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Generalised Absolute Stability and Sum of Squares

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Abstract— **This paper introduces a general framework for analysing systems that have non-polynomial, uncertain or high order nonlinearities. It decomposes the vector field using Lur'e type feedback into a system with a polynomial or rational vector field and a nonlinear memoryless feedback term, which is bounded by polynomial or rational functions. This decomposition can be used to model uncertainty in the nonlinear term or to bound difficult to analyse nonlinearities by simpler polynomial or rational functions. Conditions for stability are found using Lyapunov functions which are generalisations of those used for the derivation of the multivariable circle and Popov criteria. These conditions can be given in terms of polynomial inequalities and so Sum of Squares techniques can be used to efficiently analyse these systems. An example shows how the techniques can be applied to uncertain coupled genetic circuits and a pendulum, where the nonlinearity is bounded by polynomial functions. The technique is also applied to show global stability of a system in which classical absolute stability is inconclusive.**

I. INTRODUCTION

Many systems are described by inherently nonlinear models and so require nonlinear systems tools for their analysis. Often the nonlinearities in these models are saturating rational functions. For example, in a biological system, Michaelis Menten Kinetics and Hill function type terms are commonly used to describe the dynamics of the system, for example, in metabolic, gene regulatory and signalling networks within the cell [12], [1]. This paper uses these functions as motivation to develop a general framework for analysing nonlinearities using the concepts of absolute stability and sector conditions.

Absolute stability is a traditional systems and control technique for dealing with stability of nonlinear systems [11], [7], [13]. In absolute stability, a nonlinear system is decomposed into a linear system with non-linear memoryless feedback, also known as the Lur'e problem. A quadratic inequality is used to bound the non-linear feedback between two linear bounds. This inequality, together with a Lyapunov function, allows frequency domain or LMI linear systems tools to be applied. Absolute stability may be used to study saturating functions using this method (see [8], [11], [7], [10], [20], [13] and references therein). Another technique with a similar decomposition uses Integral Quadratic Constraints to bound the nonlinear feedback [16], [15], which can also be used for systems with saturating functions.

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Sum of Squares (SOS), a technique for testing non-negativity of polynomial functions, can be used to implement systems analysis tools for systems with polynomial or rational functions [19] (See [5] for a review). For this reason, Sum of Squares is well suited to implement a range of systems analysis techniques for particular applications, such as stability and robust stability of gene regulatory networks [6], [4]. However, this technique does not scale well to large system sizes.

To generalise absolute stability, we decompose the system into a polynomial or rational system with nonlinear memoryless feedback, described here as Lur'e type feedback. Quadratic type inequalities are used to bound the nonlinear feedback with two polynomial/rational functions. This compares with a generalisation which bounds the nonlinear feedback with two piecewise linear functions [9] or two sector conditions [2]. Generalised Absolute Stability gives a simple framework to robustly analyse models with uncertain vector fields, noting that this differs from models with uncertain parameters. Conditions in the theorems can be tested using Sum of Squares techniques. In this paper we give an example of a network of coupled genetic circuits, where the coupling is uncertain.

The inequality may also be used to bound complicated or difficult to analyse nonlinearities with simpler nonlinearities. For example, it is possible to cover a high order Hill function with a lower order rational function, or to use rational functions to cover a non-rational function, such as those derived with generalised mass action kinetics [12]. Using a bound, rather than an approximation, gives conservative rigorous conditions, where the degree of conservatism depends upon how closely the nonlinearity is bound. Generalised absolute stability provides a convenient framework for implementing Lyapunov function theory using SOS for any systems with vector fields that are not polynomial or rational. Previously, Sum of Squares could only be used to implement non-polynomial stability problems by either approximating non-polynomial functions with truncated Taylor series and tracking the error [3] or by recasting non-polynomial systems as polynomial systems [18]. We use the example of a pendulum to show the ability of the technique to bound nonlinearities with simpler nonlinearities. Moreover, we apply the generalized absolute stability theory developed to the robust stability analysis of a biological system.

