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Adaptive Control of Piecewise Linear Systems: the Output Tracking Case

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***Abstract*—This paper studies the adaptive control problem of single-input, single-output (SISO) piecewise linear systems, a class of linear systems with switched parameters. A direct state feedback model reference adaptive control (MRAC) scheme is developed for such systems to achieve closed-loop signal boundedness and asymptotic output tracking performance. Simulation results on linearized NASA GTM models are presented to demonstrate the effectiveness of the proposed scheme.**

I. Introduction

Adaptive control of plants that are modeled by linear time-invariant (LTI) systems has been studied extensively in the literature [4], [7], [11]. Yet in many practical applications, an LTI system model is insufficient to describe the actual plant due to the ever increasing performance requirements over a wide range of operating conditions. Linear time-varying (LTV) system models are thus considered in such applications.

Adaptive control designs have been proposed for LTV models with slowly varying parameters [6] and a more general class of LTV models [12], [13]. However, the assumption of smoothness of the parameter variations excludes an important class of LTV plants with switched parameters, the so-called piecewise linear systems. Such systems arise in aircraft flight control applications, and a typical example is the linearized dynamics of an aircraft at some chosen operating points over its flight envelope, each of which corresponding to a set of constant parameters. With sufficient number of operating points chosen, transitions among them can be modeled as parameter switches.

When adaptive control schemes designed for LTI systems are applied to piecewise linear systems, closed-loop stability may be lost for fast parameter switches. In general, however, if the average frequency of parameter discontinuities is sufficiently low, stability can be maintained [14]. Even if this is the case, deterioration of tracking performance is almost unavoidable. Some modifications are thus made to the standard adaptive control schemes aiming at improving the tracking performance of piecewise linear system. In [1], a switching scheme is presented to deal with control of plants with abruptly jumping parameters, and a stability condition on the frequency of the parameter discontinuities is derived. A multiple model adaptive control (MMAC) approach is considered in [8] which is effective in reducing transient tracking error of a piecewise linear system. However, in these control schemes, only for a finite number of parameter switches, the closed-loop system settles down at the end and asymptotic tracking is achieved. Whenever parameter switches occur, a deviation of the plant output from the desired reference trajectory appears.

In this paper, we develop direct MRAC schemes for piecewise linear systems. It is shown that with such MRAC schemes, closed-loop stability (signal boundedness) and asymptotic tracking performance are achieved for such systems, if the occurrence frequency of parameter discontinuities is sufficiently low. The desired performance are achieved for arbitrarily frequent parameter discontinuities under certain matching conditions, e.g., piecewise linear systems in controllable canonical form (CCF). As compared to the designs in [2], [10] for state tracking, we will present the state feedback for output tracking design which has a less restrictive plant model matching condition.

The paper is organized as follows. The formulation of the adaptive control problem for piecewise linear systems is presented in section II. In section III, the non-adaptive model reference control problem is considered, and an MRAC design is proposed, with the stability results established in Section IV. Illustrative examples are presented in Section V, and some concluding remarks are given in Section VI.

II. Problem Statement

In this section, the adaptive state feedback control problem for a piecewise linear system, to make its output track a desired trajectory generated from a linear time-invariant reference model system, is formulated. To characterize the system parameter discontinuities, we introduce indicator functions, based on which an MRAC approach for such a control problem is proposed in the next section.

A. Controlled Plant

We consider a SISO piecewise linear system

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{b}(t)u(t),\\ y(t) &= \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}(t), \ \ \boldsymbol{x}(0) = \boldsymbol{x}_0,\end{aligned} \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state vector and is available for measurement, $u(t) \in \mathbb{R}$ is the control input, $y(t) \in \mathbb{R}$ is the controlled output, $\boldsymbol{A}(t) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{b}(t) \in \mathbb{R}^n$ are unknown time-varying system parameter matrices, and $\boldsymbol{c} \in \mathbb{R}^n$ is an unknown constant parameter vector. The parameters matrices $\boldsymbol{A}(t)$, $\boldsymbol{b}(t)$ vary in a piecewise linear pattern; that is, during different time periods, $(\boldsymbol{A}(t), \boldsymbol{b}(t))$ take on different values as specified by the parameter matrix sets $(\boldsymbol{A}_i, \boldsymbol{b}_i)$, called *a mode of* (1), $i \in \mathcal{I} \triangleq \{1, 2, \ldots, l\}$, where $\boldsymbol{A}_i$, $\boldsymbol{b}_i$ are unknown but constant parameter matrices representing the controlled plant *operating at the ith mode*, and l is the total number of the system modes.

To characterize such time-varying behaviors of the system, we introduce the indicator functions.

Indicator functions. The knowledge of the durations of time of the ith mode that the system assumes and the time instants at which it switches to the jth, $i, j \in \mathcal{I}$, is crucial for adaptive control design. The indicator functions $\chi_i(t)$, which contain such knowledge of the system parameter discontinuities, are assumed to be known and defined as

$$\chi_i(t) = \begin{cases} 1, & \text{if } (\boldsymbol{A}(t), \boldsymbol{b}(t)) = (\boldsymbol{A}_i, \boldsymbol{b}_i), \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

It follows that $\sum_{i=1}^{l} \chi_i(t) = 1$, $\chi_j(t)\chi_k(t) = 0$, $j \neq k$. With the indicator functions $\chi_i(t)$, the time-varying plant parameter matrices $\boldsymbol{A}(t)$, $\boldsymbol{b}(t)$ can be expressed as

$$\boldsymbol{A}(t) = \sum_{i=1}^{l} \boldsymbol{A}_i \chi_i(t), \ \boldsymbol{b}(t) = \sum_{i=1}^{l} \boldsymbol{b}_i \chi_i(t). \tag{3}$$

B. *Control Objective*

The control objective is to develop a feedback control law for the system (1) with parameter variations characterized as in (3) such that all the signals in the closed-loop system are bounded, and the plant output $y(t)$ asymptotically tracks a reference signal $y_m(t)$, i.e., $\lim_{t\to\infty}(y(t) - y_m(t)) = 0$, with $y_m(t)$ generated from a reference model system

$$y_m(t) = W_m(s)[r](t), \ W_m(s) = \frac{1}{P_m(s)}, \tag{4}$$

where $P_m(s)$, independent of the system parameters, is a desired closed-loop characteristic polynomial of degree n^*, and $r(t)$ is an external reference input signal which is bounded and piecewise continuous.

