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On Optimal Defibrillating Pulse Synthesis

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Abstract— **This article proposes a control framework suitable to study the synthesis of electrical signals that bring a fibrillating heart to a nominal (defibrillated) state so that regular, autonomous cardiac activity resumes. A parallel resistor/capacitor (single RC-circuit) and energy source has been used for over 70 years to describe the heart's defibrillated state as a condition of reaching nominal threshold potential values represented by the capacitor voltage. The bidomain model adopted in the 1990's was an improvement as it provided 2D and 3D continuum models of cardiac tissue enabling significant advances in modeling electrical cardiac activity. We have spatially discretized a version of the bidomain equations so that the temporal behavior of the transmembrane potential at a point is deduced from an infinite number of first-order differential equations. The connection to a network of RC-circuits is logical and creates opportunities for theoretical and practical modeling and control contributions. A strategy that minimizes a weighted time/energy cost for the single RC-circuit was applied to a multi-RC model. The proposed pulse is *agile* and *energy conscientious* thus outperforming the pulse developed to minimize energy consumption alone. The results can also be used to formalize ad-hoc reports that explore the time-energy tradeoff in other defibrillating pulse candidates. In addition, the multi-RC circuit may be used to explore cardiac tissue recovery and multi-path defibrillation. A minimum-time strategy is proposed for the multi-RC circuit to address fast capacitor discharge requirements. These efforts suggest new waveforms to potentially innovate defibrillator design using feedback control.**

Keywords: Defibrillation, optimal, bidomain model.

I. INTRODUCTION

Cardiovascular disease is the leading cause of death in the United States contributing to 831,000 fatalities in 2006 [1]. Cardiac arrhythmias are present in roughly half of heart disease deaths and is therefore an important disease pathology. Fibrillation is the cessation of the coordinated function in the heart and is often associated with ischemic heart disease or injury, but can be caused by a variety of acquired, genetic, or environmental factors. Abnormal automacity or re-entrant arrhythmias prevent the coherent contraction of the cardiac muscle thus limiting the pumping capacity of the heart. Prompt termination of these arrhythmias is best achieved by the application of a strong electric shock across the heart - known as defibrillation therapy, to depolarize the cardiac muscle and restore normal sinus rhythm.

The safety and efficiency of commercial defibrillators can be significantly increased by including sophisticated and automated control algorithms. Therefore, the ultimate goal is to design a controller that brings cardiac tissue back to its resting potential, until the autonomously oscillating cells located in the sinoatrial node (SA) - the heart's natural pacemaker - initiate regular wave patterns prompting normal cardiac activity. Normally, the electrical activity of cardiac tissue cells requires precise temporal and spatial electrical wave propagation so that contraction of the four chambers of the heart are in nearly perfect synchronization. This is easily detected by a regular heartbeat and may be precisely measured by an electrocardiogram. Cardiac cell contraction occurs via voltage-gated ion channel activation and exchange (largely calcium, sodium, and potassium ions) that initiates with cellular depolarization. Cellular depolarization is a change in a cell's membrane potential from resting (negative voltage) to positive voltage. Together this process is known as an action potential and in cardiac tissue it is freely propagated in three-dimensions to adjacent cells through gap junctions. The amplitude of the action potential is independent of the amplitude of the electric shock (current), hence the need for limiting the level of a defibrillating stimulus to a safe threshold value.

The use of simple models to characterize the response of heart tissue to electric stimuli started over a century ago (references 1, 2 in [26]). Since the 1930's, a parallel resistor/capacitor and electrical energy source have been the standard model in pacing and defibrillation studies. Conceptually, Figure (1) depicts the defibrillator device as a power supply (battery if implantable), a signal or wave generator and switching electronics driving an equivalent parallel R_dC_d output circuit. The device generates a waveform across its output circuit and delivers the electrical energy to the heart via a pair of electrodes. In virtually every study, the load (heart) is also modeled as a parallel R_mC_m circuit; hence, R_dC_d is made to equal R_mC_m to explore waveform synthesis. Researchers have used this model to (1) conduct theoretical investigations on the effectiveness of typical pulse-shapes in relation to circuit time constants, pulse duration, energy minimization, and others; (2) to reveal the clinical physiological relevance of the RC-circuit model; and (3) to find the practical relevance of these studies relative to the design of pacing and defibrillating devices. A short representation of articles in the last two decades reporting theoretical and experimental results that explore capacitor-size reduction, measurement of membrane time constants, energy optimization, pulse-duration selection, effectiveness of mono- and bi-phasic waveforms, dual-path defibrillation configurations, and portable and implantable defibrillators includes [2]-[26]. Our interest is in defibrillating pulse synthesis via feedback control frameworks.