In Section II, a new generalised sector conditions and a definition of generalised absolute stability is given. In Section III, stability conditions are given using polynomials Lyapunov functions, which generalise the Lyapunov functions used for the derivation of multivariable circle criterion. In

Section IV, we discuss generalised gradient and Hamiltonian functions before generalising Lur'e type Lyapunov functions. We also introduce different bounds to analyse integrals and then find stability conditions using Lur'e type Lyapunov functions, which generalise the Lyapunov functions used for the derivation of the Multivariable Popov criterion. In Section V, we apply the techniques to a number of examples, which illustrate the application of the methods to various problems as well as showing the advantage of generalised absolute stability for the case of global stability.

II. Generalised Sectors and Generalised Absolute Stability Definitions

In this section we introduce a new generalised sector constraint and show how it can be used with Sum of Squares techniques for systems analysis. We also introduce the definitions for Absolute Stability.

A. Generalised Sector Conditions

In this section we introduce a generalised sector constraint that consists of polynomial or rational functions.

We let

$$\phi(x) = \sum_{k=1}^{s} \phi^{(k)}(x)$$

A classical sector constraint for $\phi(x)$ is

$$[\phi(x) - K_1 x]^T [\phi(x) - K_2 x] \leq 0, \forall x \in D \tag{1}$$

where $K_1, K_2 \in \mathbb{R}^{n\times n}$ and $K_2 \succ K_1$. In comparison, we place a generalised sector constraint upon each $\phi^{(k)}(x)$

$$\begin{aligned}&\left[\phi^{(k)}(x) - \mu_1^{(k)}(x)\right]^T \left[\phi^{(k)}(x) - \mu_2^{(k)}(x)\right] \leq 0,\\ &\forall x \in D \text{ and } k = 1, \ldots, s\end{aligned} \tag{2}$$

where $\phi^{(k)}(x), \mu_1^{(k)}(x), \mu_2^{(k)}(x) \in \mathbb{R}^n$, $\mu_1^{(k)}(x)$, $\mu_2^{(k)}(x)$ are continuous and $\mu_1^{(k)}(0) = \mu_2^{(k)}(0) = 0$. Note: To ensure that (2) is well defined, we require that the intersection of the different constraints is non empty for all $x \in D$.

Example 1: We wish to find polynomial bounds for the function $\sin(z)$. We can base the choice of bounds on the Taylor series $\sin z = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \mathrm{O}(z^7)$ and so we can use $\mu_1 = z - \frac{1}{6}z^3$ and $\mu_2 = z - \frac{1}{10}z^3$ for $-\pi \leq z \leq \pi$. Using these bounds, which are 3rd order, simplifies analysis over a polynomial approximation with an error bound, which is 5th order.

Note: Basing the bounds on the Taylor series gives us tight bounds near $z = 0$, but we can also use bounds for which the difference between the bounds is more uniform for $-\pi \leq z \leq \pi$.

B. Generalised Sector Conditions and Sum of Squares

We can test whether a function $V(x, \phi(x)) \geq 0$ for $x \in D$ by using Sum of Squares programming techniques. To do this we can use Positivestellensatz, where $V(x, \phi)$ is a polynomial in x and ϕ, which are treated as independent variables, and the constraints are the domain and the generalised sector. For one sector this can be written:

$$V(x,\phi) + w(x,\phi)\,[\phi - \mu_1(x)]^T\,[\phi - \mu_2(x)] + p(x)^T a(x) \geq 0 \tag{3}$$

where $D = \{x \in \mathbb{R}^n | a_i(x) \leq 0, j = 1, \ldots, c\}$ and $w(x,\phi)$,$p_j(x), j = 1, \ldots, c$ are Sum of Squares. Note: If the sector domain D is global then we can set $a(x) = 0$. If (3) is a polynomial of order $2d$ then we can write (3) in quadratic form:

$$Z(x,\phi)^T Q Z(x,\phi) \geq 0 \tag{4}$$

where $Z(x,\phi)$ is a vector composed of monomials of x and ϕ up to order d, and Q is non-unique and so contain 'slack' variables. $w(x,\phi)$ and $p_j(x)$ can be written in similar quadratic forms. If $Q \succeq 0$ then the LHS of (3) is a Sum of Squares and so (3) holds. This, together with Sum of Squares $w(x,\phi)$ and $p_j(x)$, give sufficient conditions for $V(x,\phi) \geq 0$ for $x \in D$ and $\phi(x)$ in the generalised sector. These LMIs can be solved efficiently using Semidefinite programming, where the 'slack' variables and unknown parameters become decision variables in the LMI. The process of converting a SOS problem into an LMI and solving with SDP can be handled in one step with software tools such as SOSTOOLS [21]. The technique described above can easily be modified for multiple sectors or rational functions.

