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Complexity Reduction of Robust Model Predictive Controller for Uncertain Piecewise Affine Systems

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Abstract— **This paper considers discrete-time, uncertain Piecewise Affine (PWA) systems affected by both polytopic parameter variations and bounded disturbances. We are interested in the Robust Model Predictive Control (RMPC) for uncertain PWA systems where the uncertainty can be presented in polytopes framework. RMPC is known as a complex problem and indeed for PWA systems, the on-line computation becomes computationally burdensome and inapplicable. In this paper we develop a new algorithm that consists of three different phases, based on the system states location. The proposed algorithm gives a simple and fast sub-optimal solution which considerably reduces the on-line computation and guarantee to drive the system states to the target region in spite of the considered uncertainties. The proposed algorithm is applied in simulation to a two tanks example.**

I. Introduction

PIECEWISE Affine Systems (PWA) [1] is a powerful framework that can model a broad class of hybrid systems and nonlinear systems where nonlinearities can be represented by a set of linear models around different operating modes or different state conditions such as saturation or dead zones. Uncertainties could arise from different sources like model simplification, limited system knowledge, and changes of the components value. Thus, control robustness becomes mandatory, so that performances of the systems are preserved in spite of these different causes of uncertainty.

Robust controls for uncertain linear systems are commonly used in the literature through a min-max control problem, but it is known that the min-max control is a complex problem and computationally burdensome. Also, Robust Model Predictive Control (RMPC) is presented in the literature as an affective technique for constrained uncertain discrete-time linear systems and for perturbed continuous PWA systems [2], [3], respectively, where a control technique based on minimizing the worst-case cost function is proposed. Some new techniques using parametric programming for linear systems are proposed in the literature to reduce the computation load [4], [5].

In [6] a RMPC for UPWA systems with polytopic parameter uncertainty is presented. The paper considers the case of unconstrained PWA systems. The authors show how to transfer the min-max control problem to a LMI problem. The technique developed on this paper is to drive the system states to the origin. The developed algorithm aims at reducing the computational load. It consists of two different parts; the first based on an on-line optimization problem, and then in the second part, when the state enters a computed attraction domains [6], an explicit feedback control law is applied. The algorithm reduces the computational time, but in the first part of the algorithm an on-line heavy computational load still exists. An extension to drive the system states to a reference point different from the origin and to include input and output constraints is not straightforward and more work needs to be done.

This paper examines a class of discrete-time Uncertain Piecewise Affine (UPWA) systems, where the uncertainties are coming from parameter variations and bounded disturbances. Constraints on control signals and measured outputs are taken into account. We consider a constant set-point tracking problem where the set-point in general could be different from the origin.

For this class of systems, solutions and analysis are proposed in [7]. An attainability checking that employs the predecessor operator is presented. Then the original min-max optimization problem is reduced to a linear programming problem. A Matlab toolbox 'HyStar' for this technique is developed and presented in [8]. The technique developed in [7] and programmed in [8] is aiming at driving the system states to the origin; however a straightforward modification to drive the states to a constant reference point can be done. A comparison between this approach and the approach developed on this paper shows that the developed technique offers a slightly better performance while reduces considerably the computation time.

Based on algorithms to compute reachable regions, cyclic invariance and invariant sub-sets for UPWA systems that presented in [9], a MPC algorithm is presented in this paper as a simple and fast suboptimal robust controller for the considered systems. The proposed algorithm consists of three different phases, based on the state location, as will be explain in the next sections. It is shown that the proposed technique significantly reduces the on-line computational load since one or just few QP have to be solved at each time step.

The paper is organized as follows. A brief description of UPWA systems and the considered class is given in section 2. Section 3 summarizes the attainability technique developed in [9] with some modifications. A fast and

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suboptimal robust MPC algorithm for the considered class is then developed in section 4. An application of the proposed technique to a two-tank example is presented in section 5. Finally conclusions and some remarks are given in section 6.