Typical *assumed* temporal forms of the signal applied by

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the defibrillator have included square, triangular, trapezoidal, damped sinusoids, and various truncated decaying exponential functions. In practice, simple and easy-to-generate waveforms have been chosen especially for lightweight implantable devices. As cited in [16], studies have hinted that damped sinusoidal waveforms may not be optimal (not in a mathematical sense) due to possible adverse effects caused by the waveform's trailing edge. Article [17] describes the derivation of the *optimal* waveshape - an *ascending* exponential function of time, that minimizes a measure of the energy spent in the *charge-banking* and *charge-burping* hypotheses of defibrillation of cardiac tissue using the single RC-circuit. The minimization of energy is deemed important because it impacts size reduction, input energy availability, and longevity of implantable cardiac defibrillators. The optimal waveform provides a pulse that synchronizes the majority of the cells during the *charge-banking* phase, and returns the myocardium to a defibrillated state in a *charge-burping* phase in a given time interval. This two-phase waveform is called a biphasic pulse and it has been found to accomplish better defibrillating results than the single phase or monophasic pulse. In computer simulations of the charge-banking phase the myocardium is at rest modeled by the voltage across the capacitor equal to zero. Then, the electric shock steers the voltage to a known threshold in finite time. In the charge-burping phase, the capacitor voltage is brought to a lower potential (perhaps zero) preferably as quickly as possible [16]. As reported in [25], the single RC-circuit has predicted major defibrillation aspects that have been experimentally verified in both animal and human studies.

Earlier studies [9], [10], [11], investigated the use of smaller capacitors delivering a *shorter duration* waveform as a possible means for reducing the size of implantable defibrillators. However, it is also reported that a smaller capacitor results in a *reduction of available energy output*. In [2], the effect of the pulse duration is explicitly investigated for monophasic pulses ranging from 1.0 to 20.0 msec and biphasic pulses with a fixed-duration first phase, and a second phase also ranging from 1.0 to 20 msec. Fundamentally, these are explorations into the tradeoff between the time duration and energy content of the defibrillating pulse. Motivated by these works and [17], we subsequently explored the shape of a single-path pulse (one input) that minimizes a weighted penalty of energy and time requirements also for the single RC-model [27]. Although this problem statement and its solution are well known in the control community both as an open-loop or as a closed-loop strategy, electrophysiologists and defibrillator manufacturers will find it important to consider a waveform that is both *agile* and *energy conscientious* that outperforms the pulse developed to minimize energy consumption alone. In addition, the proposed waveform gives the opportunity to reconcile the recommendations published over the years that deal with the conflicting requirements of simultaneous time and energy minimization cast as a problem of capacitor sizing and pulse duration selection which have been studied by computer simulations.

During the 1990's, the electrophysiology community adopted the *bidomain* model - a set of 3D nonlinear partial differential equations developed in the 1970's that is currently considered to be the most complete model description of three-dimensional cardiac electrical activity. The bidomain model has been used extensively but primarily as a computational tool to simulate for example how lethal arrhythmias may initiate and how electric shocks may be used to terminate them [28]. A perturbation expansion method was used in [29] to obtain approximate solutions to the bidomain equations. To our knowledge, [30], [31] were the first to present an infinite-mode expansion of the bidomain model leading to a truncated set of decoupled first-order ordinary differential equations. Each equation can be viewed as a single RC-circuit and by aggregation, the set of equations provides a multi-RC view of cardiac tissue that is suitable for further studies, including modeling, control, sensing, and experimentation. Also in [31], the optimal waveform presented earlier [27] is applied to an RC-circuit comprised of three parallel RC-circuits to illustrate one possible application of the model and pulse design. One goal of this article is to propose a modified multi RC-circuit model from [31] that we hope will enable further studies on modeling and control, and perhaps experimental verifications by the electrophysiology community. We believe our efforts address the concern that from the point of view of practical defibrillator design, defibrillation efficacy will not improve substantially without radical changes in waveform design [25].