III. Adaptive Control Design

We propose a new state feedback controller structure and the adaptive laws for the piecewise linear plant (1) to achieve closed-loop stability and asymptotic output tracking in this section. The non-adaptive model reference control problem is considered first, and a gradient design is presented to solve the adaptive control problem.

Assumptions. Suppose for the ith mode the transfer function of the system is

$$G_i(s) = \boldsymbol{c}^\top (s\boldsymbol{I} - \boldsymbol{A}_i)^{-1}\boldsymbol{b}_i = \frac{k_{pi} Z_i(s)}{P_i(s)}, \tag{5}$$

with $k_{pi} \neq 0$ a constant and $P_i(s) = \det(s\boldsymbol{I} - \boldsymbol{A}_i)$, $Z_i(s)$ being monic polynomials with unknown constant coefficients and of degrees n and m, respectively. To design an adaptive state feedback control law for output tracking, the following assumptions are made for $i \in \mathcal{I}$:

(A1) $(\boldsymbol{A}_i, \boldsymbol{b}_i, \boldsymbol{c}^\top)$ is stabilizable and detectible;
(A2) The zeros of $Z_i(s)$ are stable;
(A3) The degree m of $Z_i(s)$ is known;
(A4) The degree of $P_m(s)$ is $n^* = n - m$;
(A5) The sign of k_{pi}, $\text{sign}[k_{pi}]$, is known.

A. *Nominal Controller Scheme*

If $\boldsymbol{A}_i$ and $\boldsymbol{b}_i$, $i \in \mathcal{I}$, are known, we propose the state feedback model reference control law

$$u(t) = \boldsymbol{k}_x^{*\top}(t)\boldsymbol{x}(t) + k_r^*(t)r(t) \tag{6}$$

with the controller parameters

$$\boldsymbol{k}_x^*(t) = \boldsymbol{k}_{x1}^*\chi_1(t) + \boldsymbol{k}_{x2}^*\chi_2(t) + \cdots + \boldsymbol{k}_{xl}^*\chi_l(t), \tag{7}$$

$$k_r^*(t) = k_{r1}^*\chi_1(t) + k_{r2}^*\chi_2(t) + \cdots + k_{rl}^*\chi_l(t), \tag{8}$$

where $\boldsymbol{k}_{xi}^* \in \mathbb{R}^n$ and $k_{ri}^* \in \mathbb{R}$ are defined to satisfy

$$\det(s\boldsymbol{I} - \boldsymbol{A}_i - \boldsymbol{b}_i\boldsymbol{k}_{xi}^{*\top}) = P_m(s)Z_i(s), \ k_{ri}^* = \frac{1}{k_{pi}}. \tag{9}$$

The existence of such $\boldsymbol{k}_{xi}^*$ and k_{ri}^* is guaranteed by Assumption (A1)–(A2). From (9), we have

$$\begin{aligned} \boldsymbol{c}^\top(s\boldsymbol{I} - \boldsymbol{A}_i - \boldsymbol{b}_i\boldsymbol{k}_{xi}^{*\top})\boldsymbol{b}_i k_{ri}^* &= \frac{k_{pi}Z_i(s)k_{ri}^*}{\det(s\boldsymbol{I} - \boldsymbol{A}_i - \boldsymbol{b}_i\boldsymbol{k}_{xi}^{*\top})} \\ &= W_m(s), \end{aligned} \tag{10}$$

such that when the model reference controller (6) is applied to the plant (1), the closed-loop system becomes

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{l} \left((\boldsymbol{A}_i + \boldsymbol{b}_i\boldsymbol{k}_{xi}^{*\top})\chi_i\boldsymbol{x}(t) + \boldsymbol{b}_i k_{ri}^*\chi_i r(t)\right) \tag{11}$$

$$y(t) = \boldsymbol{c}^\top\boldsymbol{x}(t). \tag{12}$$

Let the increasing sequence $\{t_i\}_{i=1}^{\infty}$ denote the time instants at which system mode switches occur. With (9)–(12), the output tracking error is $e(t) = y(t) - y_m(t) = \epsilon_0(t)$, where $\epsilon_0(t) = \boldsymbol{c}^\top\boldsymbol{\Phi}(t, t_0)\boldsymbol{x}(t_0)$, and $\boldsymbol{\Phi}(t, t_0)$ is the state transition matrix associated with the homogeneous system of (11)–(12):

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{A}_m(t)\boldsymbol{z}(t), \tag{13}$$

where $\boldsymbol{A}_m(t) = \sum_{i=1}^{l}\boldsymbol{A}_{mi}\chi_i(t)$ with $\boldsymbol{A}_{mi} \triangleq \boldsymbol{A}_i + \boldsymbol{b}_i\boldsymbol{k}_{xi}^{*\top}$ being stable. Exponential stability of (13) is sufficient for stability of (11), which has been studied in [3]. It is well known that for (13) to be exponentially stable, the time interval between two consecutive mode switches should be long enough. Let $T_0 = \min_{k\in\mathbb{Z}^+}\{t_k - t_{k-1}\}$, where $\mathbb{Z}^+$ stands for all positive integers, and $\boldsymbol{P}_{mi}$, $\boldsymbol{Q}_{mi} \in \mathbb{R}^{n\times n}$ be symmetric, positive definite matrices satisfying

$$\boldsymbol{A}_{mi}^\top\boldsymbol{P}_{mi} + \boldsymbol{P}_{mi}\boldsymbol{A}_{mi} = -\boldsymbol{Q}_{mi}, \ \ i \in \mathcal{I}. \tag{14}$$

