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On feedback stabilization of a class of stochastic nonlinear systems with delays

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Abstract— **In this paper, the problem of feedback stabilization of stochastic differential delay systems is considered. The systems under study are nonlinear and nonaffine. By using a LaSalle-type theorem for stochastic systems, general conditions for stabilizing the closed-loop system with delays are obtained. In addition, stabilizing state feedback control laws are proposed.**

I. INTRODUCTION

During the last decades, the problems of stabilization and controller design for linear systems with delays has been extensively studied and is still under investigation (see [6], [8], [11], [18], [20], [23]). In practice, many control processes involve delays (often due to transmission or transportation phenomenons). Delays may significantly affect the closed-loop performances or even be a source of instability.

In the case of nonlinear systems with delays, the problem of stabilization is more complex. This is mainly due to the infinite dimensionality of the system state combined with the nonlinear structure of the differential equations.

In ([1]-[4]) we investigated the problem of stabilization of nonlinear, nonaffine systems involving delays in both continuous and discrete-time cases.

However, the presence of delays and nonlinearities are not the only sources of complexity. Indeed, various disturbances that are not measurable may arise which, in turn, limit the application of classical control systems design. This motivates the study of the stabilizability and the control design in a stochastic framework, where the state equation is described by an Itô differential delay equation driven by Wiener noise.

The stability analysis of the equilibrium positions of stochastic differential equations with delays has been extensively studied (see for instance [12],[15],[19]). In the case of linear stochastic systems with delays, some results on stabilization have been proposed (in [7], [22], for instance). However, the stabilization of nonlinear stochastic systems with delays still remains an open problem.

In this paper, the state feedback stabilizability problem of equilibrium positions of stochastic nonlinear systems with delays is considered. The class of systems under study is nonaffine in control. Moreover, the system without drift - in other words, the related autonomous system - also involves delays. By combining a suitable mathematical formalism and a LaSalle-type theorem dedicated to stochastic systems ([16], [17]), sufficient conditions guaranteeing the stability of the closed-loop system are developed and feedback controllers for these systems are proposed. The approach adopted in this paper allows considering a rather large class of nonlinear stochastic systems. Moreover, the autonomous system ($u = 0$) as well as the controlled part are affected by a noise.

The organization of the paper is as follows. In Section 2 the class of systems considered is presented and some basic notions are recalled . In Section 3, the main results are given and proved. Finally, Section 4 gives conclusions.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following class of systems:

$$\begin{cases} dx(t) = f(x(t), x(t-\tau), u)dt + g(x(t), x(t-\tau), u)d\xi(t) \\ x(t) = \phi(t), \quad t \in [-\tau, 0] \end{cases} \tag{1}$$

where f and g are smooth vector field such that $f(0,0,0) = g(0,0,0) = 0$. In the following, $x(t) \in \mathbb{R}^n$ is the state vector and $u \in \mathbb{R}$ is the input vector. τ is a positive scalar that represents the delay.

The function $\phi(t) \in \mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ represents the initial condition. $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ is the banach space of continuous function mapping $[-\tau, 0]$ into $\mathbb{R}^n$, with the norm $\|\phi\| = \sup_{t\in[-\tau,0]} |\phi(t)|$ where $|\phi(t)|$ stands for the Euclidean norm of $\phi(t) \in \mathbb{R}^n$.

$\{\xi(t), t \geq 0\}$ is a standard Wiener process defined on the usual complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ with $(\mathcal{F}_t)_{t\geq 0}$ being the complete right-continuous filtration generated by ξ and $\mathcal{F}_0$ contains all P-null sets.

Let $\mathcal{C}^b_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n)$ be the set of all $\mathcal{F}_0$-measurable bounded $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$-valued random variables ϕ.