II. UNCERTAIN PIECEWISE AFFINE SYSTEMS

Piecewise affine systems are powerful tools for describing or approximating both nonlinear and hybrid systems, and represent a straightforward extension from linear to hybrid systems [1]. This paper focuses on the class of uncertain discrete-time PWA systems subject to parameter variations and bounded disturbances, defined as:

$$S^i : \left\{ \begin{aligned} &\mathbf{x}_{t+1} = \mathbf{A}^i(w_t)\mathbf{x}_t + \mathbf{B}^i(w_t)\mathbf{u}_t + \mathbf{f}^i(w_t) + \mathbf{C}^i\mathbf{d}_t \\ &\text{for } \begin{bmatrix}\mathbf{x}_t \\ \mathbf{u}_t\end{bmatrix} \in \chi_i \end{aligned} \right\} \tag{1}$$

where $\mathbf{x}_t \in \mathbf{X}, \mathbf{u}_t \in \mathbf{U}, w_t \in \mathbf{W}$ and $\mathbf{d}_t \in \mathbf{D}$ denote the system state, the control input, the uncertainty and the disturbance vector, respectively, at time instant t (for the i-th mode) with $\mathbf{X}, \mathbf{U}, \mathbf{W}, \mathbf{D}$ assigned polytopes, where $\mathbf{D}$ contains the origin. $\{\chi_i\}_{i=1}^{s}$ is the polyhedral coverage of the state and input spaces $\mathbf{X} \times \mathbf{U}$, s being the number of subsystems (modes). Each χ_i is given by:

$$\chi_i = \left\{ \begin{bmatrix}\mathbf{x}_t \\ \mathbf{u}_t\end{bmatrix} \,\middle|\, \mathbf{Q}^i \begin{bmatrix}\mathbf{x}_t \\ \mathbf{u}_t\end{bmatrix} \le \mathbf{q}^i \right\} \subset \Re^{n+m} \tag{2}$$

where n, m are the dimensions of state and input vectors, respectively. In this formalism, the existence of logical decision variables is taken into account by developing an affine model (1) with constraints (2) for each possible combination.

Exact state measurement $\mathbf{x}$ is supposed to be available. An important point has to be clarified for the dependence on w_t for the transition matrices in (1). In the following we describe this dependence through the vertices $\mathbf{A}^{ij}, \mathbf{B}^{ij}, \mathbf{f}^{ij}$, the parameter w playing the role of weighting between them leading in a mathematical sense to a convex combination:

$$\mathbf{A}^i(\mathbf{w}) = \sum_j^v w^j \mathbf{A}^{ij},\ \mathbf{B}^i(w) = \sum_j^v w^j \mathbf{B}^{ij},\ \mathbf{f}^i(w) = \sum_j^v w^j \mathbf{f}^{ij},$$
$$\text{where}: w^j \ge 0,\ \sum_{j=1}^v w^j = 1 \tag{3}$$

$\left(\mathbf{A}^{ij}, \mathbf{B}^{ij}, \mathbf{f}^{ij}\right)$ is the *j*-th vertex of the *i*-th model, v being the number of vertices. The matrices $\left(\mathbf{A}^i(w), \mathbf{B}^i(w), \mathbf{f}^i(w)\right)$ represent the model subject to uncertainty, described by the polytopic set $Convhull\left\{(\mathbf{A}^{ij}, \mathbf{B}^{ij}, \mathbf{f}^{ij}), j = 1, \cdots, v\right\}$ for each mode $i \in I$. The coefficients w^j are unknown and possibly time varying. In this way, for each polyhedral region χ_i, the model is affected by polytopic uncertainty.

III. ATTAINABILITY: A POLYHEDRAL APPROACH

The goal of the control policy is: drive the system from an initial point in $\mathbf{X}$ to a given target region $\mathbf{X}_f \subset \mathbf{X}$. The first issue is to determine the maximal subset of $\mathbf{X}$ for which this problem is well posed. This can be answered through a reachability analysis. In this section we summarize, with some modifications, the algorithms to compute the reachable regions, the cyclic invariance and invariant subsets that presented in [9].

A. Reachable Set

Let us consider the region $\mathbf{X}_f$, as a given target region in the global state space $\mathbf{X}$, with the initialization $\mathbf{R}_k = \mathbf{X}_f$. The following algorithm construct the robust N-steps ahead reachable region $\mathbf{R}_{k-N}$ defined as the region in the state space for which there exist a feasible mode (1) and an admissible control sequence able to drive the states from $\mathbf{R}_{k-N}$ into $\mathbf{R}_k$ in N-steps despite all allowable disturbances and parameter variations.

