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An Algebraic Approach to Design Observers for Delay-Independent Stability of Systems with Single Output Delay

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Abstract— **This paper presents an algebraic approach to design the control law of a LTI observer used to stabilize a LTI plant with an output delay. Different than the existing work, we use the observer gains to influence the plant stability. This becomes possible simply by removing the delay terms from the observer part. Given the plant controller gains, our approach can find the parametric regions with respect to the observer controller gains so that gains selected from these regions make the combined plant-observer system asymptotically stable independent of the amount of delay in the plant. An example with simulations is provided to demonstrate the advantages of the proposed observer design.**

Keywords: **Time-Delay System, Delay-free Observer, Algebraic approach, Delay-independent Stability.**

I. INTRODUCTION

Time-delay systems (TDS) describe a wide range of dynamical systems [1], [2], [3], [4], and availability of system states is crucial for designing controllers for these systems [5], [6], [7]. In many cases, however, not all state variables are available for measurements. Therefore, a state observer is usually used in order to estimate the unavailable state variables [8]. Observer design for LTI time-delay systems (TDS) has attracted considerable interest over the years, and several design methods have been proposed [9], [10], [11], [12], [13], [14], including coordinate-change approach [15], reducing transformation technique [16], polynomial approaches [17], [18], frequency domain approaches [19], [20], and the factorization approach [21]. There also exist studies that propose to make observer states *unaffected* from delays [22], [23]. Another useful technique is to incorporate the delay of the plant into the observer [24]. Adding the delay in the observer dynamics allows an elegant decoupling of the stability properties of the plant and observer. Consequently, one can independently design the plant and observer controllers [24], [25]. This decoupling idea, which can be implemented in frequency domain, is practical as it simplifies the control design.

In this paper, however, we do the contrary; we remove the decoupling property. This in turns allows influencing the LTI plant stability by using the LTI observer controller gains. In other words, when the decoupling property is removed by not allowing the delay term in the observer, the observer becomes a stability facilitator for the plant. This approach not only leads to the design of a stable observer, but it also uses the observer gains advantageously to make the plant dynamics *delay-independent stable*. More specifically, given the plant controller gains, our approach can find the parametric regions with respect to the observer controller gains so that gains selected from these regions make the combined plant-observer system asymptotically stable independent of the amount of delay in the plant. That is, the approach reveals the gain space that achieves the stability of the combined system, no matter how large the delay is. Moreover, the approach is able to express the boundaries of these regions algebraically as a function of observer controller gains. The delay-independent stability (DIS) analysis becomes possible with the results of Descartes on algebra [26], [27], which allow designing the observer controller gains with sufficient stability conditions.

The rest of the paper is organized as follows. In Section II, we introduce the main problem, the preliminaries on the stability of TDS, and Descartes rule of signs. In Section III, the main results are presented, and a case study including time-domain simulations and spectrum computations are given in Section IV. Section V ends the paper with conclusions.

II. PRELIMINARIES

A. Delay-free observer design

The focus in this paper is on the control of the following LTI plant with a single output delay $\tau \geq 0$,

$$\begin{aligned} \dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}u(t), \\ y(t) &= \mathbf{C}x(t-\tau), \\ u(t) &= \mathbf{K}\tilde{x}(t), \end{aligned} \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times m}$ are the constant system and control matrices, respectively, $\mathbf{C} \in \mathbb{R}^{q\times n}$ is the output matrix, $x(t) \in \mathbb{R}^n$ is the system state vector, $u(t)$ is the control input to the plant, $y(t) \in \mathbb{R}^q$ is the output vector, and $\mathbf{K} = \{k_{iv}\}$ is the plant control law, where $k_{iv} \in \mathbb{R}$ are the controller gains of the plant. In (1), the system states $x(t)$ are unavailable, hence the prediction of the states $\tilde{x}(t) \in \mathbb{R}^n$ is used to control the plant. The prediction $\tilde{x}(t)$ can be obtained from the *delay-free* observer,

$$\begin{aligned} \dot{\tilde{x}}(t) &= \mathbf{A}\tilde{x}(t) + \mathbf{B}u(t) + \tilde{\mathbf{K}}(\tilde{y}(t) - y(t)), \\ \tilde{y}(t) &= \mathbf{C}\tilde{x}(t), \end{aligned} \tag{2}$$

where $\tilde{y}(t) \in \mathbb{R}^q$ is the observer output vector, and $\tilde{\mathbf{K}} = \{\tilde{k}_{iv}\}$ is the observer control law, where $\tilde{k}_{iv} \in \mathbb{R}$ are the controller gains of the observer. Different from the literature,