We now recall some basic notions of stability that will be used latter. In order to set the ideas, let us consider the differential stochastic system of the general form

$$\begin{cases} dx(t) = F(t, x_t)dt + G(t, x_t)d\xi(t) \\ x(t) = \phi(t), \quad t \in [-\tau, 0] \end{cases} \tag{2}$$

where $F, G : [0, \infty) \times \mathcal{C}([-\tau, 0], \mathbb{R}^n) \mapsto \mathbb{R}^n$ is continuous with respect to the first argument, locally Lipschitz with respect to the second and satisfy $F(t, 0) = G(t, 0) = 0$ for all $t \geq 0$.

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For $t \geq \sigma - \tau$, we denote by $x(\sigma, \phi)(t)$ its solution at time t with initial data ϕ, specified at time σ, i.e., $x(\sigma, \phi)(\sigma+\theta) = \phi(\theta)$, $\forall \theta \in [-\tau, 0]$. For $\theta \in [-\tau, 0]$,

$$x_t(\theta) = x(t+\theta)$$

and represents the state of the delay system.

Under the previous assumptions on F and G, it is known (see e.g. [15], [19], [21]) that equation (2) has a unique solution $x(\sigma, \phi)(t)$ for $t \geq \sigma - \tau$.

Let us introduce δ, the delay operator defined for any function $a(.)$ by :

$$\delta a(t) = a(t-\tau). \tag{3}$$

For further simplification, we will use, for any function $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the following type of notation indexed by δ,

$$h_\delta(x(t)) = h(x(t), \delta x(t)) = h(x(t), x(t-\tau)). \tag{4}$$

Consider system (2) in the particular case where it contains discrete delays, i.e. the system is of the form :

$$\begin{cases} dx(t) = F(t, x(t), x(t-\tau))dt + G(t, x(t), x(t-\tau))d\xi(t) \\ x(t) = \phi(t), \quad t \in [-\tau, 0] \end{cases} \tag{5}$$

The infinitesimal generator associated to (5), obtain by differentiating V in the sense of Itô, is given by

$$\mathcal{L}_\delta V(t,x) = \frac{\partial V(t,x)}{\partial t} + < F(t,x,\delta x), \nabla V(t,x) > \\ + \frac{1}{2}\mathrm{Tr}\Big([G(t,x,\delta x)G^T(t,x,\delta x)]\nabla^2 V(t,x)\Big) \tag{6}$$

where ∇ denotes the gradient and $\langle .,. \rangle$ designates the scalar product. The matrix $\nabla V(t,x) = \dfrac{\partial^2 V(t,x)}{\partial x^2}$ is the Hessian matrix of the second order partial derivatives.
The notation Tr(.) designates the trace of a matrix. In the following, we will also use Ker(.) to designate the Kernel of a matrix or a function and $d(x, D)$ will represents the Haussdorf semi-distance between a point $x \in \mathbb{R}^n$ and a set D (i.e. $d(x, D) = \inf_{y \in D} |x-y|$.) For any matrix M, M^T denotes its transpose.

We shall now recall some notions of stability of equilibrium solution of stochastic differential delay equations that generalize the notions initially dedicated to stochastic differential systems (cf. [5], [10], [13]).

DEFINITION 1:
The equilibrium solution, $x \equiv 0$ of the stochastic differential delay equation (2) is said to be

1) stable in probability, if for any $\sigma \in \mathbb{R}$, $\varepsilon > 0$, there is a $\beta = \beta(\varepsilon, \sigma)$ such that $\|\phi\| < \beta$ implies $P\big(\sup_{t \geq \sigma} \|x_t(\sigma, \phi)\| > \varepsilon\big) = 0$.
2) uniformly stable in probability, if the number β is independent of σ.

For all $\beta > 0$, let us denote by $\mathcal{B}(0, \beta)$, the ball

$$\mathcal{B}(0, \beta) = \{\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n) : \|\phi\| < \beta\}.$$

DEFINITION 2:
The equilibrium solution, $x \equiv 0$ of the stochastic differential delay equation (2) is said to be asymptotically stable in probability, if it is stable and there exists $b_0 = b_0(\sigma) > 0$ such that $\phi \in \mathcal{B}(0, b_0)$ implies $P\big(\lim_{t \to +\infty} \|x_t(\sigma, \phi)\| = 0\big) = 1$.