Algorithm 3.1 ($\mathbf{R}_k$ ***computation)***

1. Define the target region $\mathbf{X}_f$, the disturbance polytope $\mathbf{D}$ and let $\mathbf{R}_k = \mathbf{X}_f$,
2. For $z = 1 : N_{\max}$, $N_{\max}$: maximum number of iteration
3. for each region 'm' inside $\mathbf{R}_k$

   3.1) For i = 1,…, s

   3.1.1) $\widetilde{\mathbf{R}}_k = Difference(\mathbf{R}_k, \mathbf{C}^i\mathbf{D})$

   3.1.2) Compute $\mathbf{F}, \mathbf{g}$, where: $\widetilde{\mathbf{R}}_k = \{x \,|\, \mathbf{F}x \le \mathbf{g}\}$

   3.1.3) Calculate the robust region under mode i:

$$\mathbf{R}_{k-1}^i = \left\{ \Pr_{\mathbf{X}}\left(\bigcap_{j=1}^{v} \left(\begin{bmatrix} \mathbf{F}\mathbf{A}^{ij} & \mathbf{F}\mathbf{B}^{ij} \\ \mathbf{Q}^i & \end{bmatrix} \begin{bmatrix}\mathbf{x} \\ \mathbf{u}\end{bmatrix} \le \begin{bmatrix} \mathbf{g} - \mathbf{F}\,\mathbf{f}^{ij} \\ \mathbf{q}^i \end{bmatrix} \right) \right) \cap \mathbf{X} \right\}$$

   3.1.4) Store $I_{seq_{k-z}}^i$

   end

   end

4. $\mathbf{R}_{k-z} = \bigcup_{i=1}^{s} \mathbf{R}_{k-1}^i$
5. $\mathbf{R}_k = \mathbf{R}_{k-z}$

End

In an explicit formulation this comes to:

$$\mathbf{R}_{k-1} = \left\{ \begin{aligned} &\mathbf{x} \in \mathbf{X} \,\middle|\, \exists i, \mathbf{u}_{t-1} \in \mathbf{U}, \begin{bmatrix}\mathbf{x} \\ \mathbf{u}\end{bmatrix} \in \chi_i \\ &s.t.: \left\{\mathbf{A}^i(w_t)\mathbf{x}_t + \mathbf{B}^i(w_t)\mathbf{u}_t + \mathbf{f}^i(w_t) + \mathbf{C}^i\mathbf{d}_t\right\} \subset \mathbf{R}_k \end{aligned} \right\} \tag{4}$$

Remark 1: The geometrical operations used here can be efficiently implemented using standard computational geometry software such as [10-12].

Remark 2: The region $\mathbf{R}_{k-z} = \bigcup_r \mathbf{R}_{k-z}^r$, where $\mathbf{R}_{k-z}^r$ are convex, but $\mathbf{R}_{k-z}$ is not necessary convex.

B. Safety and Cyclic Invariance

Considering that the system states are driven to the target region, the states will be retained inside the target region if the target region is defined as safe or invariance set.

Safety, the set $\mathbf{R}_k$ is safe if and only if $\mathbf{R}_k \subseteq \mathbf{R}_{k-1}$ [13].

Cyclic Invariance, The control invariance (safety) of the region $\mathbf{R}_k$ often turns out to be a strong condition, hard to be satisfied in a robust manner with available discrete-time control actions. A relaxed safety condition can be defined as following:

- The existence of $l \geq 1$ such that $\mathbf{R}_k \subseteq \mathbf{R}_{k-l}$, called safety cycle (or cyclic invariance) of length l for region $\mathbf{R}_k$

This means that all the states inside the target region $\mathbf{R}_k$, or inside $\mathbf{R}_{k-l}$ will be driven to $\mathbf{R}_k$ in l steps.

To find the minimum cyclic invariance l, algorithm 3.1 can be run with the following condition:

If $\mathbf{X}_f \subseteq \mathbf{R}_{k-z}$ stop; and $l = z$.