The remainder of the article is organized as follows: in Section II, the bidomain equations are described leading to an improved multi-RC model over the one we presented in [31]. Section III reviews single-RC and multi-RC models while Section IV describes various optimal pulse-shapes. Section V concludes and offers several directions for continued research.

II. BIDOMAIN EQUATIONS

The bidomain equations are a bulk model of electrical activity in cardiac cells that describe the transmission of electrical current across the multiple cell membranes. Each cardiac cell is viewed as a cubical region comprised of intracellular and extracellular domains that are separated by a membrane. Contiguous cells then connect with each other at the sides of the cube [33]. Under certain conditions, the 3D bidomain equations may be expressed as a set of second order partial differential equations, governing the intracellular V_i and extracellular V_e electrical potentials in the myocardium [29]:

$$\frac{\partial^2\Phi_m}{\partial x^2}+\frac{\partial^2\Phi_m}{\partial y^2}+\frac{\partial^2\Phi_m}{\partial z^2}-\Phi_m-\frac{\partial\Phi_m}{\partial t}=-\beta_1\frac{\partial^2\Psi}{\partial z^2}+\kappa_1\gamma_e-\kappa_3\gamma_i \tag{1}$$

$$\beta_2\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}\right)+\beta_3\frac{\partial^2\Psi}{\partial z^2}=\beta_4\left(\frac{\partial^2\Phi_m}{\partial x^2}+\frac{\partial^2\Phi_m}{\partial y^2}\right)-\kappa_2\gamma_e-\kappa_2\gamma_i \tag{2}$$

where $\Phi_m(x,y,z,t) = V_i - V_e$ is the transmembrane potential, $\Psi = \Phi_i + (g_{eL}/g_{iL})\Phi_e$ is an auxiliary potential, $g_{eL},\ g_{iL}$ are conductivity parameters in the direction parallel to the fiber axis, $\beta_1 : \beta_4$ and $\kappa_1 : \kappa_3$ are constants, and $\gamma_i,\ \gamma_e$ represent external source terms. Despite the computational complexities associated with the model, it has been recognized as a novel framework in which to study the generation of membrane potentials and the transmission of electric currents as a result of external excitation - hence the interest in defibrillating applications [28].

We used separation of variables to express the trans-membrane potential $\Phi_m(x, y, z, t)$ as an infinite sum of the product of spatial eigenfunctions and temporal modes each of which satisfies a first-order differential equation [31]. Then, by keeping N terms in the expansion, one can assemble a finite dimensional system of first-order *uncoupled* differential equations of the form

$$\dot{X}(t) = AX(t) + BU(t); \quad Y(t) = CX(t) \qquad (3)$$

where X is the state vector in $\Re^n,\ n = N^3$; U denotes the external input (or inputs), Y is an output (or outputs), and matrices $A,\ B,\ C$ are appropriately dimensioned. The model (3) can be used for example to generate plots of the transmembrane potential $\Phi_m(x, y, z, t)$ caused by external stimuli; to make a formal connection to the widely used single RC-circuit model of cardiac tissue; and also to suggest a multi RC-circuit model due to the fact that the system matrix A in (3) is diagonal $A = diag(a_i)$ [31]. The two-dimensional bidomain model was approximated by an RC-circuit in [32] where the intracellular and extracellular spaces are represented by resistor grids, while the membrane connecting the spaces is modeled by a parallel RC-circuit. Recently, [26] indicated the possibility of estimating the effect of an electric shock on many small sections of the heart each modeled as a separate RC circuit.