In practice, we can use the following LaSalle-type Theorem for stochastic differential delay system (cf. [16], [17]). This is a stochastic version of the well-known LaSalle Theorem (cf. [9], [14]).

THEOREM 1:
Assume that there are functions $V \in \mathcal{C}^{1,2}([0,\infty) \times \mathbb{R}^n, [0,\infty))$, $\gamma \in L^1([0,\infty),[0,\infty))$ and $w_1, w_2 \in \mathcal{C}(\mathbb{R}^n, [0,\infty))$ such that

$$\mathcal{L}V(t,x,y) \leq \gamma(t) - w_1(x) + w_2(y), \quad x, y \in \mathbb{R}^n, t \geq 0 \\ w_1(x) \geq w_2(x) \quad x \in \mathbb{R}^n \\ \text{and} \quad \lim_{|x| \to \infty} \inf_{0 \leq t < \infty} V(x,t) = \infty \tag{7}$$

Then, $Ker(w_1 - w_2) \neq \emptyset$ and

$$\lim_{t \to \infty} d(x(t,\phi), Ker(w_1 - w_2)) = 0 \quad \text{a.s (almost surely)}$$

for every $\phi \in \mathcal{C}^b_{\mathcal{F}_0}([-\tau, 0], \mathbb{R}^n)$.

REMARK 1: The infinitesimal generator associated to system (5) is defined, in this theorem, by

$$\mathcal{L}V(t,x,y) = \frac{\partial V(t,x)}{\partial t} + < F(t,x,y), \nabla V(t,x) > \\ + \frac{1}{2}\mathrm{Tr}\Big([G(t,x,y)G^T(t,x,y)]\nabla^2 V(t,x)\Big)$$

and $L^1([0,\infty),[0,\infty))$ are the functions $\gamma : [0,\infty) \to [0,\infty)$, firstly integrable i.e. such that $\int_0^\infty \gamma(t)dt < \infty$.
Note that the notation $\mathcal{L}_\delta V(t,x)$ which will be used throughout the paper and defined by (6), corresponds to $\mathcal{L}V(t,x,\delta x)$ in Theorem 1.

We shall now state and prove our main results.

III. MAIN RESULTS

We consider system (1) where we assume that f can be developed in the form :

$$f(x(t), x(t-\tau), u) = f_0(x(t), x(t-\tau)) \\ + u\ f_1(x(t), x(t-\tau)) + u^2\ f_2(x(t), x(t-\tau), u).$$

and

$$g(x(t),x(t-\tau),u)=g_0(x(t),x(t-\tau))$$
$$+u\;g_1(x(t),x(t-\tau))+u^2\;g_2(x(t),x(t-\tau),u).$$

f_0 and g_0 are functions defined by :

$$f_0(x(t),x(t-\tau))=f(x(t),x(t-\tau),0)$$

and

$$g_0(x(t),x(t-\tau))=g(x(t),x(t-\tau),0).$$

f_i and g_i $(i=1,2)$ are suitable smooth functions.

Note that since f and g are smooth, such expansion are possible where

$$f_1(x(t),x(t-\tau))=\frac{\partial f}{\partial u}(x(t),x(t-\tau),0)$$

and $f_2(x(t),x(t-\tau),u)$ correspond to the rest of the Taylor expansion. A similar remark stands for g as well.