C. Invariant Subset inside the Target Region

Starting from the hypothesis that there exists $l \geq 1$ such that $\mathbf{X}_f = \mathbf{R}_k \subseteq \mathbf{R}_{k-l}$, the goal is to define the invariant subset inside X_f defined as once the system states enter this subset, the states will remain inside this subset while keeping the same mode switch sequence fixed over the l steps. Here the mode switching sequence I_{seq} includes the active mode i at each sampling instant over the horizon.

The idea is to initialize a new reachability analysis with the nonempty target regions among the following:

$$\mathbf{R}_k^m = \mathbf{R}_k \cap \mathbf{R}_{k-l}^m, \tag{5}$$

$\forall m = 1,2,\cdots,index_{k-l}$ (No. of regions al level *k-l*)

The numerical procedure follows the following steps:

A. *For each* $m = 1,2,\cdots,index_{k-l}$,
compute $\mathbf{D}_0^m = \mathbf{R}_k \cap \mathbf{R}_{k-l}^m$

B. *For j=1,..,l*
Compute $\mathbf{D}_{k-j}^{I_{seq_l}^m(k-j)}(\mathbf{D}_{k-j+1}^m)$
end

C. $\mathbf{D}_0^m = \mathbf{D}_0^m \cap \mathbf{D}_{k-l}^{I_{seq_l}^m(k-l)}$

D. If $\mathbf{D}_0^m$ evolves (beyond a given tolerance) go to (B)

E. *end*

The output of this procedure is a collection of regions inside X_f considered as invariant subsets, each with its fixed mode switch sequence $I_{seq_l}^m$ over l steps.

IV. ROBUST MPC ALGORITHM

MPC has proved to efficiently control a wide range of applications in industry for non-hybrid systems as well as hybrid systems [14-16], and it is also used as robust control for constrained linear systems [3]. In this paper we develop a new algorithm based on MPC for PWA systems with quadratic cost function. The developed algorithm offers a fast suboptimal robust solution.

In this section we develop an algorithm that benefits from the results of the previous section, i.e. given $x_o \in \mathbf{R}_{k-N}$ insure that there is a feasible mode (1) and an admissible control sequence able to drive the states from $\mathbf{R}_{k-N}$ into $\mathbf{R}_k$ in N-steps. Moreover the associated mode sequence index $I_{seq_k}^m$ to each region is known. Thus, the goal for the closed-loop system trajectories is that starting from the given initial state x_o inside a reachable region $\mathbf{R}_{k-N}$ it should go through a finite sequence of regions $\mathbf{R}_{k-N+1}, \mathbf{R}_{k-N+2}, \cdots, \mathbf{R}_k$ and finally reach the target region.

The proposed control strategy consists of three different phases as follows:

Phase one: variable control horizon,

Phase two: fixed (classical) predictive control,

Phase three: invariant control, with fixed mode switching sequence (constant I_{seq}^m)

In **phase one**, a predictive control problem with variable control horizon is considered. The main idea is to use the minimum control horizon (minimum number of steps) that can drive the states to the target region $\mathrm{X}_f = \mathbf{R}_k$. In this phase the MPC control horizon starts from N_r and decreases to N_l, where N_r is equal to the minimum horizon N such that $x_0 \in \mathbf{R}_{k-N}$, and where $N_l = l$ the cyclic invariance horizon.

Considering that the current instant is the ($k-N$) instant, the following optimization problem has to be solved at each sampling instant:

$$\min_{\mathbf{u}_{k-N}^{k-1}} J(\mathbf{u}_{k-N}^{k-1}, \mathbf{x}_{k-N}) = \sum_{j=1}^{N} \left\| \mathbf{x}_{k-N+j} - \mathbf{x}_e \right\|_{\Lambda}^2 + \sum_{j=0}^{N-1} \left\| \mathbf{u}_{k-N+j} \right\|_{\Gamma}^2 \tag{6}$$