III. RC MODELS

The differential equation governing the capacitor voltage $V_m(t) = x(t)$ in Figure 1 may be written as follows

$$\dot{x}(t) + \frac{1}{R_m C_m} x(t) = b_s \imath_s(t); \quad y(t) = x(t) = V_m(t) \quad (4)$$

where R_m and C_m model the myocardial resistance and capacitance, respectively. The single RC-circuit is a lumped, input-output idealization of a region of cardiac tissue. The defibrillator provides an ideal current source $\imath_s(t)$ that is distributed throughout the heart by the shock pulse between the electrodes. It is assumed that only a portion of this current reaches the modeled myocardial cells, hence, b_s is another modeling parameter. The N-mode expansion (3) is derived from a distributed parameter model where the transmembrane potential Φ_m is a function of spatial coordinates and time. Therefore, the temporal behavior of Φ_m at a 3D point P_0 can be thought of as a linear combination of capacitor voltages of an RC-circuit model weighted by the eigenfunctions Υ_i evaluated at the point P_0:

$$\Phi_m(P_0, t) = \sum_{i=1}^{N} \Upsilon_i(P_0) x_i(t) \ .$$

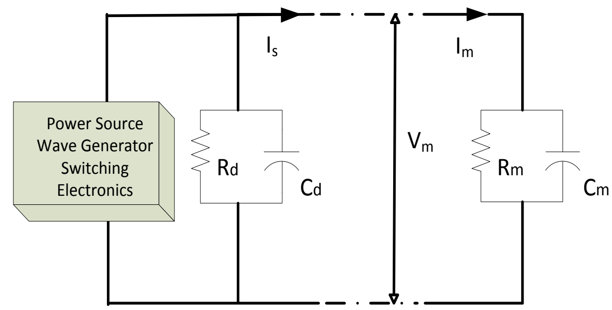

Fig. 1. Single RC-Circuit Idealization for Defibrillation Control.

The simplest case of the single-input, single-output, one-mode expansion is of the same form as (4)

$$\dot{x}_1(t) = -a_1 x_1(t) + b_1 u(t); \quad y_1(t) = c_1 x_1(t) \ . \qquad (5)$$

In the single RC-circuit, the time constant is $\tau_m = R_m C_m$ and many publications deal with RC-parameter selection using simulations and also to match experimental data. The i^{th}-mode in the bidomain model (3) has a time constant $\tau_i = 1/a_i$. A typical set of parameter values [34] was used in a numerical example [30] using the three-dimensional bidomain model. It was found that $\tau_1 = 6.2\ msec$ and $\tau_{1000} = 2.5\ msec$ which is in agreement with the widely reported cardiac tissue time constant range of $\tau_m = 1.5$ to around $\tau_m = 5\ msec$.

Motivated by these developments, we are prompted to propose a multi-RC circuit as shown in Figure 2. The defibrillator produces a defibrillating pulse $u(t)$ that will exhibit certain desired characteristics across the electrodes. Defibrillator models that generate monophasic (one capacitor) or bi-phasic (two capacitors) pulses were depicted in [16]. On the load side, the bidomain equations are interpreted as a large (theoretically infinite) array of single-RC circuits each providing a path for current to flow. The resistors R_F and R_T are simply idealizations of energy loss elements. The ideal switches (shown open) are normally closed. Conditions of damaged tissue (e.g., due to ischemic heart disease) may be modeled by open switches at the input, output or both sides. Invariably, one truncates the expansion to a large but finite number N, so that each RC-circuit could model individual cardiac cells, groups of cells, or even finite regions within the heart responsible for the propagation of action potentials. Letting $x_i(t)$ denote the voltage in the i^{th}-capacitor and $u(t) = v_o(t)$ the voltage applied at the electrodes, the following system of equations follows for the case of all switches in the closed position:

$$\dot{x}_i(t) = -a_i x_i(t) + b_i u(t); \quad i = 1, 2, \ \ldots \ N \qquad (6)$$

$$y(t) = \sum_{i=1}^{N} \gamma_i x_i(t) = V_m(t) \qquad (7)$$

where letting τ_i denote the $(R_i C_i)$-time constant, τ_{Fi} denote the $(R_{Fi} C_i)$-time constant, and τ_{Ti} denote the $(R_{Ti} C_i)$-time constant, then $a_i = \frac{1}{\tau_i} + \frac{1}{\tau_{Fi}} + \frac{1}{\tau_{Ti}}$; $b_i = \frac{1}{\tau_{Fi}}$ and