Due to the stability of $\boldsymbol{A}_{mi}$, there exist $a_{mi}, \lambda_{mi} > 0$ such that $\|e^{\boldsymbol{A}_{mi}t}\| \leq a_{mi}e^{-\lambda_{mi}t}$. Define $a_m = \max_{i\in\mathcal{I}} a_{mi}$, $\lambda_m = \min_{i\in\mathcal{I}}\lambda_{mi}$, $\alpha = \max_{i\in\mathcal{I}}\lambda_{\max}[\boldsymbol{P}_{mi}]$, and $\beta = \min_{i\in\mathcal{I}}\lambda_{\min}[\boldsymbol{P}_{mi}]$, where $\lambda_{\min}[\cdot]$ and $\lambda_{\max}[\cdot]$ denote the minimum and maximum eigenvalues of a matrix. The following lemma gives a lower bound on T_0 that ensures exponential stability of (13), thus the stability of (11):

Lemma 1. The homogeneous system (13) is exponentially stable with decay rate $\sigma \in (0, 1/2\alpha)$ if the minimum switching time interval T_0 is such that

$$T_0 \geq \frac{\alpha}{1-2\sigma\alpha} \ln(1+\mu\Delta_{\boldsymbol{A}_m}),\ \mu = \frac{a_m^2}{\lambda_m \beta} \max_{i\in\mathcal{I}} \|\boldsymbol{P}_{mi}\|,\quad (15)$$

where $\Delta_{\boldsymbol{A}_m}$ stands for the largest difference between any two modes of $\boldsymbol{A}_m(t)$, i.e., $\Delta_{\boldsymbol{A}_m} = \max_{i,j\in\mathcal{I}} \|\boldsymbol{A}_{mi} - \boldsymbol{A}_{mj}\|$.

Proof: Let $\boldsymbol{A}_{m(k-1)}$ denote the mode of $\boldsymbol{A}_m(t)$ over $[t_{k-1}, t_k)$, $\boldsymbol{A}_{m(k-1)} \in \{\boldsymbol{A}_{m1}, \dots, \boldsymbol{A}_{ml}\}$, $k \in \mathbb{Z}^+$. Consider a mode switch at $t = t_k$. Due to the stability of $\boldsymbol{A}_{m(k-1)}$, $\boldsymbol{A}_{m(k)}$ and without loss of generality, there exist symmetric, positive definite $\boldsymbol{P}_{m(k-1)}$, $\boldsymbol{P}_{m(k)} \in \mathbb{R}^{n\times n}$ such that

$$\boldsymbol{A}_{m(k-1)}^\top \boldsymbol{P}_{m(k-1)} + \boldsymbol{P}_{m(k-1)} \boldsymbol{A}_{m(k-1)} = -\boldsymbol{I}_n,$$
$$\boldsymbol{A}_{m(k)}^\top \boldsymbol{P}_{m(k)} + \boldsymbol{P}_{m(k)} \boldsymbol{A}_{m(k)} = -\boldsymbol{I}_n,$$

where $\boldsymbol{I}_n$ denotes the n-dimensional identity matrix. With $\Delta\boldsymbol{A}_{m(k)} = \boldsymbol{A}_{m(k)} - \boldsymbol{A}_{m(k-1)}$, $\Delta\boldsymbol{P}_{m(k)} = \boldsymbol{P}_{m(k)} - \boldsymbol{P}_{m(k-1)}$, we have

$$\boldsymbol{A}_{m(k)}^\top \Delta\boldsymbol{P}_{m(k)} + \Delta\boldsymbol{P}_{m(k)} \boldsymbol{A}_{m(k)} = -\boldsymbol{S}_{(k)},\quad (16)$$
$$\boldsymbol{S}_{(k)} = \Delta\boldsymbol{A}_{m(k)}^\top \boldsymbol{P}_{m(k-1)} + \boldsymbol{P}_{m(k-1)} \Delta\boldsymbol{A}_{m(k)}.\quad (17)$$

Since $\boldsymbol{A}_{m(k)}$ is stable, the solution of (16) is

$$\Delta\boldsymbol{P}_{m(k)} = \int_0^\infty e^{\boldsymbol{A}_{m(k)}^\top t} \boldsymbol{S}_{(k)} e^{\boldsymbol{A}_{m(k)} t} dt.$$

It follows from (17) that $\|\boldsymbol{S}_{(k)}\| \leq 2\|\boldsymbol{P}_{m(k-1)}\| \|\Delta\boldsymbol{A}_{m(k)}\|$, and with $\|e^{\boldsymbol{A}_{m(k)} t}\| \leq a_m e^{-\lambda_m t}$, we have

$$\|\Delta\boldsymbol{P}_{m(k)}\| \leq \frac{a_m^2}{\lambda_m} \|\boldsymbol{P}_{m(k-1)}\| \|\Delta\boldsymbol{A}_{m(k)}\|.\quad (18)$$

Consider the piecewise continuous Lyapunov function $V = \boldsymbol{z}^\top(t) \sum_{i=1}^l \boldsymbol{P}_{mi}\chi_i \boldsymbol{z}(t)$ with $\boldsymbol{P}_{mi}$ satisfying (14) for $\boldsymbol{Q}_{mi} = \boldsymbol{I}_n$ so that $\boldsymbol{P}_{m(k-1)}, \boldsymbol{P}_{m(k)} \in \{\boldsymbol{P}_{m1}, \dots, \boldsymbol{P}_{ml}\}$. At $t = t_k$, (18) and the fact that $\beta\|\boldsymbol{z}(t)\|^2 \leq V$ lead to

$$V(t_k) - V(t_k^-) = \boldsymbol{z}^\top(t_k) \Delta\boldsymbol{P}_{m(k)} \boldsymbol{z}(t_k)$$
$$\leq \frac{a_m^2}{\lambda_m \beta} \|\boldsymbol{P}_{m(k-1)}\| \|\Delta\boldsymbol{A}_{m(k)}\| V(t_k^-)$$