C. Generalised Absolute Stability Definitions

In order to deal with models that have nonlinearities and uncertainty in this form, we study systems which are decomposed into a polynomial or rational system with a nonlinear or uncertain feedback:

$$\begin{aligned}\dot{x} &= f(x) + B(x)u\\ y &= h(x), \quad u = -\phi(t,y)\end{aligned} \tag{5}$$

where $x \in D \subset \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^r$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $B : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ and $h : \mathbb{R}^n \to \mathbb{R}^r$ are linear, polynomial or rational functions, $\phi(x) : \mathbb{R} \times \mathbb{R}^r \to \mathbb{R}^p$ is a non-linear function.

In this paper we look at the case where $y = x$,$u \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times n}$ is constant and ϕ is not time varying. Therefore we look at the case where (5) can be written as:

$$\dot{x} = f(x) - B\phi(x) \tag{6}$$

where $f(x)$, $\phi(x)$ are sufficiently smooth and $f(0) = \phi(0) = 0$. We can express $f(x) = \frac{f_n(x)}{f_d(x)}$ where f_n, f_d are polynomials and $f_d(x) > 0$ for $x \in D$.

Definition 2: Consider system (6) where $\phi(x)$ satisfies (2) in D. The system is 'absolutely stable with a finite domain D' if the origin is asymptotically stable for any non-linearity in (2).

The paper finds stability conditions for the above defined problem for the case $s = 1$ and polynomial or rational bounding functions:

$$\mu_1(x) = \frac{r_1(x)}{q_1(x)}, \quad \mu_2(x) = \frac{r_2(x)}{q_2(x)} \tag{7}$$

where $r_1(x), r_2(x) \in \mathbb{R}^n[x]$, $q_1(x), q_2(x) \in \mathbb{R}[x]$, $r_1(0) = r_2(0) = 0$ and $q_1(x), q_2(x) > 0$ for all $x \in D$. Note: $\mathbb{R}^n[x]$ represents an n^{th} order vector field composed of polynomials in x.

III. Polynomial Lyapunov functions

We first generalise the use of the circle criterion for absolute stability. We use a Lyapunov function candidate $V(x)$ which is a polynomial of degree $2d$ and required to be a positive definite function. The multivariable circle criterion can be obtained by setting $d = 1$ and so $V(x) = x^T P x$ is a quadratic function. In the following proof, we use Lyapunov theory which can be found in the Appendix. We also use multipliers in the derived Lyapunov theory, such as $\tau(x)$ which is useful for transforming rational conditions to polynomial conditions and so allows implementation.

Theorem 3: System (6) is absolutely stable with domain $D = \{x \in \mathbb{R}^n | a_j(x) \le 0 \text{ for } j = 1, \ldots, c\}$ and a generalised sector defined by (2) if for all x, ϕ there exists continuously differentiable $V(x)$, continuous $\tau(x) \in \mathbb{R}$, continuous $p(x) \in \mathbb{R}^c$, continuous $w(x) \in \mathbb{R}^s$ and continuous positive definite $\varphi_1(x)$, $\varphi_2(x)$ such that

$$V(x) \ge \varphi_1(x) \text{ and } V(0) = 0 \quad (8)$$

$$\begin{aligned} &-\tau(x)\nabla V(x)^T[f(x) - B\phi] + p(x)^T a(x) \\ &+ \sum_{k=1}^{s} w_k(x) \left[\phi^{(k)} - \mu_1^{(k)}(x)\right]^T \left[\phi^{(k)} - \mu_2^{(k)}(x)\right] \\ &\ge \varphi_2(x) \end{aligned} \quad (9)$$

$$p_j(x) \ge 0, j = 1, \ldots, c, \quad w_k(x) \ge 0, k = 1, \ldots, s \quad (10)$$