Let us denote by $L_{0,\delta}$ and $L_{1,\delta}$ the second order differential operators defined for all $\Xi\in\mathcal{C}^2(\mathbb{R}^n)$ by :

$$L_{0,\delta}\Xi(x(t))\;=\;<f_{0,\delta}(x(t)),\nabla\Xi(x(t))>$$
$$+\;\frac{1}{2}\mathrm{Tr}\big(g_{0,\delta}(x(t))\,g_{0,\delta}^T(x(t))\nabla^2\Xi(x(t))\big)$$

and

$$L_{1,\delta}\Xi(x(t))=\;<f_{1,\delta}(x(t)),\nabla\Xi(x(t))>$$
$$+\frac{1}{2}\mathrm{Tr}\big([g_{0,\delta}(x(t))\,g_{1,\delta}^T(x(t))$$
$$+g_{1,\delta}(x(t))\,g_{0,\delta}^T(x(t))]\nabla^2\Xi(x(t))\big).$$

Moreover, we introduce $H(x(t),x(t-\tau),u)$ defined by

$$\begin{aligned}&H(x(t),x(t-\tau),u)=\\&<f_{2,\delta}(x(t),u),\nabla\Xi(x(t)>\\&+\frac{1}{2}\,\mathrm{Tr}\,\Big(\{[g_{0,\delta}(x(t))\,g_{2,\delta}^T(x(t),u)+g_{2,\delta}(x(t),u)\,g_{0,\delta}^T(x(t))]\\&+u[g_{1,\delta}(x(t))\,g_{2,\delta}^T(x(t),u)+g_{2,\delta}(x(t),u)\,g_{1,\delta}^T(x(t))]\\&+g_{1,\delta}(x(t))\,g_{1,\delta}^T(x(t))\\&+u^2g_{2,\delta}(x(t),u)\,g_{2,\delta}^T(x(t),u)\}\nabla^2\Xi(x(t)\,\Big).\end{aligned}\tag{8}$$

where we use the notation (4) and set

$$f_{2,\delta}(x(t),u)=f_2(x(t),x(t-\tau),u)$$

and

$$g_{2,\delta}(x(t),u)=g_2(x(t),x(t-\tau),u),$$

in order to get more compact expressions.

We suppose that there exists a Lyapunov function $V\in\mathcal{C}^2(\mathbb{R}^n,[0,\infty))$ and functions α and $\alpha_d\in\mathcal{C}(\mathbb{R}^n,[0,\infty))$ such that

$$L_{0,\delta}V(x(t))\le-\alpha(x(t))+\alpha_d(x(t-\tau))\tag{9}$$

where

$$\alpha_d(x)\le\alpha(x)\quad\forall x\in\mathbb{R}^n.$$

Let us denote by M the set :

$$\mathcal{M}=Ker(\alpha-\alpha_d)\cap Ker(L_{1,\delta}V)\tag{10}$$

We then have the following result :

THEOREM 2: Let

$$u(x(t),x(t-\tau))=-\psi(x(t),x(t-\tau))\,L_{1,\delta}V(x(t))\tag{11}$$

where $\psi\in\mathcal{C}^\infty(R^n\times R^n;]0,\infty))$ is a function satisfying :

$$\psi(x(t),x(t-\tau))\le\frac{1}{\sup\limits_{|u|\le1}|H_\delta(x(t),u)|^2+|L_{1,\delta}V(x(t))|^2+2}\tag{12}$$

and $H_\delta(x(t),u_\delta(x(t)))=H\big(x(t),x(t-\tau),u_\delta(x(t))\big)$ with $u_\delta(x(t))=u(x(t),x(t-\tau))$.

If the set $\mathcal{M}$ defined by (10) is reduced to the origin, then the trivial solution of the system (1) with (11) is almost surely asymptotically stable.