With $\mu = \frac{a_m^2}{\lambda_m \beta} \max_{i\in\mathcal{I}} \|\boldsymbol{P}_{mi}\|$, we have for $k \in \mathbb{Z}^+$

$$V(t_k) \leq (1+\mu\Delta_{\boldsymbol{A}_m}) V(t_k^-).$$

In addition, with the fact $V \leq \alpha\|\boldsymbol{z}(t)\|^2$, we have the time derivative of V over $[t_{k-1}, t_k)$ satisfies $\dot{V} \leq -V/\alpha$, and

$$V(t) \leq e^{-\frac{1}{\alpha}(t-t_0)} (1+\mu\Delta_{\boldsymbol{A}_m})^{k-1} V(t_0),\quad t \in [t_{k-1}, t_k),$$

which leads to

$$\|\boldsymbol{z}(t)\| \leq \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}} e^{-\frac{1}{2\alpha}(t-t_0) + \frac{k-1}{2}\ln(1+\mu\Delta_{\boldsymbol{A}_m})} \|\boldsymbol{z}(t_0)\|.\quad (19)$$

Furthermore, with T_0 being the minimum switching time interval, we have $t - t_0 \geq (k-1)T_0$ for $t \in [t_{k-1}, t_k)$, which together with the condition (15), leads to

$$-\frac{1}{2\alpha}(t-t_0) + \frac{k-1}{2}\ln(1+\mu\Delta_{\boldsymbol{A}_m}) \leq -\sigma(t-t_0),\quad (20)$$

and exponential stability can thus be concluded. ∇

Remark 1: In this paper, we consider a constant system output vector $\boldsymbol{c}$ as in (1), by which (4), (11), (12) lead to

$$e(t) = y(t) - y_m(t) = \epsilon_0(t)\quad (21)$$

under the piecewise plant-model matching condition (9). For $\boldsymbol{c}$ piecewise constant, we have $e(t) = \boldsymbol{c}^\top(t)\boldsymbol{x}(t) - y_m(t)$ with $\boldsymbol{c}(t) = \sum_{i=1}^l \boldsymbol{c}_i\chi_i(t)$. Here $\boldsymbol{x}(t)$ and $y_m(t)$ are continuous, while $\boldsymbol{c}(t)$ is not; in particular, whenever a system mode switch occurs, a discontinuity in $y(t)$ appears. Since $y_m(t)$ is continuous, we can conclude that asymptotic tracking cannot be achieved with a switching $\boldsymbol{c}$ matrix. □

Remark 2: In the state feedback state tracking design for piecewise linear systems [10], a plant-model matching condition, $\boldsymbol{A}_i + \boldsymbol{b}_i\boldsymbol{k}_{xi}^{*\top} = \boldsymbol{A}_{mi}, \boldsymbol{b}_i k_{ri}^* = \boldsymbol{b}_{mi}$, is crucial and certain structural information about the plant parameter matrices $\boldsymbol{A}_i$, $\boldsymbol{b}_i$ are needed for the specification of $\boldsymbol{A}_{mi}$, $\boldsymbol{b}_{mi}$ for a choice of the reference model system. In the output tracking case, such restrictive matching conditions are relaxed; in particular, the triple $(\boldsymbol{A}_i + \boldsymbol{b}_i\boldsymbol{k}_{xi}^{*\top}, \boldsymbol{b}_i k_{ri}^*, \boldsymbol{c}^\top)$ here is a state space realization of $1/P_m(s)$ to ensure input-output, piecewise plant-model matching, which can always be satisfied under the stated assumptions. In other words, the existence of the parameter vectors $\boldsymbol{k}_{xi}^{*\top}$, k_{ri}^* is guaranteed. □

B. *Adaptive Control Scheme*

Since $\boldsymbol{A}_i$, $\boldsymbol{b}_i$, $i \in \mathcal{I}$, are unknown, the nominal parameters $\boldsymbol{k}_{xi}^*$, k_{ri}^* are also unknown, and the nominal control law (6) cannot be implemented. An adaptive control law with its parameters updated from some adaptive laws is needed.

Controller structure. The adaptive control law

$$u(t) = \boldsymbol{k}_x^\top(t)\boldsymbol{x}(t) + k_r(t)r(t),\quad (22)$$

is applied, where $\boldsymbol{k}_x(t)$ and $k_r(t)$ are defined as

$$\boldsymbol{k}_x(t) = \sum_{i=1}^l \boldsymbol{k}_{xi}(t)\chi_i(t),\ k_r(t) = \sum_{i=1}^l k_{ri}(t)\chi_i(t).$$

The parameters $\boldsymbol{k}_{xi}(t)$, $k_{ri}(t)$ are the adaptive estimates of $\boldsymbol{k}_{xi}^*(t)$, $k_{ri}^*(t)$, respectively, and are updated from some adaptive laws to be developed in the following subsections.

By applying the adaptive control law (22) to the plant (1), and defining $\tilde{\boldsymbol{k}}_x(t) = \boldsymbol{k}_x(t) - \boldsymbol{k}_x^*(t)$, $\tilde{k}_r(t) = k_r(t) - k_r^*(t)$, $\tilde{\boldsymbol{k}}_{xi}(t) = \boldsymbol{k}_{xi}(t) - \boldsymbol{k}_{xi}^*$, $\tilde{k}_{ri}(t) = k_{ri}(t) - k_{ri}^*$, we have

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^l \left((\boldsymbol{A}_i + \boldsymbol{b}_i\boldsymbol{k}_{xi}^{*\top})\chi_i(t)\boldsymbol{x}(t) + \boldsymbol{b}_i k_{ri}^*\chi_i(t)r(t)\right)$$
$$+ \sum_{i=1}^l \boldsymbol{b}_i k_{ri}^* \frac{1}{k_{ri}^*} \left(\tilde{\boldsymbol{k}}_{xi}^\top(t)\chi_i(t)\boldsymbol{x}(t) + \tilde{k}_{ri}(t)\chi_i(t)r(t)\right),$$
$$y(t) = \boldsymbol{c}^\top\boldsymbol{x}(t).\quad (23)$$

In view of (4) and (10), the tracking error equation follows:

$$e(t) = W_m(s)\left[\sum_{i=1}^l \frac{1}{k_{ri}^*}(\tilde{\boldsymbol{k}}_{xi}^\top\chi_i\boldsymbol{x} + \tilde{k}_{ri}\chi_i r)\right](t) + \epsilon_0(t),\quad (24)$$

where $\epsilon_0(t)$ is an initial condition related term.