$$\tau(x) > 0 \text{ for } x \in D \quad (11)$$

Proof: We wish to apply Theorem 10 for the constraints $a_j(x) \le 0$ and (2). We therefore require positive definite $V(x)$ which is met by (8). Substituting the constraints $a_j(x) \le 0$ and (2) for $b_j(x)$ in Theorem 10, we require

$$\begin{aligned} &-\tau(x)\nabla V(x)^T[f(x) - B\phi(x)] + \sum_{j=1}^{m} q_j(x) b_j(x) - \varphi_2(x) \\ &= -\tau(x)\nabla V(x)^T[f(x) - B\phi] \\ &+ \sum_{k=1}^{s} w_k(x) \left[\phi^{(k)} - \mu_1^{(k)}(x)\right]^T \left[\phi^{(k)} - \mu_2^{(k)}(x)\right] \\ &+ p(x)^T a(x) - \varphi_2(x) \ge 0 \end{aligned}$$

which is met by (9). We also require that $\tau(x) > 0$ and $q_i(x) \ge 0$ which are met by (10) and (11). Therefore the conditions of Theorem 10 are met and system (6) is absolutely stable with domain D. ■

Corollary 4: System (6) is absolutely stable on domain $D = \{x \in \mathbb{R}^n | a_j(x) \le 0 \text{ for } j = 1, \ldots, c\}$ with a generalised sector defined by (2) and (7) if for all x, θ there exist continuously differentiable $V(x)$, continuous $p(x) \in \mathbb{R}^c$, constant $\omega \ge 0$ and continuous positive definite $\varphi_1(x)$, $\varphi_2(x)$ such that

$$V(x) - \varphi_1(x) \ge 0 \text{ and } V(0) = 0 \quad (12)$$

$$p_j(x) \ge 0 \text{ for } j = 1, \ldots, c \quad (13)$$

$$\begin{aligned} &-\nabla V(x)^T[g(x) - B\theta] + p(x)^T a(x) \\ &+ \omega\left[\theta - \eta_1(x)\right]^T\left[\theta - \eta_2(x)\right] - \varphi_2(x) \ge 0 \end{aligned} \quad (14)$$

where

$$g(x) = f_n(x) q_1(x) q_2(x)$$

$$\eta_1 = f_d(x) q_2(x) r_1(x), \quad \eta_2 = f_d(x) q_1(x) r_2(x).$$

Proof: We apply Theorem 3 for constraints ((2)–(7)) and $s = 1$. We set $\tau(x) = f_d(x) q_1(x) q_2(x) > 0$, $w_1(x) = \omega f_d^2(x) q_1(x)^2 q_2(x)^2 \ge 0$, which imply (10) and (11) hold. If $\theta = f_d q_1(x) q_2(x) \phi$ then (14) implies (9) holds. Therefore the conditions for Theorem 3 are met. ■

IV. Lur'e type Lyapunov functions

In this section we first describe generalised Hamiltonian functions and the related Lur'e type Lyapunov functions. We generalise the Lur'e type Lyapunov function for system (6) by linking it to existing concepts from Generalised Hamiltonian theory. We also introduce a new bound in order to find less conservative positive definite conditions for Lur'e type Lyapunov functions. We then use the introduced concepts for generalised absolute stability.

A. Generalised Hamiltonian and Lur'e type Lyapunov functions

In [14],[23] (see [22] and references therein for Port Hamiltonian Theory) the differential equations are assumed to be of the form:

$$\dot{x} = f(x) = -L(x)\nabla V(x) \quad (15)$$

where $f(x)$ represents the system dynamics, $\nabla V(x)$ is a gradient vector field and $L(x)$ is arbitrary. In order to study stability, we can test whether $V(x)$ is a Lyapunov function; since $\dot{V}(x) = -\nabla V(x)^T[L(x) + L(x)^T]\nabla V(x)$ it is required to find $L(x)$ such that $V(x)$ is positive definite and $L(x) + L(x)^T \succeq 0$. It can be noted that $L(x)$ is typically split into the symmetric an asymmetric components where only the symmetric component affects $\dot{V}(x)$ e.g. a Hamiltonian System, which has $\dot{V} = 0$, also has $L(x) + L(x)^T = 0$ when placed in the form (15).

