mathbf{z} = \mathbf{0}$.

A PBH test is then carried out to detect the observability of $(\tilde{\mathbf{A}}, \bar{\mathbf{C}})$, using $-\mu$ as the eigenvalue:

$$\begin{bmatrix} \bar{\mathbf{C}} \\ -\mu\mathbf{I} - \tilde{\mathbf{A}} \end{bmatrix}_{(4n+2q)\times(4n+q)} = \begin{bmatrix} \bar{\mathbf{C}} \\ \bar{\mathbf{S}}^{-1}\bar{\mathbf{A}} \end{bmatrix}.$$

Obviously, the possible maximum rank of the matrix $4n+q$ only depending on its dimension, whereas the actual rank is at most $2n+2q$, which is less than $4n+q$ under the assumption $q < 2n$. In other words, when $q < 2n$, $(\tilde{\mathbf{A}}, \bar{\mathbf{C}})$ is not observable.

Theorem 6.O1 in [12] states that with Hurwitz $\tilde{\mathbf{A}}$, observable $(\tilde{\mathbf{A}}, \bar{\mathbf{C}})$ exists if $\bar{\mathbf{P}}$ is positive definite. Equivalently, when $\tilde{\mathbf{A}}$ is Hurwitz and $(\tilde{\mathbf{A}}, \bar{\mathbf{C}})$ is unobservable, $\bar{\mathbf{P}}$ is not positive definite. In other words, any $\mathbf{C}_0$ with $q < 2n$ cannot generate a positive definite $\bar{\mathbf{P}}$.

Note that the PBH test also applies to $\mathbf{C}_0$ with row number $q \geq 2n$ but rank $= q_{real} < 2n$, as in these cases $\mathrm{rank}(\begin{bmatrix} \bar{\mathbf{C}}^T & (\bar{\mathbf{S}}^{-1}\bar{\mathbf{A}})^T \end{bmatrix}^T) \leq 2n + q + q_{real} < 4n + q$ so that $(\tilde{\mathbf{A}}, \bar{\mathbf{C}})$ is still unobservable. In sum, any $\mathbf{C}_0$ with rank $< 2n$ cannot provide a positive definite $\bar{\mathbf{P}}$, i.e., each positive definite $\bar{\mathbf{P}}$ with Hurwitz $\tilde{\mathbf{A}}$ means a $\mathbf{C}_0$ with rank $2n$.

### B. Proof of "⇒"

Firstly we consider the case $q = 2n$ and $\mathrm{rank}(\mathbf{C}_0) = 2n$, i.e. $\mathbf{C}_0$ is invertible. The PBH test with an arbitrary eigenvalue $\lambda$ of $\tilde{\mathbf{A}}$ becomes

$$\begin{bmatrix} \bar{\mathbf{C}} \\ \lambda\mathbf{I} - \tilde{\mathbf{A}} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_0 & \mathbf{0} & \mathbf{I}_{2n} \\ (\lambda+\mu)\mathbf{I} + \mathbf{A}_0 & \mathbf{I}_{2n} & \mathbf{0} \\ \mathbf{0}_{2n\times 2n} & (\lambda+\mu)\mathbf{I} & \mathbf{0}_{2n\times 2n} \\ -\mathbf{C}_0\mathbf{A}_0 & -\mathbf{C}_0 & (\lambda+\mu)\mathbf{I} - \mathbf{M}^{-1} \end{bmatrix}.$$

A $6n \times 6n$ submatrix is then formed with the $(4n+1)^{\mathrm{th}}$ to $6n^{\mathrm{th}}$ rows removed. It can be decomposed into the product of two invertible matrix, indicating its full rank and thus the observability of $(\tilde{\mathbf{A}}, \bar{\mathbf{C}})$. The detailed decomposition is not listed here. Provided observable $(\tilde{\mathbf{A}}, \bar{\mathbf{C}})$ and stable $\tilde{\mathbf{A}}$, [12] shows the Lyapunov equation $\tilde{\mathbf{A}}^T\bar{\mathbf{P}} + \bar{\mathbf{P}}\tilde{\mathbf{A}} = -\bar{\mathbf{C}}^T\bar{\mathbf{C}}$ has a positive definite solution $\bar{\mathbf{P}}$.

As for the cases that $q > 2n$ and $\mathrm{rank}(\mathbf{C}_0) = 2n$, $\mathbf{C}_0$ has $2n$ irrelevant rows. The $2n \times 2n$ square matrix formed by cascading these rows will pass the PBH test as proved above, so will $\mathbf{C}_0$ itself. Hence, $(\tilde{\mathbf{A}}, \bar{\mathbf{C}})$ is observable, and all the subsequent results can be proved following the procedures above. In sum, any $\mathbf{C}_0$ with rank $= 2n$ can provide a unique positive definite $\bar{\mathbf{P}}$ as the solution of the Lyapunov equation.

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