Error model. To derive an estimation error equation for adaptive law design, we denote $\boldsymbol{\theta}_i(t) = [\boldsymbol{k}_{xi}^\top(t), k_{ri}(t)]^\top$, $\boldsymbol{\theta}_i^* = [\boldsymbol{k}_{xi}^{*\top}, k_{ri}^*]^\top$, and define the auxiliary signals $\boldsymbol{\omega}(t) = [\boldsymbol{x}^\top(t), r(t)]^\top$, $\boldsymbol{\zeta}_i(t) = W_m(s)[\boldsymbol{\omega}\chi_i](t)$, $\xi_i(t) = \boldsymbol{\theta}_i^\top(t)\boldsymbol{\zeta}_i(t) - W_m(s)[\boldsymbol{\theta}_i^\top\boldsymbol{\omega}\chi_i](t)$ and the estimation error signal

$$\epsilon(t) = e(t) + \sum_{i=1}^{l} \rho_i(t)\xi_i(t), \tag{25}$$

where $\rho_i(t)$ is an estimate of $\rho_i^* = k_{pi}^*$. With (24) in (25) and $\epsilon_0(t)$ ignored, we have the estimation error model

$$\epsilon(t) = \sum_{i=1}^{l} \left(\rho_i^*\tilde{\boldsymbol{\theta}}_i^\top(t)\boldsymbol{\zeta}_i(t) + \tilde{\rho}_i(t)\xi_i(t)\right), \tag{26}$$

which is linear in parameter errors $\tilde{\boldsymbol{\theta}}_i(t) = \boldsymbol{\theta}_i(t) - \boldsymbol{\theta}_i^*$, $\tilde{\rho}_i(t) = \rho_i(t) - \rho_i^*$, and is suitable for adaptive control design.

Adaptive laws. Based on the error model (26), we propose gradient adaptive laws to update $\boldsymbol{\theta}_i(t)$ and $\rho_i(t)$, $i \in \mathcal{I}$:

$$\dot{\boldsymbol{\theta}}_i(t) = -\frac{\text{sign}[k_{pi}]\boldsymbol{\Gamma}_i\boldsymbol{\zeta}_i(t)\epsilon(t)}{m^2(t)}, \;\; \boldsymbol{\Gamma}_i = \boldsymbol{\Gamma}_i^\top > 0, \tag{27}$$

$$\dot{\rho}_i(t) = -\frac{\gamma_i\xi_i(t)\epsilon(t)}{m^2(t)}, \;\; \gamma_i > 0, \;\; t \geq 0 \tag{28}$$

with arbitrary initial estimates $\boldsymbol{\theta}_i(0) = \boldsymbol{\theta}_{i0}$, $\rho_i(0) = \rho_{i0}$, $\boldsymbol{\Gamma}_i \in \mathbb{R}^{(n+1)\times(n+1)}$, $\gamma_i \in \mathbb{R}$, and the normalizing signal $m^2(t) = 1 + \sum_{i=1}^{l}\left(\boldsymbol{\zeta}_i^\top(t)\boldsymbol{\zeta}_i(t) + \xi_i^2(t)\right)$.

IV. Stability Analysis

We analyze the stability and asymptotic tracking performance of the closed-loop system with the controlled plant (1), the reference model (4), and the adaptive controller (22) updated from the adaptive laws (27)–(28). Some desired properties of the adaptive laws are presented first, which will then be used to establish the asymptotic tracking performance.

The adaptive laws (27)–(28) have the following desired properties for $i \in \mathcal{I}$:

Lemma 2. *The adaptive laws (27)–(28) ensure that* $\boldsymbol{\theta}_i(t), \rho_i(t) \in \mathcal{L}^\infty$, *and* $\frac{\epsilon(t)}{m(t)}, \dot{\boldsymbol{\theta}}_i(t), \dot{\rho}_i(t) \in \mathcal{L}^2 \cap \mathcal{L}^\infty$.

Proof: Consider the Lyapunov function candidate

$$V(\tilde{\boldsymbol{\theta}}_i, \tilde{\rho}_i) = \frac{1}{2}\sum_{i=1}^{l}\left(|\rho_i^*|\tilde{\boldsymbol{\theta}}_i^\top\boldsymbol{\Gamma}_i^{-1}\tilde{\boldsymbol{\theta}}_i + \gamma_i^{-1}\tilde{\rho}_i^2\right).$$

Its time derivative along the trajectories of (27)–(28) is

$$\begin{aligned}\dot{V}(\tilde{\boldsymbol{\theta}}_i, \tilde{\rho}_i) &= \sum_{i=1}^{l}\left(|\rho_i^*|\tilde{\boldsymbol{\theta}}_i^\top\boldsymbol{\Gamma}_i^{-1}\dot{\boldsymbol{\theta}}_i + \gamma_i^{-1}\tilde{\rho}_i\dot{\rho}_i\right)\\ &= -\frac{\epsilon^2(t)}{m^2(t)} \leq 0.\end{aligned}$$

Hence $\boldsymbol{\theta}_i(t), \rho_i(t) \in \mathcal{L}^\infty$, $\frac{\epsilon(t)}{m(t)} \in \mathcal{L}^2 \cap \mathcal{L}^\infty$. From (27)–(28) and $\frac{\boldsymbol{\zeta}_i(t)}{m(t)}, \frac{\xi_i(t)}{m(t)} \in \mathcal{L}^\infty$, we have $\dot{\boldsymbol{\theta}}_i(t), \dot{\rho}_i(t) \in \mathcal{L}^2 \cap \mathcal{L}^\infty$. ∇

With Lemma 2, we can establish the following result:

Theorem 1. *All signals in the closed-loop system with the plant (1), the reference model (4), and the controller (22) updated by the adaptive laws (27)–(28) are bounded, and the tracking error* $e(t) = y(t) - y_m(t)$ *satisfies*

$$\lim_{t\to\infty} e(t) = 0, \;\; e(t) \in \mathcal{L}^2, \tag{29}$$

if the minimum switching time interval satisfies (15).