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the observer output $\tilde{y}(t)$ in (2) is not affected by τ. The dynamics of the combined system then becomes

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\tilde{x}}(t) \end{bmatrix} = \mathbf{A}^* \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} + \mathbf{B}^* \begin{bmatrix} x(t-\tau) \\ \tilde{x}(t-\tau) \end{bmatrix}, \qquad (3)$$

with

$$\mathbf{A}^* = \begin{bmatrix} \mathbf{A} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A}+\mathbf{BK}+\tilde{\mathbf{K}}\mathbf{C} \end{bmatrix} \text{ and } \mathbf{B}^* = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\tilde{\mathbf{K}}\mathbf{C} & \mathbf{0} \end{bmatrix}, \qquad (4)$$

where $\mathbf{0}$ is an $n \times n$ matrix with all its entries zero. Since we propose a delay-free observer, the observer dynamics cannot be decoupled from that of the plant. Therefore, the combined system in (3) should be considered as a whole. It must be noted that the selected LTI plant and its observer do not have disturbances and uncertainties. These conditions can be relaxed in future work.

Remark 1: It is important to note that system (3) can be considered as the homogenous part of an output tracking problem, in which input reference will not influence stability. Once the stability of (3) is achieved, tracking performance measured by transient and steady state characteristics can be studied with respect to $\tilde{k}_{iv}$ and τ.

B. Stability of the single-delay system in (3)

Stability of (3) is studied over its characteristic equation given by

$$f(s,\tau,\tilde{k}_{iv}) = |s\mathbf{I} - \mathbf{A}^* - \mathbf{B}^* e^{-\tau s}| = 0, \qquad (5)$$

where k_{iv} is assumed to be known, τ and $\tilde{k}_{iv}$ are unknown parameters, $\mathbf{I}$ is the $2n \times 2n$ identity matrix, s is the Laplace variable, and delay τ appears in the exponent in Laplace domain. The characteristic equation (5) can be written as

$$f(s,\tau,\tilde{k}_{iv}) = \sum_{r=0}^{c} P_r(s,\tilde{k}_{iv}) e^{-r\tau s} = 0, \qquad (6)$$

where P_r are polynomials in terms of s and the entries of $\tilde{\mathbf{K}}$, and c represents the order of commensuracy of τ, which is $c = rank(\mathbf{B}^*)$ [28].

Stability analysis requires to detect critical values τ^* for which at least one characteristic root of (6) lies on the imaginary axis, $s = j\omega$, where $\omega \geq 0$ without loss of generality [29]. For finding the stability *transitions* of the system with respect to the delay parameter, the movement of $s = j\omega$ across the imaginary axis should be checked as τ^* infinitesimally increases. Roots crossing the imaginary axis from left to right (or from right to left) favor instability (or stability). This information can be used to account for the unstable roots of the system, leading to the stability analysis with respect to τ [30].

Solving $s = j\omega$ roots from the transcendental characteristic equation (6) is cumbersome [29], yet several enabling manipulations can be done. For instance, we can convert the infinite dimensional characteristic equation (6) to a finite dimensional characteristic equation that has continuous coefficients. This conversion does not lose the infinite dimensional nature of the problem, and can be done via the exact Rekasius transformation [31],

$$e^{-\tau s} \equiv \frac{1-Gs}{1+Gs},\ G \in \mathbb{R},\ s = j\omega,\ \omega \geq 0, \qquad (7)$$

which has a back-transformation rule from (G,ω) domain to τ domain [28] via the following

$$\tau = \frac{2}{\omega}[\arctan(\omega G) \pm L\pi], \qquad (8)$$

where $L = 0, 1, 2, \ldots$, and $0 \leq \arctan(\cdot) < \pi$. With the Rekasius substitution of (7) into (6), and with a manipulation to remove the fractions, we obtain the transformed characteristic equation on the imaginary axis $s = j\omega$ as

$$g(j\omega, G, \tilde{k}_{iv}) = \left(f \Big|_{e^{-j\omega\tau} = \frac{1-Gj\omega}{1+Gj\omega}} \right) (1+Gj\omega)^c = 0, \qquad (9)$$

where g is a polynomial in terms of its arguments.