For (5) we can write the Lur'e type form as

$$\dot{x} = f(x) - B(x)\phi(x)$$

where $y = x$ and $\phi(x)$ is not a function of time. The Lur'e type Lyapunov function we propose takes the form:

$$V(x) = \sigma(x) + \gamma \int_0^x \phi(z) dz$$

where $\sigma(x) : \mathbb{R}^n \to \mathbb{R}$ and $\phi(x)$ is a gradient vector field. Using a function of this form has benefits in finding a function $V(x)$ which meets the condition that $\dot{V}(x)$ is negative definite. The positive definite condition on $V(x)$ can be found by using the bound:

$$x^T \phi(x) \ge x^T \rho(x)$$

where $\rho(x)$ is a gradient vector field and so

$$x^T[\phi(x) - \rho(x)] \geq 0 \Rightarrow \int_0^x \phi(z) - \rho(z)dz \geq 0 \quad (16)$$

$$V(x) \geq \sigma(x) + \gamma \int_0^x \rho(z)dz = \sigma(x) + \xi(x)$$

where $\nabla\xi(x) = \gamma\rho(x)$. In the following section we look at the case where $\sigma(x)$ is a polynomial function and we analyse (6) where B is constant.

B. Absolute Stability using Lur'e type functions

We next find generalised Absolute stability conditions using the generalised Lur'e type Lyapunov function as well as by using a bound for the integral.

Theorem 5: System (6) is absolutely stable with domain $D = \{x \in \mathbb{R}^n | a_j(x) \leq 0 \text{ for } j = 1, \ldots, c\}$ and a generalised sector defined by (2) if for all x, ϕ there exists continuously differentiable $\sigma(x)$, continuous $\tau(x)$, continuous $p(x) \in \mathbb{R}^c$, continuous $w(x) \in \mathbb{R}^s$, continuous positive definite $\varphi_1(x)$, $\varphi_2(x)$ and gradient vector field $\rho_1(x)$ such that

$$x^T\phi(x) \geq x^T\rho_1(x), \forall x \in D \quad (17)$$

where $\phi(x)$ is a gradient vector field,

$$\begin{array}{l}\sigma(x) + \xi_1(x) + p(x)^T a(x) - \varphi_1(x) \geq 0 \\ \text{and } \sigma(0) = \xi_1(0) = 0\end{array} \quad (18)$$

where $\xi_1(x) = \gamma \int_0^x \rho_1(z)dz$ and

$$\begin{aligned}&- \tau(x)\left[\nabla\sigma(x) + \gamma\phi\right]^T\left[f(x) - B\phi\right] + p(x)^T a(x) \\ &+ \sum_{k=1}^{s} w_k(x)\left[\phi^{(k)} - \mu_1^{(k)}(x)\right]^T\left[\phi^{(k)} - \mu_2^{(k)}(x)\right] \quad (19)\\ &- \varphi_2(x) \geq 0\end{aligned}$$

$$p_j(x) \geq 0,\ j = 1, \ldots, c, \quad w_k(x) \geq 0,\ k = 1, \ldots, s \quad (20)$$

$$\tau(x) > 0 \text{ for } x \in D. \quad (21)$$

Proof: We wish to apply Theorem 10. Therefore we require positive definite $V(x)$. If $\rho_1(x)$ is a gradient vector field then $\xi_1(x) = \int_0^x \rho_1(z)dz$. Using (17) and noting (16) we obtain

$$V(x) = \sigma(x) + \gamma \int_0^x \phi(z)dz \geq \sigma(x) + \xi_1(x)$$

Together with (18), this proves that $V(x)$ is positive definite. Substituting the constraints $a_j(x) \leq 0$ and (2) for $b_j(x)$ in Theorem 10, we require

$$\begin{aligned}&- \tau(x)\left[\nabla\sigma(x) + \gamma\phi\right]^T\left[f(x) - B\phi\right] + p(x)^T a(x) \\ &+ \sum_{k=1}^{s} w_k(x)\left[\phi^{(k)} - \mu_1^{(k)}(x)\right]^T\left[\phi^{(k)} - \mu_2^{(k)}(x)\right] \\ &- \varphi_2(x) \geq 0\end{aligned}$$

which is met by (19). We also require that $\tau(x) > 0$ and $q_i(x) \geq 0$ which are met by (20), (21). Therefore the conditions of Theorem 10 are met and system (6) is absolutely stable with domain D. ■