Proof :

With the control law (11), the closed-loop system is of the form :

$$\begin{aligned}dx(t)&=f_0(x(t),x(t-\tau))\,dt\\&+u(x(t),x(t-\tau))\,f_1(x(t),x(t-\tau))\,dt\\&+u^2(x(t),x(t-\tau)).\\&\qquad f_2\big(x(t),x(t-\tau),u(x(t),x(t-\tau))\big)\,dt\\&+g_0(x(t),x(t-\tau))\,d\xi(t)\\&+u(x(t),x(t-\tau))\,g_1(x(t),x(t-\tau)))\,d\xi(t)\\&+u^2(x(t),x(t-\tau)).\\&\qquad g_2\big(x(t),x(t-\tau),u(x(t),x(t-\tau)\big)\,d\xi(t).\end{aligned}\tag{13}$$

The infinitesimal generator associated to this system (13) is given by :

$$\mathcal{L}_\delta V(x(t)) =< f_{0,\delta}(x(t)), \nabla V(x(t)) >$$
$$+u_\delta(x(t)) \ < f_{1,\delta}(x(t)), \nabla V(x(t)) >$$
$$+u_\delta^2(x(t)) \ < f_2\big(x(t), \delta x(t), u_\delta(x(t))\big), \nabla V(x(t)) >$$
$$+\frac{1}{2}\mathrm{Tr}\Big([g_{0,\delta}(x(t)) + u_\delta(x(t))\, g_{1,\delta}(x(t))$$
$$+u_\delta^2(x(t))\, g_2\big(x(t), \delta x(t), u_\delta(x(t))\big)].$$
$$[g_{0,\delta}(x(t)) + u_\delta(x(t))\, g_{1,\delta}(x(t))$$
$$+u_\delta^2(x(t))\, g_2\big(x(t), \delta x(t), u_\delta(x(t))\big)]^T \ \nabla^2 V(x(t))\Big) \quad (14)$$

Using (11)(12) and the definition of H given by (8), we get

$$\mathcal{L}_\delta V(x(t)) = L_{0,\delta} V(x(t)) + u_\delta(x(t)) \ L_{1,\delta} V(x(t))$$
$$+u_\delta^2(x(t)) \ H\big(x(t), x(t-\tau), u_\delta(x(t))\big) \quad (15)$$

It is easy to check that u satisfies the following inequalities

$$u(x(t), x(t-\tau)) L_{1,\delta} V(x(t)) \le 0,$$
$$|u(x(t), x(t-\tau))| \le \frac{1}{2}$$
$$\text{and} \ \ |\psi(x(t), x(t-\tau)) \ H_\delta(x(t), u_\delta(x(t)))| \le \frac{1}{2}, \quad (16)$$
$$\text{with} \quad u_\delta(x(t)) = u(x(t), x(t-\tau)).$$

Then

$$\mathcal{L}_\delta V(x(t)) = L_{0,\delta} V(x(t)) - \psi(x(t), x(t-\tau))(L_{1,\delta} V(x(t)))^2$$
$$+\psi^2(x(t), x(t-\tau))(L_{1,\delta} V(x(t)))^2.$$
$$H_\delta\big(x(t), -\psi(x(t), x(t-\tau))(L_{1,\delta} V(x(t)))\big) \quad (17)$$

By definition of ψ with (12) and (16),

$$\big[\ 1 - \psi(x(t)), x(t-\tau))\, H_\delta(x(t), u_\delta(x(t)))\ \big] \ge \frac{1}{2} > 0$$

and

$$\mathcal{L}_\delta V(x(t)) = L_{0,\delta} V(x(t)) - \psi(x(t), x(t-\tau))\big(L_{1,\delta} V(x(t))\big)^2$$
$$.\big[\ 1 - \psi(x(t)), x(t-\tau))\, H_\delta(x(t), u_\delta(x(t)))\,\big]$$
$$\le L_{0,\delta} V(x(t)) - \frac{\Psi(x(t), x(t-\tau))}{2}\big(L_{1,\delta} V(x(t))\big)^2$$
$$\le \ 0. \quad (18)$$

Therefore, we can deduce that the system (1)(11) is stable in probability.