Property 1: It is crucial to note that (7) is not a Padé approximation, but it is an exact transformation for the imaginary roots of (6). The imaginary roots $s = j\omega^*$ of f and the imaginary roots $s = j\omega^*$ of g are identical to each other, and they are finite numbered. That is, each and every $s = j\omega^*$ root of f is also a root of g, and vice versa. All $G = G^* \in \mathbb{R}$ values that are solutions of g for $s = j\omega^*$ have a mapping to τ domain: $(G^*, \omega^*) \longrightarrow \tau^*$ [32]. This is a one-to-infinity mapping as can be seen from (8).

Remark 2: For delay-independent stability of TDS, it is necessary that the uncontrolled system and the delay-free controlled system are Hurwitz stable [14], [30]. This automatically guarantees that $\omega = 0$ cannot be a feasible solution of the characteristic equation for both $\tau = 0$ and any finite τ. Hence, $\omega = 0$ solutions are ignored in the rest of the text.

C. On algebraic polynomials

We present an interesting result from the theory of polynomials. The theoretical development in the main results section will reformulate the stability problem introduced above as the analysis of a single-variable algebraic polynomial. At that point, the following definition will be essential to establish the delay-independent stability property of (3).

Property 2: (Descartes Rule of Signs [26], [27]): For any non-zero real polynomial A, the number of sign variations, $var(A)$, in the coefficient sequence of the polynomial is less than or equal to the number of positive real zeros of A, counting multiplicities, by a non-negative even integer [33]. That is, if $var(A) = 0$, then the polynomial A is guaranteed to have no positive real zeros, and if $var(A)$ is odd, then A is guaranteed to have at least one positive real zero. When $var(A)$ is even, Descartes rule of signs is however inconclusive on the number of positive real zeros of A. In such a case, A has either no positive real zeros, or has even number of positive real zeros.

III. Main Results

In this section, we develop an algebraic approach to select the observer controller gains $\tilde{k}_{iv}$. For this objective, we connect the transformed equation (9) to the results of Descartes. Recall that the imaginary spectrum of (3) is converted to $s = j\omega$ roots of g in (9). This is where we start. We first represent g as follows,

$$g(j\omega, G, \tilde{k}_{iv}) = g_{\mathbb{R}}(\omega, G, \tilde{k}_{iv}) + j g_{\mathbb{I}}(\omega, G, \tilde{k}_{iv}) = 0, \qquad (10)$$

where $g_{\mathbb{R}}$ and $g_{\mathbb{I}}$ are respectively the real and imaginary parts of g, and are given by,

$$g_{\mathbb{R}}(\omega, G, \tilde{k}_{iv}) = \sum_{r=0}^{c} \alpha_r(\omega, \tilde{k}_{iv}) G^r, \tag{11}$$

$$g_{\mathbb{I}}(\omega, G, \tilde{k}_{iv}) = \sum_{r=0}^{c} \beta_r(\omega, \tilde{k}_{iv}) G^r. \tag{12}$$