Corollary 6: System (6) is absolutely stable with domain $D = \{x \in \mathbb{R}^n | a_j(x) \leq 0 \text{ for } j = 1, \ldots, c\}$ and a generalised sector defined by (2) and (7) if for all x, θ there exists continuously differentiable $\sigma(x)$, continuous $p(x) \in \mathbb{R}^c$, constant $\omega \geq 0$, constant $\alpha \geq 0$, continuous positive definite $\varphi_1(x)$, $\varphi_2(x)$ and gradient vector field $\rho_1(x)$ such that

$$x^T\phi(x) \geq x^T\rho_1(x), \forall x \in D \quad (22)$$

where $\phi(x)$ is a gradient vector field,

$$\begin{array}{l}\sigma(x) + \xi_1(x) + p(x)^T a(x) - \varphi_1(x) \geq 0 \\ \text{and } \sigma(0) = \xi_1(0) = 0\end{array} \quad (23)$$

$$\begin{aligned}&- \left[h(x) + \gamma\theta\right]^T\left[g(x) - B\theta\right] + p(x)^T a(x) \\ &+ \omega\left[\theta - \eta_1(x)\right]^T\left[\theta - \eta_2(x)\right] - \varphi_2(x) \geq 0\end{aligned} \quad (24)$$

$$p_j(x) \geq 0, j = 1, \ldots, c \quad (25)$$

where $\xi_1(x) = \gamma \int_0^x \rho_1(z)dz$

$$g(x) = q_1(x)q_2(x)f_n(x), h(x) = f_d(x)q_1(x)q_2(x)\nabla\sigma(x)$$

$$\eta_1 = f_d(x)q_2(x)r_1(x), \quad \eta_2 = f_d(x)q_1(x)r_2(x).$$

Proof: We apply Theorem 5 on ((2)–(7)) and $s = 1$. We set $\tau(x) = f_d(x)^2 q_1(x)^2 q_2(x)^2 > 0$, $w_1(x) = \omega f_d^2(x) q_1(x)^2 q_2(x)^2 \geq 0$, which imply (20) and (21) hold. If $\theta = f_d q_1(x) q_2(x)\phi$ then (24) implies (19) holds. Therefore the conditions for Theorem 5 are met. ■

If we assume that $a_j(x)$, $p_j(x)$, $\varphi_1(x)$, $\varphi_2(x)$ and $\rho_1(x)$ are polynomials, then conditions (23), (25), (24) are all polynomials and all hold if they are SOS. Therefore the theorem can be implemented using SOS techniques.

V. Examples using Polynomial and Lur'e type Lyapunov functions

In this section we provide examples which illustrate the application of the generalised absolute stability results that are presented in this paper.

Example 7: The first example represents an uncertain, nonlinear biological system and is used to illustrate the application of the technique for uncertain nonlinear models. The model can represents two uncertain coupled genetic circuits, where a repressor is connected in feedback with an activator and for simplicity it is assumed that the steady state is known and has been transformed to the origin; the uncertainty is only in the dynamics. We let

$$f = \begin{pmatrix} -x_1 \\ -2x_2 \end{pmatrix}, B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \phi(x) = \begin{pmatrix} \phi_1(x_1) \\ \phi_2(x_2) \end{pmatrix}$$

$$\frac{r_1}{q_1} = \begin{pmatrix} \frac{c_1 x_1}{1+2x_1} \\ \frac{b_1 x_2}{1+x_2} \end{pmatrix}, \quad \frac{r_2}{q_2} = \begin{pmatrix} \frac{c_2 x_1}{1+2x_1} \\ \frac{b_2 x_2}{1+x_2} \end{pmatrix}$$

where $c_1 = 0.9$, $c_2 = 1.1$, $b_1 = 0.9$, $b_2 = 1.1$. Applying SOSTOOLS, for 2^{nd} order $a(x)$ and 6^{th} order $p(x)$ we obtain:

$$V(x) = 0.50031x_1^2 + 0.12179x_1x_2 + 0.42617x_2^2$$

which is valid for the domain $-0.2 \leq x_1 \leq 0.2$ and $-0.2 \leq x_2 \leq 0.2$.