By applying the stochastic version of LaSalle Theorem (cf. [16], [17]), we can deduce that the solution $x(t)$ tend to the set asymptotically $\mathcal{N} = \{x \in \ker(\mathcal{L}_\delta V)\}$ with probability one.
If x is an element of $\mathcal{N}$ then with (18), and the fact that

$$\big[\ 1 - \psi(x(t)), x(t-\tau))\, H_\delta(x(t), u_\delta(x(t)))\ \big] > 0$$

we can deduce that

$$L_{0,\delta} V(x) = 0 \quad \text{and} \quad L_{1,\delta} V(x) = 0.$$

From the condition (9) on $L_{0,\delta} V(x)$, it follows that :

$$\alpha(x) - \alpha_d(x) = 0 \quad \text{and} \quad L_{1,\delta} V(x) = 0.$$

Therefore, x is an element of $\mathcal{M}$. Since $\mathcal{M} = \{0\}$, the almost sure attractivity of the origin is proved. Consequently, the closed-loop system is almost surely asymptotically stable.
This completes the proof of Theorem 2.

REMARK 2: For sake of simplicity our main result is given for $u \in \mathbb{R}$. A similar result for $u \in \mathbb{R}^p \, (p > 1)$ can be analogously established.

REMARK 3: When the system is affine in control, i.e., of the form :

$$dx(t) = [f_0(x(t), x(t-\tau)) + u\, f_1(x(t), x(t-\tau))\,]dt$$
$$+[g_0(x(t), x(t-\tau)) + u\, g_1(x(t), x(t-\tau)))]\, dw(t) \quad (19)$$

we have the following result :

COROLLARY 1: Suppose there exists a Lyapunov function $V \in \mathcal{C}^2(\mathbb{R}^n, [0, \infty))$ and functions α and $\alpha_d \in \mathcal{C}(\mathbb{R}^n, [0, \infty))$ such that condition (9) is satisfied.
If the set

$$\mathcal{M} = Ker(\alpha - \alpha_d) \cap Ker(L_{1,\delta} V)$$

is reduced to $\{0\}$, then the system (19) is almost surely stabilizable by means of the feedback law

$$u(x(t), x(t-\tau)) = -\kappa\, L_{1,\delta} V(x(t)) \quad (20)$$

with $\kappa > 0$.

Example 1

Let us consider the following simple example, described by a one-dimensional stochastic differential system with a single delay

$$\begin{cases} dx(t) = -2x^3(t)dt + ux(t)dt + 2x^2(t-\tau)d\xi(t) \\ x(t) = \phi(t), \quad t \in [-\tau, 0] \end{cases} \quad (21)$$

Let us consider the Lyapunov function $V(x) = x^2$. This function satisfy the condition $\lim\limits_{|x|\to\infty} V(x) = \infty$.

The infinitesimal generator associated to this system is given by

$$\mathcal{L}_\delta V(x(t)) = -4x^4(t)+4x^4(t-\tau)+2u(x(t),x(t-\tau))x^2(t) \tag{22}$$

Here $H_\delta(x(t),u_\delta(x(t)) = 0$ and $L_{1,\delta}V(x(t)) = 2x^2(t)$.

The operator $L_{0,\delta}V(x(t))$ is given by

$$L_{0,\delta}V(x(t)) = -4x^4(t) + 4x^4(t-\tau).$$

We can notice that condition (9) is satisfied with

$$\alpha(x(t)) = 4x^4(t) \quad \text{and} \quad \alpha_d(x(t-\tau)) = 4x^4(t-\tau).$$

By Theorem 2, if we choose a control

$$u(x(t),x(t-\tau)) = -\Theta(x(t),x(t-\tau))\, L_{1,\delta}V(x(t))$$
$$= -2\Theta(x(t),x(t-\tau))\, x^2(t).$$

then

$$\mathcal{L}_\delta V(x(t)) = -4x^4(t)+4x^4(t-\tau)-4\Theta(x(t),x(t-\tau))\, x^4(t). \tag{23}$$

with $\Theta(x(t),x(t-\tau) \geq 0$. It suffices to take Θ equal to a positive constant.
We can remark that $\mathcal{M} = 0$. Thus, according to Theorem 2, the solution of the closed-loop system verify

$$\lim_{t\to\infty} x(t) = 0 \;a.s \text{ (almost surely).}$$

We can also remark that $\ker(\alpha - \alpha_d)$ is not reduced to zero. Thus, concerning the open-loop system, we can only conclude that it is stable in probability but not asymptotically stable in probability.