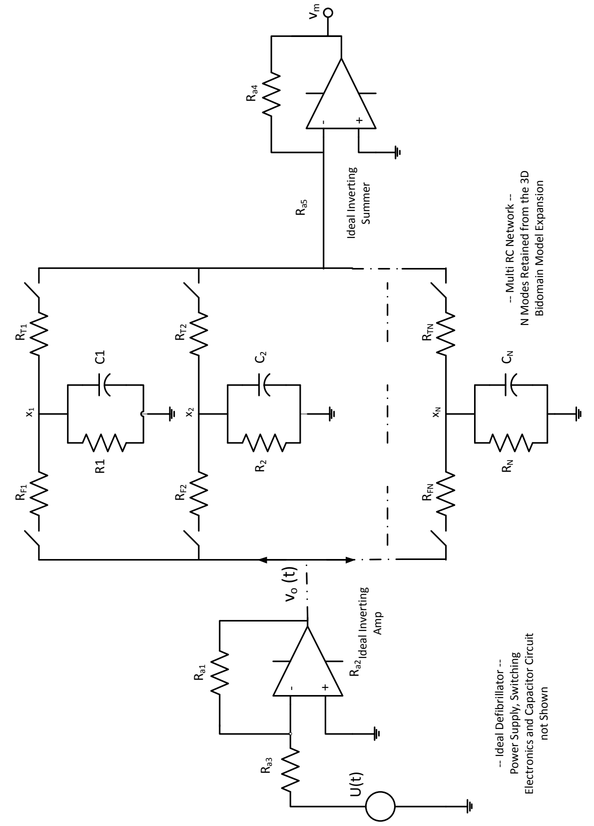

Fig. 2. Multi RC-Circuit Idealization for Defibrillation Control.

γ_i are expressions involving the resistors of the summing amplifier and resistors R_{Ti} that could be selected to match the bidomain eigenfunction values $\Upsilon_i(P_0)$.

IV. ENERGY AND TIME OPTIMIZATION

A. Minimum Energy: Single RC-Circuit

This section addresses the question of synthesizing a waveform that exhibits certain characteristics deemed important for achieving defibrillation. The development in [17], considers the single RC-circuit, a *fixed* final time t_f, and finds the shape of the optimal current $u^*(t)$ that minimizes a measure of the total energy delivered in the fixed interval $t \in [0,\ t_f]$ given by

$$J_1 \ = \ \int_0^{t_f} u^2(t)dt \ . \tag{8}$$

The qualitative exponentially ascending behavior of the optimal current was a new and beneficial contribution to the defibrillation literature. It also became apparent that higher values of cardiac tissue time-constants require higher levels of input energy for a fixed interval. Next, [17] proceeds to determine the optimal current that discharges the capacitor also with minimum energy consumption. The result is again an ascending exponential, and taken together, the optimal *biphasic* current is a pair of ascending exponentials executing the charge banking and charge burping hypothesis of defibrillation.

B. Minimum Weighted Time-Energy: Single RC-Circuit

Subsequently, in [27], [31] we reported on the synthesis of the optimal waveform for the single RC-circuit that minimizes a weighted measure of spent energy and elapsed time given by

$$J_2 = \int_0^{t_f} \left[\rho + u^2(t)\right] dt \tag{9}$$

where the final time t_f is unknown. The limiting cases of $\rho \to 0$ (and fixed final time) and $\rho \to \infty$ (free final time) correspond to minimum-energy and minimum time problems, respectively. The simulations were done for a generic LTI first-order differential equation:

$$\frac{dV_m(t)}{dt} \ + \ a_0 V_m(t) \ = \ b_1 \frac{du(t)}{dt} \ + \ b_0 u(t) \ , \tag{10}$$

which reduces to (4) when setting $b_1 = 0$. The optimal policy $u^*(t)$ is given in [27], and a representative plot for the charge banking case and various ρ values is shown in Figure 3. The optimal one-mode policy was also applied to a 3-mode