Proof: The homogeneous part of the closed-loop system (23) is (13), and as shown in the proof of Lemma 1, with a mode switching time interval $T > T_0$, (13) is exponentially stable. Therefore, the initial condition related term expressed as $\epsilon_0(t) = \boldsymbol{c}^\top\boldsymbol{\Phi}(t,0)\boldsymbol{x}(0)$ is decaying exponentially to zero, and thus can be ignored in a gradient design.

Closed-loop signal boundedness can be proved by first using a reduced-order state observer design of the piecewise linear system (1) to parameterize the state feedback controller structure in (6) into an output feedback form:

$$u(t) = \bar{\boldsymbol{\theta}}_1^\top\bar{\boldsymbol{\omega}}_1(t) + \bar{\boldsymbol{\theta}}_2^\top\bar{\boldsymbol{\omega}}_2(t) + \bar{\theta}_{20}y(t) + \bar{\theta}_3 r(t), \tag{30}$$

$$\bar{\boldsymbol{\omega}}_1(t) = \frac{\boldsymbol{a}(s)}{\Lambda(s)}[u](t), \; \bar{\boldsymbol{\omega}}_2(t) = \frac{\boldsymbol{a}(s)}{\Lambda(s)}[y](t), \tag{31}$$

with $\bar{\boldsymbol{\theta}}_i = \sum_{j=1}^{l}\bar{\boldsymbol{\theta}}_{ij}(t)\chi_j(t)$, $i = 1, 2, 20, 3$, $\Lambda(s)$ being a monic stable polynomial of degree $n-1$, $\bar{\boldsymbol{\omega}}_1(t), \bar{\boldsymbol{\omega}}_2(t) \in \mathbb{R}^{n-1}$, and $\boldsymbol{a}(s) = [1, s, \ldots, s^{n-2}]^\top$. With $\bar{\boldsymbol{\theta}}^i \triangleq [\bar{\boldsymbol{\theta}}_{1i}^\top, \bar{\boldsymbol{\theta}}_{2i}^\top, \bar{\theta}_{20i}, \bar{\theta}_{3i}]^\top$ and $\bar{\boldsymbol{\omega}} \triangleq [\bar{\boldsymbol{\omega}}_1^\top, \bar{\boldsymbol{\omega}}_2^\top, y, r]^\top$, the adaptive control design in Section III with $\boldsymbol{\theta}_i$ replaced by $\bar{\boldsymbol{\theta}}^i$, $\boldsymbol{\omega}$ by $\bar{\boldsymbol{\omega}}$, results in the desired signal properties as stated in Lemma 1: $\bar{\boldsymbol{\theta}}^i(t), \rho_i(t) \in \mathcal{L}^\infty$, and $\frac{\epsilon(t)}{m(t)}, \dot{\bar{\boldsymbol{\theta}}}^i(t), \dot{\rho}_i(t) \in \mathcal{L}^2 \cap \mathcal{L}^\infty$.

A filtered system output $y(t)$, i.e., $z(t) \triangleq \frac{1}{s+a_0}[y](t)$ is then expressed in a feedback framework that is suitable for the application of the small gain theorem and with the signal properties as above [11] to conclude signal boundedness. In particular, with $z_0(t) \triangleq \frac{1}{s+a_0}[u](t)$ for $a_0 > 0$, it can be shown [11, Theorem 5.4] that

$$z_0(t) = T_1(s,\cdot)[z](t) + b_0(t), \;\; b_0(t) \in \mathcal{L}^\infty, \tag{32}$$

for some stable and proper operator $T_1(s,\cdot)$. Furthermore,

$$|\epsilon(t)| \leq \frac{|\epsilon(t)|}{m(t)}\left(1 + \sum_{i=1}^{l}\left(\|\boldsymbol{\zeta}_i(t)\| + |\xi_i(t)|\right)\right),$$

$$z(t) = \frac{1}{s+a_0}[y_m](t) + \frac{1}{s+a_0}\left[\epsilon - \sum_{i=1}^{l}\rho_i\xi_i\right](t).$$

It follows that $|z(t)| \leq x_0(t) + T_2(s,\cdot)[x_1 T_3(s,\cdot)[|z|]](t)$ for some $x_0(t) \in \mathcal{L}^\infty$, $x_1 \in \mathcal{L}^\infty \cap \mathcal{L}^2$, stable and strictly proper operator $T_2(s,\cdot)$, and stable and proper operator $T_3(s,\cdot)$, where we have used the fact that $|y(t)\chi_i(t)| \leq |y(t)|$, $|r(t)\chi_i(t)| \leq |r(t)|$. With $z_1(t) \triangleq T_3(s,\cdot)[|z|](t)$, we have

$$z_1(t) \leq b_1 + b_2\int_0^t e^{-\alpha(t-\tau)}x_1(\tau)z_1(\tau)d\tau \tag{33}$$

for some $\alpha, b_1, b_2 > 0$. By applying the small gain theorem, we can conclude $z_1(t), z(t), z_0(t) \in \mathcal{L}^\infty$, thus $u(t)$, $y(t) \in \mathcal{L}^\infty$, $\boldsymbol{\zeta}_i(t), \xi_i(t) \in \mathcal{L}^\infty$, and $\epsilon(t), \dot{\epsilon}(t) \in \mathcal{L}^\infty$ from (26). By the