Given $\tilde{k}_{iv}$, in order to solve (ω, G) pairs from (10), one should guarantee that $g_{\mathbb{R}} = 0$ and $g_{\mathbb{I}} = 0$, concurrently. At this step, we can eliminate G from these two equations to simplify the analysis. The elimination can be done by using the resultant theory [34]. A $2c$-order Sylvester matrix, $\mathbf{S}$, is constructed by using the coefficients of G in (11) and (12). In other words, the unknowns in $\mathbf{S}$ are ω and $\tilde{k}_{iv}$. For the common ω solutions of (11) and (12), Sylvester matrix $\mathbf{S}$ is singular (but *not vice versa*). That is, the resultant $R_G(\omega, \tilde{k}_{iv})$, which is the determinant of $\mathbf{S}$, should be studied for its zeros,

$$\det(\mathbf{S}) = R_G(\omega, \tilde{k}_{iv}) = \begin{vmatrix} \alpha_c & \alpha_{c-1} & . & .\,. & \alpha_0 & 0 & 0 & 0 \\ 0 & \alpha_c & \alpha_{c-1} & .\,. & \alpha_1 & \alpha_0 & 0 & 0 \\ . & . & . & .\,. & . & . & . & . \\ . & . & . & .\,. & . & . & \alpha_1 & \alpha_0 \\ \beta_c & \beta_{c-1} & . & .\,. & \beta_0 & 0 & 0 & 0 \\ 0 & \beta_c & \beta_{c-1} & .\,. & \beta_1 & \beta_0 & 0 & 0 \\ . & . & . & .\,. & . & . & . & . \\ . & . & . & .\,. & . & . & \beta_1 & \beta_0 \end{vmatrix}. \tag{13}$$

Excluding $\omega = 0$ zeros of R_G, it is easy to see that R_G is an even polynomial with respect to ω. Hence, we can express R_G by defining a new variable y with $y = \omega^2$, where the number of real zeros of R_G is equal to the number of positive real zeros of the new polynomial $\Phi(y, \tilde{k}_{iv})$. Hence, we can inspect the existence of the positive real roots of $\Phi(y, \tilde{k}_{iv}) = 0$ for a given set of observer gains. If no roots are positive, this implies that no admissible $y = \omega^2 > 0$ roots exist, hence a stability switch is impossible. If, by construction, the conditions in Remark 2 also hold, we can then conclude that the system remains stable independent of the amount of delay τ. Presence of at least one positive real zero of $\Phi(y, \tilde{k}_{iv})$, however, requires checking whether $G \in \mathbb{R}$ solution exists satisfying (10). If such a G exists, then delay-independent stability can not be rendered. If such a G does not exist, then we can still declare delay-independent stability.

As reviewed in the previous section, Descartes rule of signs is a method that can assess the *number* of positive real roots of a polynomial with real coefficients, without solving for these roots. The following theorem establishes the connections between delay-independent stability (DIS), the resultant, and Descartes rule of signs.

Theorem 1: LTI-TDS in (3) is DIS in the delay parameter space τ with observer gain $\tilde{\mathbf{K}}$, if

i) $|s\mathbf{I} - \mathbf{A}^*|$ is Hurwitz stable,

ii) $|s\mathbf{I} - \mathbf{A}^* - \mathbf{B}^*|$ is Hurwitz stable,

iii) $var(\Phi) = 0$.

Proof: For the system in (3) to be delay-independent stable, it is necessary that the uncontrolled system and the delay-free controlled system ($\tau = 0$) are Hurwitz stable as per Remark 2. These conditions correspond to (i)-(ii) in Theorem 1. Furthermore, a system can never change its stability/instability behavior with respect to delay τ if it does not possess any characteristic roots on the imaginary axis. That is, $s = j\omega$, $\omega \in \mathbb{R}$, should not be a root of the corresponding characteristic equation f. Based on the derivations above, $s = j\omega$ root of f does not exist if $y = \omega^2 > 0$ roots of $\Phi = 0$ do not exist, which is guaranteed by (iii) of Theorem 1 as per Descartes rule of signs. ∎

We note that, given $\mathbf{A}^*$, $\mathbf{B}^*$, $\tilde{\mathbf{K}}$, delay-independent stability test can be completed by checking only the conditions (ii)-(iii) of Theorem 1, since these conditions guarantee that condition (i) holds. Notice also that application of Descartes rule of signs leads to sufficient conditions for delay-independent stability regions in the observer gain space [35]. Study of delay-independent stability with both necessary and sufficient conditions is left to a future study.

Theorem 2: There exists at least one observer control law $\tilde{\mathbf{K}}$ that makes the combined system in (3) delay-independent stable if $\mathbf{A}^*$ is Hurwitz.