$$\text{s.t.}: \begin{cases} \mathbf{Q}^i \begin{bmatrix} \mathbf{x}_{k-N+j} \\ \mathbf{u}_{k-N+j} \end{bmatrix} \leq \mathbf{q}^i \text{ , } \mathbf{u}_{k-N+j-1} \in \mathbf{U}, \text{ for } j = 1,2,\cdots,N & (7) \\ \left(\mathbf{A}^{ic}\mathbf{x}_{k-N} + \mathbf{B}^{ic}\mathbf{u}_{k-N} + \mathbf{f}^{ic} \right) \in \widetilde{\mathbf{R}}_{k-N+1}^i \text{ , for } c = 1,2,\cdots v & (8) \\ \mathbf{x}_{k+1}^* = \mathbf{A}^{i*}\mathbf{x}_k + \mathbf{B}^{i*}\mathbf{u}_k + \mathbf{f}^{i*}, & (9) \\ \mathbf{x}_{k+j}^* \in \widetilde{\mathbf{R}}_{k+j}^i, \quad \text{for } j = 1,2,\cdots,N & (10) \end{cases}$$

where $\mathbf{x_e}$ is the states reference, Λ, Γ are the weighting matrices $\|x\|_{\Lambda}^2 = x^T \Lambda x$.

The constraints aim at the following: (7) should guarantee the mode and control constraints, (8) should insure the robustness of moving to region $\mathbf{R}_k$ despite the parameter uncertainty w, and (9-10) should improve the performance, where $\mathbf{x}_{k+1}^*$ is the state update according to the nominal model $\left(\mathbf{A}^{i*}, \mathbf{B}^{i*}, \mathbf{f}^{i*} \right)$ at mode *i*, as it is known that most of the time the uncertain system evolves around it. The nominal model could be for example the average or the centred model. The optimization problem (6-10) is solved according to the following algorithm:

Algorithm 4.1 (Variable Control Horizon)

1- For L= 1: $N_{\max}$

1.1 if $x_0 \in R_{k-L}$

1.2 For each $m = 1,2\cdots,index_{k-L}$

1.3 If $x_0 \in R_{k-L}^m$

1.3.1 solve the QP optimisation problem (6-10), with $N=L$ and with the memorized switching mode sequence $I^m_{seq_{k-L}}$

end

end

1.4 chose the optimal solution among 1.3.1 solutions

break

end

end

2- If $N = l$, stop (the end of phase one),

Else (go to step 1 in the next sampling instant)

Step 1.4 will insure the moving of the states from level $k-L$ to level $(k-L+1)$, but it is often possible that regions at level $(k-h)$ cover the new states (x_{k+1}), where $h < L-1$. That is why the check in step 1 has to be done at each time step, and this accelerates the movements towards the cyclic level l.

Phase two, When the new states reach the cyclic invariance region (i.e. $x_{k+1} \in R_{k-l}$), we solve the optimization problem (6-10) with a constant control horizon equal to the cyclic invariance horizon (i.e. $N = l$). At each sampling instant we identify the region inside R_{k-l} that includes the current state and use its associated mode sequence $I^m_{seq_{k-l}}$ to solve (6-10).

Phase three, Once the current state enters any of the sub-invariant sets, fix the mode sequence to the mode sequence of this invariant sub-set ($I^m_{seq_{inv}}$), and solve the optimization problem (6-9) with $N = l$ and with the following additional constraints:

$$\mathbf{x}^*_{k-l+j} \in \widetilde{\mathbf{R}}^r_{inv}, \quad \text{where } \widetilde{\mathbf{R}}^r_{inv} = \mathbf{R}^r_{inv} - \mathbf{C}^r\mathbf{D}, \qquad \text{for } j = 1,2,\cdots,l, \quad I_{seq} = I^r_{inv}, \quad i = 1,2,\cdots,s \tag{11}$$

where r is the index of the sub-invariant set that includes the current state.

The proposed control technique with the three different phases is not computationally burdensome. In phase one and two only the feasible regions that include the current states with their prefixed mode sequence is considered (only a small limited number of QP's have to be solved at each sampling time), and in phase three only the invariant mode sequence is considered (only one QP has to be solved at each sampling time).

Furthermore, the proposed control strategy guarantee the feasibility and the stability of the solution if $x_o \in \mathbf{R}_{k-N}$. The control strategy forces the system states to go through the pre-computed reachable regions, and finally remains at the target region (or periodically remains there, if there is no sub-invariant set in the target region).