Barbălat Lemma, it follows that $\lim_{t\to\infty}\epsilon(t)=0$, $\xi_i(t)\in\mathcal{L}^2$ and $\lim_{t\to\infty}\xi_i(t)=0$, and from (25), we have (29). ∇

Remark 3: The minimum switching time interval requirement, $T>T_0$ for some $T_0>0$, is for ensuring internal stability in the presence of system mode switches. It can be relaxed to $T>0$, i.e., arbitrarily fast mode switches, when $\boldsymbol{A}_i+\boldsymbol{b}_i\boldsymbol{k}_{xi}^{*\mathsf{T}}=\boldsymbol{A}_m$, $i\in\mathcal{I}$, in (9) for some stable matrix $\boldsymbol{A}_m$, which implies that $Z_i(s)=Z(s)$, $i\in\mathcal{I}$, for some stable polynomial $Z(s)$ specifying system zeros. Systems in certain special forms, e.g., controllable canonical form, fit in this context. An illustrative simulation example in this regard is given in Section V. □

Remark 4: In the output tracking design, the analysis method used is analogous to the conventional state feedback output tracking design for an LTI plant. An estimation error $\epsilon(t)$ (along with some auxiliary signals) is defined as in (25) and the proposed gradient adaptive laws are such that the desired signal properties in Lemma 2, i.e., the boundedness of parameter estimates, and $\dot{\boldsymbol{\theta}}_i,\dot{\rho}_i,\epsilon/m\in\mathcal{L}^2\cap\mathcal{L}^\infty$ remain in spite of the presence of mode switches, and these properties help establish signal boundedness and the asymptotic tracking performance in a feedback framework for which small gain theorem can be applied. The only difference in the analysis with respect to the conventional output tracking design is the requirement on a minimum mode switching time interval T_0, which is needed to ensure the exponential decaying of the initial condition related term $\epsilon_0(t)$ and internal stability of the closed-loop system.

On the other hand, in the state feedback state tracking design for the general case [10], an analogous Lyapunov analysis to the conventional state tracking design cannot be carried out due to the nonexistence of a common $\boldsymbol{P}$ matrix for $\boldsymbol{A}_{mi}$, in general. A piecewise Lyapunov function was adopted, instead. However, the minimum switching time interval requirement for ensuring a stable reference model system cannot ensure bounded parameter estimates, thus a parameter projection algorithm is needed. Additional switching time interval requirements are imposed for establishing the boundedness of $\boldsymbol{e}(t)$ with the boundedness of parameter estimates. Asymptotic state tracking performance cannot be concluded due to the loss of $\boldsymbol{e}(t)\in\mathcal{L}^2$ property. □

V. Illustrative Examples

A simulation study is first performed for a piecewise linear system in controllable canonical form (CCF), then the proposed design is applied to the linearized NASA GTM models to demonstrate its effectiveness, as compared with the conventional MRAC design for LTI systems.

A. Simulation for Systems in CCF

We consider a piecewise linear system (1) with $n=3$, $l=3$: $\boldsymbol{A}_1=[0,1,0;0,0,1;2,-3,-5]$, $\boldsymbol{b}_1=[0,0,1]^\mathsf{T}$, $\boldsymbol{A}_2=[0,1,0;0,0,1;-2,0,1]$, $\boldsymbol{b}_2=[0,0,2]^\mathsf{T}$, $\boldsymbol{A}_3=[0,1,0;0,0,1;0,1,-2]$, $\boldsymbol{b}_3=[0,0,0.5]^\mathsf{T}$, and $\boldsymbol{c}=[1,1,0]^\mathsf{T}$. With $m=1$ and $n^*=n-m=2$, we choose a reference model system (4) with $W_m(s)=1/(s^2+2s+1)$.

It can be verified that the matching condition (9) can be satisfied with $\boldsymbol{k}_{x1}^*=[-3,0,2]^\mathsf{T}$, $k_{r1}^*=1$, $\boldsymbol{k}_{x2}^*=[-0.5,-1.5,-2]^\mathsf{T}$, $k_{r2}^*=0.5$, $\boldsymbol{k}_{x3}^*=[-2,-8,-2]^\mathsf{T}$, $k_{r3}^*=2$. In all simulations, system mode switches in sequence every 10 seconds, i.e., $T=10$s. The simulation parameters are as follows: $\boldsymbol{x}(0)=[4,-5,3]^\mathsf{T}$, $\boldsymbol{x}_m(0)=\boldsymbol{0}$, $\boldsymbol{\theta}_i(0)=0.8\boldsymbol{\theta}_i^*$, $\rho_i(0)=0.8\rho_i^*$, $\boldsymbol{\Gamma}_i=3\boldsymbol{I}$, $\gamma_i=3$, for $i=1,2,3$.

The tracking error $e(t)$ is plotted in Fig. 1 (top) for $r(t)=2\sin(t)$. As a comparison, the tracking performance under the same conditions with the conventional state feedback MRAC scheme for output tracking [11] is also shown in Fig. 1 (bottom). It is clear that asymptotic tracking is achieved with

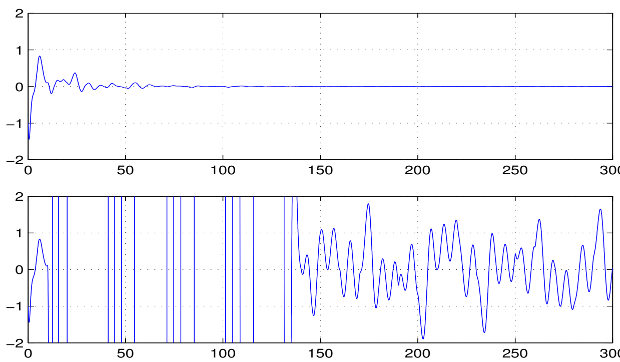

Fig. 1. Tracking error $e(t)$ with $r(t)=2\sin(t)$: the proposed scheme (top) vs. the conventional MRAC scheme (bottom).

the proposed adaptive control scheme, which is a significant performance improvement over the conventional scheme that does not take into account the parameter discontinuities.