Proof: Let $\tilde{\mathbf{K}} = \mathbf{0}$. In this case, we have $\mathbf{A}^* = \left[\begin{smallmatrix} \mathbf{A} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A+BK} \end{smallmatrix}\right]$ and $\mathbf{B}^* = \mathbf{0}$. Under this condition, the system in (3) is Hurwitz stable for a given $\mathbf{K}$, if and only if $\mathbf{A}^*$ is Hurwitz stable. The critical observation is that system in (3) is Hurwitz stable for all the delay values, $\tau \geq 0$. This is possible since $\mathbf{B}^*$ is zero, which consequently means that (3) is delay-independent stable for $\tilde{\mathbf{K}} = \{\tilde{k}_{11}, \cdots, \tilde{k}_{iv}, \cdots, \tilde{k}_{nm}\} = \mathbf{0}$ if $\mathbf{A}^*$ is Hurwitz stable. In other words, the point $\{\tilde{k}_{11}, \cdots, \tilde{k}_{iv}, \cdots, \tilde{k}_{nm}\} = \mathbf{0}$ lies in the DIS region in the parameter space of the observer controller gains. It is then straightforward to show from the continuity of the characteristic roots of the system [36] that there exist observer gains $\tilde{k}_{iv} = \tilde{\varepsilon}_{iv}$, $|\tilde{\varepsilon}_{iv}| \ll 0$ such that the combined system (3) remains DIS. ∎

Theorem 2 gives the rule by which the plant controller $\mathbf{K}$ can also be designed along with $\tilde{\mathbf{K}}$. Moreover, Theorem 2 indicates that as long as $\mathbf{A}^*$ is Hurwitz stable, we will always find an admissible parametric region in observer controller gain space, where the combined system (3) is DIS. Notice however that since Descartes rule of signs is based on sufficient conditions, this rule cannot identify the DIS regions when $var(R_G)$ is an even number in the DIS region neighboring $\tilde{\mathbf{K}} = \mathbf{0}$. Consideration of this problem is left to a future study.

IV. Case Study

Consider the plant dynamics

$$\mathbf{A} = \begin{bmatrix} -27 & 3.6 & 6 \\ 9.6 & -12.5 & 0 \\ 0 & 9 & -5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0.26 & 0 \\ 0.9 & 0.8 \\ 0 & -0.18 \end{bmatrix}, \tag{14}$$
$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

Let us set the plant and observer controllers as,

$$\mathbf{K} = \begin{bmatrix} 0 & -10 & 0 \\ -10 & -50 & -20 \end{bmatrix}, \quad \tilde{\mathbf{K}} = \begin{bmatrix} -\tilde{k}_1 \\ -\tilde{k}_2 \\ -10 \end{bmatrix}. \tag{15}$$

$\mathbf{A}^*$ and $\mathbf{B}^*$ matrices are suppressed here as they can be constructed using (14) and (15) in (4). In this problem, we have $n = 3$ and $c = rank(\mathbf{B}^*) = 1$. The objective is to find the regions in the parameter space of $\tilde{k}_1 - \tilde{k}_2$ so that a pair ($\tilde{k}_1$, $\tilde{k}_2$) selected from these regions guarantees delay-independent stability of (3).

Following the procedure explained in the previous section, we start with the characteristic equation of the coupled plant-observer system given by

$$f(s,\tau,\tilde{k}_1,\tilde{k}_2) = P_0 + P_1 e^{-\tau s} = 0, \tag{16}$$

where P_0 and P_1 are polynomials in s with coefficients in terms of $\tilde{k}_1$ and $\tilde{k}_2$. After the Rekasius substitution in (16), and substituting $s = j\omega$, the real and imaginary parts of the transformed characteristic equation are found as,

$$g_{\mathbb{R}}(\omega,G,\tilde{k}_1,\tilde{k}_2) = \left(\sum_{\ell=2}^{6} a_{1\ell}(\tilde{k}_1,\tilde{k}_2)\omega^\ell\right) G + \left(\sum_{\ell=0}^{6} a_{0\ell}(\tilde{k}_1,\tilde{k}_2)\omega^\ell\right), \tag{17}$$