We next use Corollary 6 for the same system, where ϕ is a gradient vector field. Now using the Taylor series as a starting point, consider the bound

$$\rho = [c_1 x_1(1-2x_1), b_1 x_2(1-x_2)]^T.$$

This leads to

$$\xi_1 = \gamma c_1 \left(\frac{1}{2}x_1^2 - \frac{2}{3}x_2^3\right) + \gamma b_1 \left(\frac{1}{2}x_2^2 - \frac{1}{3}x_2^3\right)$$

where γ is an undetermined constant. The returned Lyapunov function is composed of

$$\sigma = 2.91x_1^2 + 1.11x_2x_1 + 2.34x_2^2$$
$$\gamma = 0.86$$

Therefore, both techniques can be used to show absolute stability for this example.

Example 8: We next study a nonlinear pendulum using generalised absolute stability, which illustrates the use of the techniques when bounding nonlinear functions with polynomials. Here, a transcendental sine function is bounded by two 3^{rd} order polynomials so as to simplify analysis. This example also shows a new, simple method of applying SOS techniques to analyse stability of non-polynomial systems.

Consider a pendulum

$$\begin{aligned}\dot{x}_1 &= -\alpha x_1 - \sin x_2 \\ \dot{x}_2 &= x_1\end{aligned} \tag{26}$$

If we place this system in the form (6) then we obtain

$$f = \begin{pmatrix} -\alpha x_1 \\ x_1 \end{pmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \phi = \begin{bmatrix} 0 \\ \sin(x_2) \end{bmatrix}.$$

From Example 1 we use the bounds

$$r_1 = \begin{pmatrix} 0 \\ x_2 - \frac{1}{6}x_2^3 \end{pmatrix}, r_2 = \begin{pmatrix} 0 \\ x_2 - \frac{1}{10}x_2^3 \end{pmatrix}$$

where $D = \{x \in \mathbb{R}^2 | -\pi \leq x_1 \leq \pi\}$.

Applying SOSTOOLS, for 2^{nd} order $a(x)$ and 4^{th} order $p(x)$ we obtain:

$$V(x) = 26.583x_1^2 + 22.582x_2x_1 + 13.684x_2^2$$

which is valid for the domain $-0.75\pi \leq \theta \leq 0.75\pi$. In this case, the size of the domain is constrained by the bound in $r_1(x)$ for $\sin x$.

We next apply Corollary 6 where

$$\rho = \left[0, x_2 - \frac{1}{6}x_2^3\right]^T, \quad \xi_1 = \gamma\left(\frac{1}{2}x_2^2 - \frac{1}{24}x_2^4\right)$$

For a 2^{nd} order $a(x)$ and a 4^{th} order $p(x)$ the returned Lyapunov function is

$$\sigma = 13.072x_1^2 + 8.9604x_1x_2 + 4.7886x_2^2$$
$$\gamma = 13.18$$

This is valid in the domain $-0.75\pi \leq x_2 \leq 0.75\pi$. It can be noted that

$$\sigma + \xi_1(x) = 13.072x_1^2 + 8.9604x_1x_2 + 11.38x_2^2 - 0.55x_2^4$$

which is positive definite on D. Once again, the size of the domain is constrained by the bound r_1.

In order to find Lyapunov functions for larger domains, either a different bound or a higher order bound for $\sin x$ needs to be used. For example, an alternate lower bound for $\sin z$ could be $\frac{4}{5}z - \frac{1}{11}z^3$.

Example 9: We next consider an example which shows an advantage of using generalised absolute stability over classical absolute stability. A simple example of this is $\dot{x} = -\phi$ where $\mu_1 = x^3$ and $\mu_2 = 2x^3$. In this, any inequality with linear bounds (1) has to include the case $\phi = 0$ locally about $x = 0$. This implies that asymptotic stability can not be shown. This is true irrespective of whether the bounds are linear or piecewise linear.

The major advantage in using generalised absolute stability is for analysing the behaviour in nonlinear regions away from the origin. Below we give an example where generalised absolute stability shows global absolute stability but where classical absolute stability is inconclusive.