In [9] a reduction algorithm that aims at reducing the number of regions at each level is developed, and it is shown that it considerably decreases the number of regions. The problem of this algorithm is, however, that through the reduction technique (eliminating the small regions that are covered by other regions and also eliminating the overlapping regions) we lose the mode switching index $I^m_{seq_k}$ associated to each region, and this will affect step 1.3.1 in the developed control algorithm. A solution for this is to apply the first feasible solutions without enumerating all possible switching sequences.

V. APPLICATION

Let us consider as application of the previous techniques a two tanks example (Figure 1). The tanks are filled by a pump acting on tank 1, continuously manipulated from 0 up to a maximum flow Q_1. A switching valve V_{12} controls the flow between the tanks. This valve is assumed to be either completely opened or closed ($V_{12} = 1$ or 0 respectively). The V_{N2} manual valve controls the nominal outflow of the second tank. It is assumed in the simulations that the manual valves, V_{N1} is always closed and that V_{N2} is open.

The liquid levels to be controlled are denoted by h_1 and h_2 for each tank respectively. The discrete time model, with a sampling time (T_s) equal 10s, for the two tanks is:

$$\begin{aligned} h_1(k+1) &= h_1(k) + \frac{T_s}{A}(Q_1(k) - k_{12}V_{12}(h_1(k) - h_2(k)) \\ h_2(k+1) &= h_2(k) + \frac{T_s}{A}(k_{12}V_{12}(h_1(k) - h_2(k)) - k_{N2}V_{N2}h_2(k)) \end{aligned} \tag{12}$$

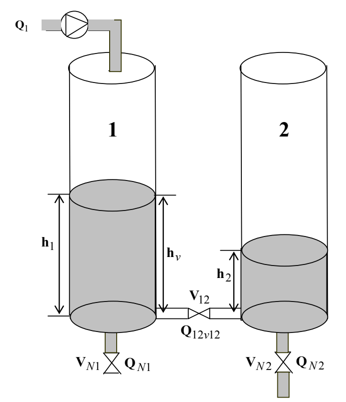

Figure 1. Two-tank benchmark.

This model can be formulated as a piecewise affine system of form (1), with two subsystems (two modes), described as follows. For mode one with $V_{12} = 1$, two vertices for the uncertainty description are considered:

$$\mathbf{A}^{11} = \begin{bmatrix} 0.9188 & 0.0812 \\ 0.0812 & 0.8377 \end{bmatrix}, \quad \mathbf{B}^{11} = \begin{bmatrix} 721.5007 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{A}^{12} = \begin{bmatrix} 0.9336 & 0.0664 \\ 0.0664 & 0.8672 \end{bmatrix}, \quad \mathbf{B}^{12} = \begin{bmatrix} 590.3188 & 0 \\ 0 & 0 \end{bmatrix}$$

For mode two with $V_{12} = 0$, two vertices for the uncertainty description are also considered:

$$\mathbf{A}^{21} = \begin{bmatrix} 1 & 0 \\ 0 & 0.9188 \end{bmatrix}, \quad \mathbf{B}^{21} = \begin{bmatrix} 721.5007 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{A}^{22} = \begin{bmatrix} 1 & 0 \\ 0 & 0.9336 \end{bmatrix}, \quad \mathbf{B}^{22} = \begin{bmatrix} 590.3188 & 0 \\ 0 & 0 \end{bmatrix}$$

The following matrices and vectors for the modes constraints are considered:

$$\mathbf{Q}^i=\begin{bmatrix}1&0&0&0\\0&1&0&0\\-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&-1&0\\0&0&0&1\\0&0&0&-1\end{bmatrix},\quad \mathbf{q}^i=\begin{bmatrix}0.62\\0.62\\0\\0\\0.0001\\0\\b\\-b\end{bmatrix}$$

where for mode one ($i=1$), $b=1$, and for mode two ($i=2$), $b=0$.