B. Simulation on NASA GTM

1) Linearized Aircraft Longitudinal Model and Reference Model System: For simplicity of presentation, we choose $l=2$, and trim the GTM at steady-state, straight, wings-level flight condition at 80 knots and 90 knots at 800 ft., respectively, to obtain a piecewise linear longitudinal system model in the form of (1), where $\boldsymbol{x}=[u,w,q,\theta]^\mathsf{T}$ with the elements being the perturbed aircraft velocity components along the x- and z-body-axis (fps), angular velocity along the y-body-axis (crad/s), and pitch angle (crad), respectively. The control input is the perturbed elevator deflection δ_e, and the parameter matrices are

$$\boldsymbol{A}_1=\begin{bmatrix}-0.0293 & 0.2460 & -0.0899 & -0.3210\\ -0.2611 & -3.0403 & 1.2973 & -0.0222\\ 1.7458 & -32.0173 & -3.8364 & 0\\ 0 & 0 & 1.0000 & 0\end{bmatrix},$$

$$\boldsymbol{A}_2=\begin{bmatrix}-0.0380 & 0.2786 & -0.0750 & -0.3213\\ -0.2440 & -3.4119 & 1.4623 & -0.0165\\ 1.3633 & -35.8069 & -4.4019 & 0\\ 0 & 0 & 1.0000 & 0\end{bmatrix},$$

$$\boldsymbol{b}_1=\begin{bmatrix}0.0031\\ -0.6953\\ -85.2589\\ 0\end{bmatrix},\boldsymbol{b}_2=\begin{bmatrix}-0.0010\\ -0.8703\\ -108.6559\\ 0\end{bmatrix},\boldsymbol{c}=\begin{bmatrix}0\\0\\0\\1\end{bmatrix},$$

with the corresponding transfer functions:

$$G_1(s) = \frac{-85.26s^2 - 239.4s - 12.66}{s^4 + 6.906s^3 + 53.62s^2 + 2.324s + 4.375},$$
$$G_2(s) = \frac{-108.7s^2 - 343.7s - 20.63}{s^4 + 7.852s^3 + 67.85s^2 + 3.157s + 4.286},$$

where we have chosen the pitch angle θ as the output, i.e., $y = \theta$. It is clear that $m = 2$, thus $n^* = n - m = 2$, and the reference model transfer function is chosen as

$$W_m(s) = \frac{1}{(s+1)(s+2)}.$$

2) Matching Condition and Nominal Parameters: It can be verified that the plant-model matching condition (9) can be satisfied with the nominal controller parameters $\boldsymbol{k}_{x1}^* = [0.0205, -0.3755, -0.0098, 0.0235]^\mathsf{T}$, $k_{r1}^* = -0.0117$, $\boldsymbol{k}_{x2}^* = [0.0125, -0.3295, -0.0129, 0.0184]^\mathsf{T}$, $k_{r2}^* = -0.0092$, and

$$\|e^{\boldsymbol{A}_{m1}t}\| \leq 12.5796e^{-0.0539t}, \quad \|e^{\boldsymbol{A}_{m2}t}\| \leq 10.6393e^{-0.0612t}.$$

With $\boldsymbol{Q}_{m1} = \boldsymbol{Q}_{m2} = \boldsymbol{I}_2$ in (14), we can compute from (15) the minimum switching time interval as $T_0 > 139.9350$s.

3) Simulation Results: In this simulation, we choose a switching time interval $T = 150$s. The initial plant state is $[5, 2, 0, 10]^\mathsf{T}$ with zero reference model initial condition, and the initial parameter estimates are chosen to be 80% of their nominal values. The adaptation gains are chosen to be unity.

Figure 2 (top) shows the output tracking error $e(t)$ with a sinusoidal reference input $r(t) = 8\sin(0.3t)$, corresponding to a desired fluctuation of the pitch angle in between $\pm 2.6°$. As a comparison, the tracking performance with the conventional SISO state feedback output tracking MRAC scheme [11] under the same conditions is plotted in Fig. 2 (bottom).

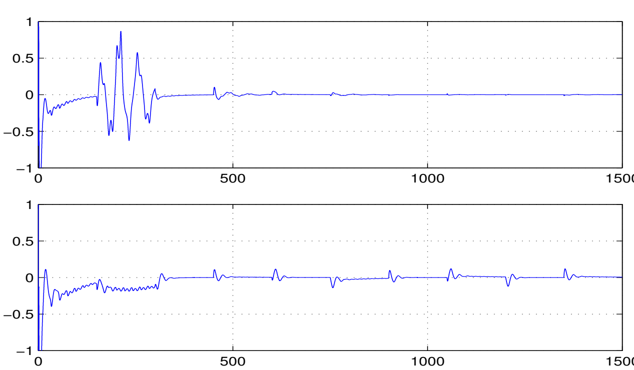

Fig. 2. Tracking error $e(t)$ with $r(t) = 8\sin(0.3t)$: the proposed scheme (top) vs. the conventional MRAC scheme (bottom).

From the simulation results, we can see that the closed-loop stability is achieved for both simulations. However, the proposed adaptive control scheme achieves asymptotic output tracking, which is a substantial improvement over the conventional MRAC scheme under the same conditions.

VI. Conclusions

A direct model reference adaptive control (MRAC) scheme is developed in this paper for SISO piecewise linear systems. The proposed control design employs the knowledge of the time instants of parameter discontinuities, which is characterized by the indicator functions. Closed-loop signal boundedness and asymptotic output tracking are achieved via state feedback for sufficiently slow system mode switches. Simulation results demonstrate the effectiveness of the proposed adaptive control scheme.

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