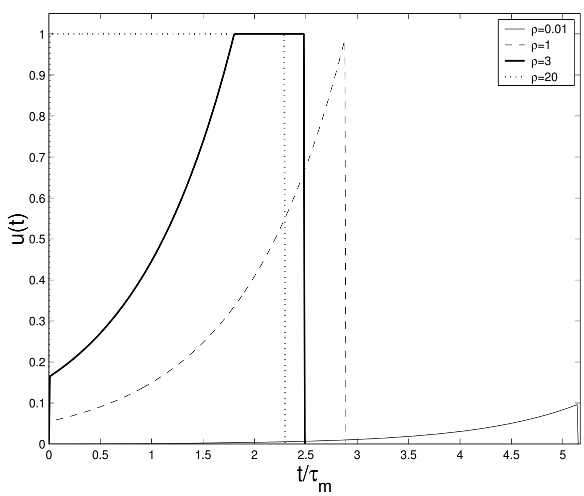

Fig. 3. Charge Banking $u^*(t)$ for Several Values of ρ.

model of the bidomain expansion, assuming a collocated one point sensor-actuator case and the input signal delivered via extracellular space [31]. As expected, the controller takes x_1 from an initial value to a close neighborhood of the specified value x_f. The effect of the other 2 modes was minimal for the simulated example. The selected modes were representative and not based on any assessment of their relative importance for the overall system response. Further analysis along this topic is needed.

The two problems and solutions presented in this section are well known in the control community (for example, see [35], [36]), but the solutions are of interest to the electrophysiology community because of the implications on the design of portable defibrillators, and because of the unexpected exponentially rising shape of the optimal waveforms which is contrary to the previously assumed square, triangular, trapezoidal, damped sinusoids, and various truncated decaying exponential functions. Here, we offer a set of new qualitative observations in an effort to spark further interest in the electrophysiologist community regarding modeling and experimental validation that we hope will lead to additional quantitative analysis relevant to defibrillator designs:

1) Contrary to the policy for cost J_1 (8), the final time t_f is not specified but rather calculated as a function of system parameters. This may be relevant to incorporating physical parameters such as gender, height, weight, and others in the design of the optimal waveform. The design parameter ρ assigns the relative importance of time minimization compared to energy consumption. A large value of ρ means that time is expensive and it should be minimized at the expense of energy.
2) The most obvious qualitative difference between the policies that achieve minimum energy expenditure (8) and weighted energy-time expenditure (9) is that in the latter $u^*(t)$ *saturates* when $\rho > 1.0$ (normalized value). That is, $u^*(t) = \pm 1.0$ for an interval of time $[t_1,\ t_f)$ where $0 \leq t_1 < t_f$. This saturation corresponds to the application of maximum available input energy. Hence the optimal policy is a combination of a rising exponential and a constant. See the case $\rho = 3.0$ in Figure 3. It can be shown that for the chosen parameter values, $\rho = 3.0$ is the best choice to minimize the total energy usage in charge banking plus charge burping.
3) For $\rho \leq 1.0$ which includes the case $\rho = 1.0$ where time and energy are equally penalized, the optimal waveform *does not* saturate. On the other hand, a large enough $\rho = \rho_m$ will result in a saturated input profile during the entire interval $[0,\ t_f]$. Values of $\rho > \rho_m$ have no effect on the shape of $u^*(t)$ and virtually no practical effect on the length of the interval $[0,\ t_f]$. See the case $\rho = 20.0$ in Figure 3, corresponding to fast performance at the expense of energy.
4) The optimal policy $u^*(t)$ is normally expressed as a nonlinear function of a so-called switching curve which depends on the capacitor voltage. It can be shown that in this problem $u^*(t)$ can be expressed as a function of time that, although easier to implement, it is an open-loop strategy.