The constraints imply limitations on the global state space: $0\le|\mathrm{X}|_{1,2}\le 0.62$, as well as on the control signal. The target region, to which system states will be driven to, is defined by the following constraints:

$$\mathrm{X}_f=\mathbf{R}_k:=\left\{x:\begin{bmatrix}1&0\\0&1\\-1&0\\0&-1\end{bmatrix}\mathbf{x}\le\begin{bmatrix}0.55\\0.25\\-0.45\\-0.15\end{bmatrix}\right\} \tag{13}$$

A polytope for a bounded disturbance is finally considered with:

$$-0.007\le|d|_{1,2}\le 0.007 \tag{14}$$

The reachable approach presented above is first applied to compute the region $\mathbf{R}_{k-N}$ in the state space which includes the states that can be driven in N steps to $\mathbf{R}_k$ for all considered disturbances and parameter variations.

Figure 2 shows the evaluation for the target region for $N\in[0,50]$ on the vertical axis.

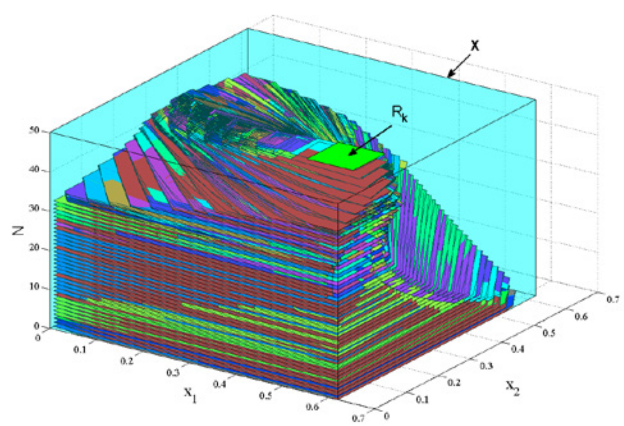

Figure 2. The region evaluation for $N\in[0,50]$.

For the two tank example the target region will be safe in 3 steps (*l*=3), $\mathbf{R}_k=\mathrm{X}_f\subseteq\mathbf{R}_{k-3}$. The number of region for *N=3* is 8.

Looking for the invariant subset inside the target region, only two modes: $I_{seq}=[1,1,2]$, and $I_{seq}=[1,2,1]$, leads to 2 non-empty regions that are two invariant subsets (see [9] for more details on safety and invariant subset results).

The RMPC algorithm presented above including the three different phases is applied, where the nominal model of the state evaluation is chosen to be the epicentre of the state matrix $0.5\left(\mathbf{A}^{i1}+\mathbf{A}^{i2}\right)$ for each mode. The diagonal weighting terms in the cost function are chosen as $\Lambda=1000\times I_2$ and $\Gamma=1$, and the state reference is $(0.5,0.2)$. The RMPC algorithm is applied several times, each with different initial states inside the feasible regions $\mathbf{R}_{k-N}$, and in each simulation a random uncertainty w is applied, and a random disturbance d is added to the system.

Figure 3 shows the results for some different initial states, where each of those initial states, for visibility purpose, is covered by the reachable regions at the 20[th] depth level ($x_0\in\mathbf{R}_{k-20}$). Thus the control algorithm starts with a control horizon of $N=20$, and decreases in different steps until $N=l=3$, and then continues with fixed horizon $N=3$, according to the technique developed in section 4.

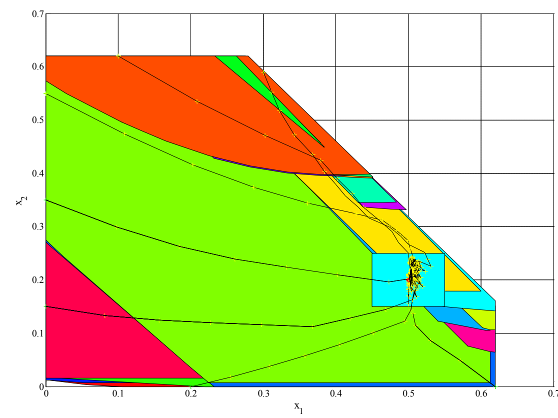

Figure 3. The states are driven to the target region

Figure 4 shows the state time evaluation with different initial states, considering random uncertainties and random disturbances as well. The green lines present the target regions. It is clear that the proposed algorithm is successful in driving the system states to the target region and keeping them inside the target region. The state trajectories do not go out of the invariant subsets once it enters any one of the two invariant subsets.