$$g_{\mathbb{I}}(\omega,G,\tilde{k}_1,\tilde{k}_2) = \left(\sum_{\ell=1}^{7} b_{1\ell}(\tilde{k}_1,\tilde{k}_2)\omega^\ell\right) G + \left(\sum_{\ell=1}^{5} b_{0\ell}(\tilde{k}_1,\tilde{k}_2)\omega^\ell\right), \tag{18}$$

where $a_{0\ell}$, $a_{1\ell}$, $b_{1\ell}$ and $b_{1\ell}$ are suppressed for conciseness. The Sylvester matrix formed by using (17) and (18), and by eliminating G is a 2×2 matrix

$$\mathbf{S} = \begin{bmatrix} \sum_{\ell=2}^{6} a_{1\ell}(\tilde{k}_1,\tilde{k}_2)\omega^\ell & \sum_{\ell=0}^{6} a_{0\ell}(\tilde{k}_1,\tilde{k}_2)\omega^\ell \\ \sum_{\ell=1}^{7} b_{1\ell}(\tilde{k}_1,\tilde{k}_2)\omega^\ell & \sum_{\ell=1}^{5} b_{0\ell}(\tilde{k}_1,\tilde{k}_2)\omega^\ell \end{bmatrix}, \tag{19}$$

from which we calculate the resultant by computing the determinant of $\mathbf{S}$. It is given by

$$\Phi(y,\tilde{k}_1,\tilde{k}_2) = \sum_{i=0}^{6} \gamma_i(\tilde{k}_1,\tilde{k}_2) y^i = 0, \tag{20}$$

where $y = \omega^2$, and $\omega = 0$ solutions are dropped as per Remark 2.

For DIS regions to exist in $\tilde{k}_1 - \tilde{k}_2$ plane, it is necessary that $\mathbf{A}^*$ is Hurwitz stable, see Theorem 1. Using the well-known Routh-Hurwitz stability criterion [8], we find that $\mathbf{A}^*$ remains stable in the light-gray-shaded region in Figure 1. Next, we find the DIS regions of the plant-observer system in $\tilde{k}_1 - \tilde{k}_2$ plane using Theorem 1 on (20).

A. Extraction of DIS regions; Sufficient conditions using Descartes rule of signs

According to Descartes rules of signs, all $\gamma_i(\tilde{k}_1,\tilde{k}_2)$ in (20) must have the same sign so that $\Phi = 0$ is guaranteed *not to have* any positive real y roots. To apply the rule, one can first draw the boundaries $\gamma_i(\tilde{k}_1,\tilde{k}_2) = 0$ in the observer gain plane. Next, by testing one point in each arising region, we can determine the parametric regions where all $\gamma_i(\tilde{k}_1,\tilde{k}_2)$ have the same sign. With the information provided in Property 2, the identification leads to three types of regions

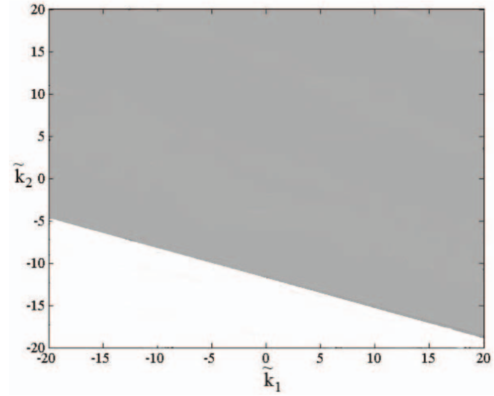

Fig. 1. Hurwitz stable region (light gray) in $(\tilde{k}_1,\tilde{k}_2)$ for the combined system when $\tau = 0$.

in Figure 2, namely, the delay-independent stable region identifiable by Descartes rule of signs (dark gray), the region where DIS is impossible since $\mathbf{A}^*$ is not Hurwitz (white region), and light gray regions where Descartes rule of signs is either inconclusive (since $var(\Phi)$ is even) or concludes that DIS property is impossible (since $var(\Phi)$ is odd).