C. Minimum Time Charge-Burping: Multi-RC Circuit

In this final section, we propose a minimum-time defibrillation policy that is applicable specifically in the multi-RC circuit to carry out the *charge burping* phase. As in the previous sections, the problem and its solution are well known to the control community but the formulation is of interest to electrophysiologists as a limiting case of fast response in emergency defibrillation cases, or to address the physiological concern of achieving fast capacitor discharge (charge burping) as advocated in [16].

Consider the multi RC-circuit state equation (6) where the control is constrained in magnitude $|u(t)| \leq U_{max}$, the constants $a_i > 0$ so the system is open-loop asymptotically stable, and $b_i \neq 0$ so the system is controllable. Then, it is known that the optimal control $u^*(t)$ that steers the states $x_i(t),\ i = 1, \ldots, N$ from arbitrary initial conditions to the origin (charge burping case) exists, it is unique, it is piece-wise constant and switches at most $(N-1)$ times. The formulation of the switching structure is fairly involved and can be found in [35]. *Such a defibrillating waveform consisting of constant values of the control at $u = +U_{max}$ or $u = -U_{max}$ for finite intervals of time is fundamentally different from any other described in the electrophysiology literature and may prove to have desirable characteristics.*

The complementary problem of steering the states to desired values (charge banking) cannot be easily treated using the minimum-time formulation, and in fact an optimal control for this case does not necessarily exist. One can attempt other strategies illustrated here for the $N = 2$ case. Write the underlying differential equation as:

$$\ddot{y} + \alpha_1 \dot{y} + \alpha_0 y = \beta_1 \dot{u} + \beta_0 u \ .$$

Then, for a constant reference y_r, the error $e(t) = y_r - y(t)$ satisfies the double integrator equation $\ddot{e} = v(t)$, where $v(t)$ is a new input and $U(s)$ is set to

$$U(s) = -\frac{\alpha_1 s + \alpha_0}{\beta_1 s + \beta_0} E(s) + \frac{\alpha_0}{\beta_1 s + \beta_0} Y_r(s) - \frac{1}{\beta_1 s + \beta_0} V(s) \ .$$

The new control $v(t)$ is then designed following the well known time-optimal solution for the double integrator that drives $e(t) \to 0$ or equivalently $y(t) \to y_r$ in minimum time with at most one switch [35], [36]. Any strategy should be tested on the multi RC-circuit model to assess its effect on un-modeled dynamics. Higher order cases, $N = 3$ [36] and $N = 4$ [35] could also be determined in a similar manner.

V. CONCLUSIONS AND FURTHER RESEARCH

The article collected various results on the problem of synthesizing optimal waveforms for a defibrillator which would bring the heart to a defibrillated state. Our approach is to utilize control design frameworks suitable for the defibrillation problem. Single and multi-RC circuits were presented as the basis for the waveform synthesis. The single RC-circuit has been in use for over 70 years; the multi-RC circuit is a natural extension to this model and it can be generated from the modal expansion solution of the 3D bidomain model of cardiac tissue. Various optimization problems were described that minimize energy consumption and/or time elapsed. Several directions for further research can be outlined:

1) The multi RC-circuit of Figure 2 may be used to explore modeling questions such as number of modes kept in the model, and time constants and correlation with known experimental values. It would also be interesting to find any correlation between the ideal switches (open or closed) and pathophysiological conditions that may lead to arrhythmias. Although we have used switches, these could be replaced by time-varying elements (resistor and/or capacitors) to simulate physiological conditions or time-dependent tissue healing scenarios [21].

2) One could generate a comprehensive simulation study of the different defibrillating waveforms available in the literature as applied to the multi RC-circuit model.
3) The multi RC circuit model suggests how to study actuator/sensor location, multi-input approaches (multiple defibrillating pulses in different locations) and the effect of mode truncation and un-modeled dynamics.
4) The control policies presented in the literature are essentially *open-loop* so the efficacy of the applied defibrillating pulse is dictated by the physiological accuracy of the RC-circuit models. Further research will reveal whether a real-time physiological signal can be provided to the defibrillator device that will allow more advanced control frameworks to be explored.
5) Fast capacitor discharge can also be achieved with deadbeat control in discrete-time [37].
6) We are interested in exploring improvements in defibrillator designs.

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