For the system:

$$f = \begin{pmatrix} -5x_1 + 10x_2 \\ -10x_1 - 2x_2 \end{pmatrix}$$

$$\frac{r_1}{q_1} = \begin{pmatrix} -\frac{5x_1}{1+x_1^2} \\ -\frac{6x_2^2}{1+x_2^2} \end{pmatrix}, \quad \frac{r_2}{q_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{27}$$

Using generalised absolute stability, this system is absolutely stable for an unrestricted domain where the returned Lyapunov function is:

$$\begin{aligned}V(x) =& 0.64012x_1^2 + 0.05352x_1x_2 + 0.63271x_2^2 \\ &- 0.22889x_1^3 - 0.00471x_1^2x_2 - 0.32305x_1x_2^2 \\ &- 0.016708x_2^3 + 0.25884x_1^4 - 0.13788x_1^3x_2 \\ &+ 0.50798x_1^2x_2^2 - 0.16723x_1x_2^3 + 0.25988x_2^4\end{aligned}$$

It can be noted that SOSTOOLS could not find a Lyapunov function which was 2^{nd} order. This Lyapunov function is radially unbounded and so the system is globally asymptotically stable.

Using classical stability then the system can be analysed locally. However, for (27) the global bounds in (1) are:

$$K_1 = \begin{pmatrix} -5 & 0 \\ 0 & -3 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$$

One case within these bounds is:

$$\dot{x} = f(x) - \phi(x) = \begin{pmatrix} 0 & 10 \\ -10 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

which is unstable. Therefore, no conditions for global stability can be found for this case using classical stability. It is not possible to use Lur'e type Lyapunov functions to conclude global stability with polynomial $\rho_1(x)$ for bounding $\phi(x)$. This restriction could be removed using rational $\rho_1(x)$, but further work is required for this case.

In the simple examples provided, no advantage has been illustrated for the use of Lur'e type over polynomial Lyapunov functions. However, it is expected that using Lur'e type

Lyapunov functions will simplify analysis for larger or higher order systems, as well as in situations where higher order polynomial Lyapunov functions are required, because of the possible encapsulation of higher order vector field terms in the integral part of the Lur'e type Lyapunov function.

VI. Conclusion

In this paper, we have provided a generalised version of absolute stability using generalised sector constraints. The stability conditions were derived with a generalised version of the Lyapunov functions used in the development of the multivariable circle and multivariable Popov criteria. These conditions were tested with Sum of Squares techniques. They were also applied to a pendulum to show an example of a non-polynomial and non-rational system which this technique can be applied to. Finally, we used our generalised absolute stability technique to conclude global asymptotic stability in an example for which classical absolute stability is inconclusive.

Future work will look at using the input-output case (5) as well as finding the region of attraction of the steady state. We will also look at system characteristics other than stability, such as performance and incorporating uncertainty in the functions describing the dynamics along with uncertainty in the steady state.

Appendix

This appendix presents a theorem for the stability of the zero equilibrium of

$$\dot{x} = f(x) \tag{28}$$

with the constraints $b_j(x) \leq 0$ for $j = 1, \ldots, m$ [17].

Theorem 10: Suppose that for the above system there exist continuous functions $\tau(x)$ and $q_j(x)$, $j = 1, \ldots, m$ continuous positive definite function $\varphi(x)$ and continuously differentiable function $V(x)$ such that

$$V(0) = 0 \text{ and } V(x) > 0, \text{ for } x \neq 0$$

$$\tau(x) > 0 \text{ in } D \text{ and } q_j(x) \geq 0 \text{ for } j = 1, \ldots, m \tag{29}$$

$$-\tau(x)\nabla V^T f(x) + \sum_{j=1}^{m} q_j(x) b_j(x) - \varphi(x) \geq 0 \text{ for } x \neq 0 \tag{30}$$

Then the origin of the system is asymptotically stable.

Proof: From $b_j(x) \leq 0$ in D, (30) and (29) it can be seen that

$$\begin{aligned}\dot{V} &= \nabla V^T f(x) \\ &\leq \frac{1}{\tau(x)} \sum_{j=1}^{m} q_j(x) b_j(x) - \frac{1}{\tau(x)} \varphi(x) < 0 \text{ for } x \neq 0\end{aligned}$$

It can be noted that $\dot{V}(0) = 0$ as $f(0) = 0$. As $V(x)$ is positive definite in D and $\dot{V}(x)$ is negative definite in D then $x = 0$ is asymptotically stable. ■

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