Example 2

Consider the following example

$$\begin{cases} dx(t){=}[-2x^3(t) + x^\mu(t)u]dt + [2x^2(t-\tau) + x(t)u]d\xi(t) \\ x(t){=}\phi(t), \quad t \in [-\tau, 0] \end{cases} \tag{24}$$

where $\mu \in \mathbb{N}, \mu \geq 1$.
We consider again the Lyapunov function $V(x) = x^2$.
Differentiating V in the sense of Itô, leads to

$$\mathcal{L}_\delta V(x(t)) = \big(-4x^4(t) + 4x^4(t-\tau)\big)$$
$$+u\big(2x^{\mu+1}(t) + 2x(t)x^2(t-\tau)\big) + u^2 H\big(x(t),x(t-\tau),u\big) \tag{25}$$

Here

$$L_{0,\delta}V(x(t)) = -4x^4(t) + 4x^4(t-\tau)$$

$$L_{1,\delta}V(x(t)) = 2x^{\mu+1}(t) + 2x(t)x^2(t-\tau).$$

and

$$H\big(x(t),x(t-\tau),u\big) = x^2(t).$$

Note that the infinitesimal generator $\mathcal{L}_\delta V(x(t))$ involve the control term u in a square form.
By Theorem 2, the equilibrium position $x \equiv 0$ of the closed-loop system (24) to which we apply the control law of the form

$$u(x(t),x(t-\tau)) = -\tilde{\Theta}(x(t),x(t-\tau))\, L_{1,\delta}V(x(t))$$
$$= -2\tilde{\Theta}(x(t),x(t-\tau))(\, x^{\mu+1}(t) + x(t)x^2(t-\tau)). \tag{26}$$

is almost surely asymptotically stable if the set $\mathcal{M} = \{0\}$.
The function $\tilde{\Theta}$ is chosen so that the condition

$$\tilde{\Theta}(x(t),x(t-\tau)) \leq \frac{1}{x^4(t) + 4(x^{\mu+1}(t) + x(t)x^2(t-\tau))^2 + 2} \tag{27}$$

is satisfied.
If we determinate

$$\mathcal{M} = \mathrm{Ker}(\alpha - \alpha_d) \cap Ker(L_{1,\delta}V)$$

we can remark that $\mathrm{Ker}(\alpha - \alpha_d) = \mathbb{R}^n$ and that $Ker(L_{1,\delta}V)$ depend on the value of μ. Indeed, if $x(t) \in \mathrm{Ker}(L_{1,\delta}V)$ then

$$x(t) = 0 \quad \text{or} \quad x^\mu(t) = -x^2(t-\tau).$$

If μ is odd then we cannot deduce directly the attractivity of the origin.
If $\mu \geq 2$ and μ is even, then $\mathcal{M} = \{0\}$ and we can deduce that the closed-loop system (24)(26) is almost surely asymptotically stable.

IV. CONCLUSIONS

In this paper, we have considered the problem of feedback stabilization of a class of nonlinear stochastic time-delayed systems. We considered the case where the systems have discrete time delay. We have used the Stochastic version of the Invariance Principle of LaSalle for stochastic differential delayed systems in order to tackle this problem. In addition, we have obtained sufficient conditions for guaranteeing the asymptotic stability of the closed-loop system and derived stabilizing state feedback control laws. The case of systems with multiple delays is currently under investigation.

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