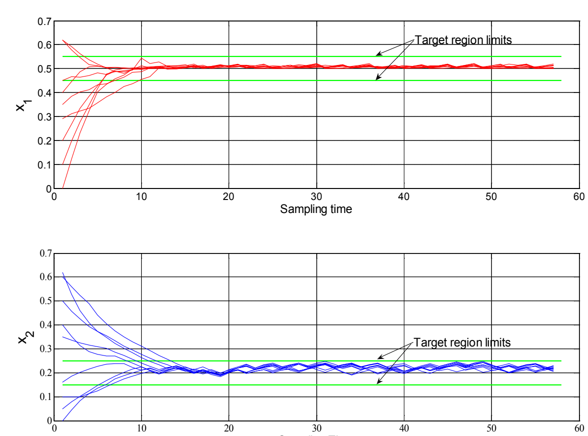

Figure 4. State evaluation with RMPC for different initial states.

As mentioned in the introduction, the approach presented in [6] can not be applied to our example, as the considered set-point differs from the origin, and also the two-tanks example has constraints over the control signals. Thus we can not compare our approach to this technique.

The approach of [7] is modified and the toolbox 'HyStar' [8] is adapted to consider the case of constant reference point different from origin. The developed control strategy in this paper is compared with the technique presented on [7] (after doing the necessary modifications) that transfers the min-max problem to a LP optimization problem. Figure 5 shows the performance for both approaches starting from $x_0 = [0.62;0.0]$, the same parameter uncertainty w and the random disturbance d is applied for both cases. The red solid line according to the LP technique [7], blue dashed line according to the developed technique and green dashed line are the target region limits. We have to mention that the reference point $xe = [0.5;0.2]$ is not an equilibrium point for the system states, thus the controller keeps regulating the system states around the desired point.

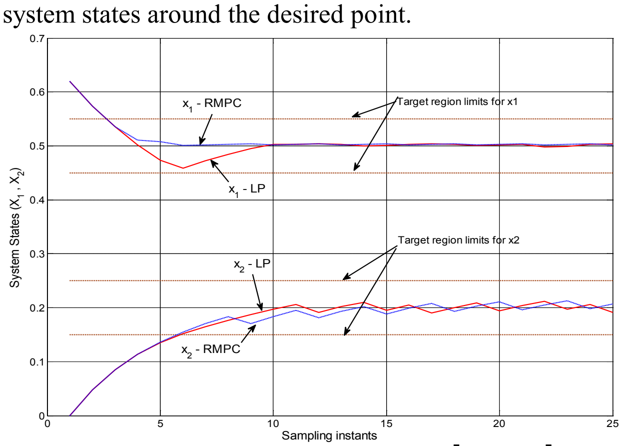

Figure 5. State evaluation starting from $x_0 = [0.62;0.0]$, red solid lines according to the LP technique [7], blue dashed line according to the developed technique (RMPC), and bruin dashed lines are the target region limits.

For many different simulations the technique presented in this paper offers a slightly better performance as it predict future performance over the N future steps, while the LP approach consider only the current instant. But in general the two performances are very close. The main difference is the computational time, while the LP approach takes in average 5.96 second for each time step to compute the solution; the technique presented in this paper takes in average 0.06s/step.

VI. CONCLUSION

This paper has examined a class of uncertain discrete-time piecewise affine systems affected by both polytopic parameter variations and bounded disturbance. Input constraints as well as output constraints are taken into account. The developed control strategy aims at driving the system states to a constant reference point that could be in general different from the origin. Based on a polyhedral technique to define the regions in the state space where a feasible robust control is assured, a new robust model predictive control technique consists of three different phases is developed. The developed technique able to drive the system states to the desired region and then keeps them inside this region (or periodically remains there, if there is no sub-invariant set in the target region) despite the considered uncertainty. Compared to the classical min-max optimization problem and also to the recent techniques presented in [7-8], the proposed technique reduces significantly the on-line computational time. Moreover, the developed technique guarantees the feasibility and the stability of the solution in driving the system state to the predefined target region. The proposed controller has been applied to a two tanks example.

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