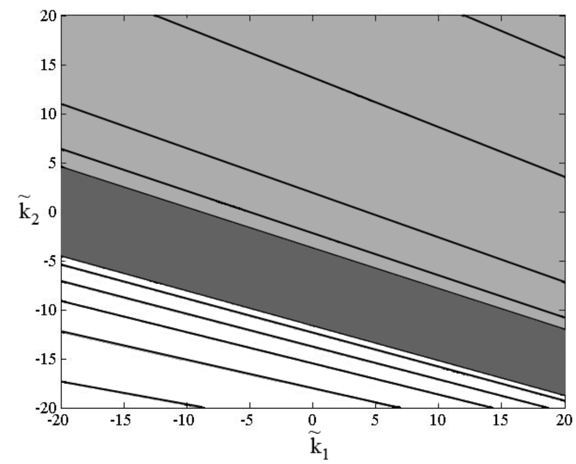

Fig. 2. DIS region of the combined system (dark gray) in $(\tilde{k}_1,\tilde{k}_2)$ found by using Theorem 2. The curves are determined by $\gamma_i = 0$.

Some interesting observations on Figure 2 are as follows. The DIS region (dark gray) can neighbor a region that does not lead to Hurwitz stability of the delay-free system (white region). The boundaries that separate such regions present significant lack of robustness. This observation also concludes that the delay-free controlled system can have stable eigenvalues that are significantly close to the imaginary axis, but the system can still be made DIS by selecting $\tilde{k}_1$ and $\tilde{k}_2$ from the dark gray region. This is again a consequence of proposing a delay-free observer, which strengthens the stability of the system against the delay τ.

B. Simulations

We implement the designed observer controller gains in time domain simulations in order to inspect the output response $y(t)$. For this, we choose $\tilde{k}_1 = -20$ and $\tilde{k}_2 = -1$, which is a point in the DIS region (dark gray) shown in Figure 2. We next simulate the combined system using MATLAB/Simulink. With appropriate settings of the numerical integration method and with different initial conditions in the plant and the observer, the simulation results are obtained, see Figure 3 and Figure 4 for $\tau = 0.5$ and $\tau = 3$ sec cases, respectively.

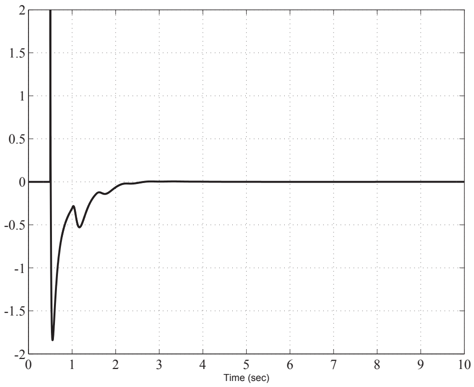

Fig. 3. Output response $y(t)$ of the coupled system model. Here $\tilde{k}_1 = -20$, $\tilde{k}_2 = -1$, and $\tau = 0.5$.

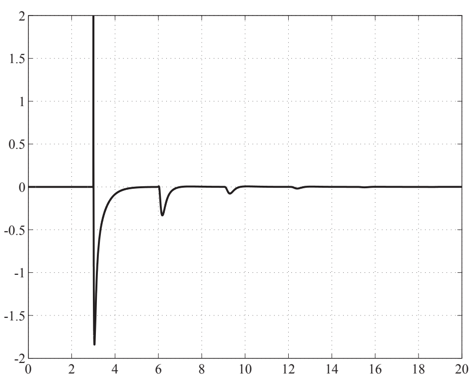

Fig. 4. Output response $y(t)$ of the coupled system model. Here $\tilde{k}_1 = -20$, $\tilde{k}_2 = -1$, and $\tau = 3$.

C. Rightmost root computation

We revisit our analytical results in Figure 2. We select points from different regions of this figure to show the spectrum of the system. Notice that detecting the spectrum of LTI time-delay systems is not a trivial task, and it is a research topic alone [24], [37].

In this paper, we use the TRACE DDE toolbox developed in [37] to compute the rightmost roots of the system at hand. The objective is to test the behavior of the rightmost roots of the system for two different choices of $(\tilde{k}_1, \tilde{k}_2)$ pairs. First pair $p_1 = (\tilde{k}_1, \tilde{k}_2) = (0, -11)$ is selected inside the DIS region (dark gray in Figure 2) and sufficiently close to the white regions (unstable system for $\tau = 0$). The second pair $p_2 = (\tilde{k}_1, \tilde{k}_2) = (0, -3.7)$ is selected in the neighborhood of

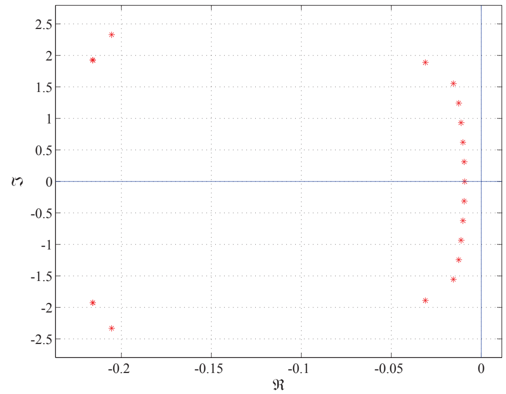

Fig. 5. Twenty rightmost characteristic roots of the combined system for $\tilde{k}_1 = 0$, $\tilde{k}_2 = -11$, and $\tau = 20$.

the DIS region but in the light gray region, where we know that DIS does not hold but the controlled system is stable for $\tau = 0$.

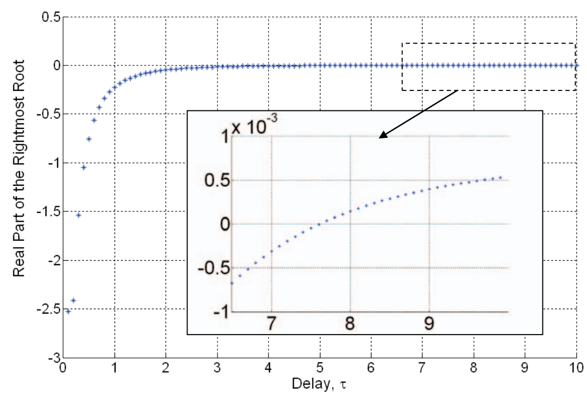

Fig. 6. Real part of the rightmost characteristic root of the combined system for delay $0 \leq \tau \leq 10$. The observer controller gains are selected at $p_2 = (\tilde{k}_1, \tilde{k}_2) = (0, -3.7)$ from the light gray region in Figure 2.

The rightmost roots of the system with $\tau = 20$ and the observer controller gains at p_1 are shown in Figure 5. It is clear that the system is stable for $\tau = 20$ since the real part of the rightmost root is negative, as expected from our DIS analysis. The information in Figure 5 is obviously inconclusive to conclude DIS property. To further test DIS, we next compute the real part of the rightmost root with respect to τ for the given gains at p_1 and for $\tau \in [0, 1000]$. The results are again consistent with our DIS analysis and thus suppressed.

We study the rightmost root behavior for the gains at p_2, Figure 6. We see that for delay values in the range of $[0, 10]$, the real part of the rightmost characteristic root will change sign past a critical delay value. That is, the system will become unstable. This result confirms the analytical findings that the point p_2 does not guarantee delay-independent stability of the system.

It is important to note that it would be hard to fully confirm the DIS property of a TDS using rightmost root computation. On the other hand, these computational tools make it possible to reliably test stability in a finite range of τ.

V. Conclusion

This paper presents an algebraic approach to design observer controller gains for stabilizing a LTI plant with an output delay. The design is achieved by constructing a delay-free observer, which then enables delay-independent stability (DIS) of the controlled plant. Using algebraic tools, we reveal the parametric space of the observer controller gains such that DIS holds, and we express the boundaries of these regions in terms of these gains. An example case study is provided to demonstrate the approach and its advantages. The designed observer can be useful in controlling nonlinear systems and for controlling systems with delays where the amount of delays are unknown.

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