2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Robust stability criteria for uncertain systems with delay and its derivative varying within intervals

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Abstract— **In this paper, stability criteria are proposed for linear systems liable to model uncertainties and with the delay and its derivative varying within intervals. The results are an improvement over previous ones due to the development of a new Lyapunov-Krasovskii functional (LKF). The analysis incorporates recent advances such as convex optimization technique and piecewise analysis method with new delay-interval-depedent LKFs terms and a novel auxiliary delayed state. Stability conditions are provided for the cases when the delay derivative is upper and lower bounded, when the lower bound is unknown, and when no restrictions are cast upon the derivative. The analysis is enriched with numerical examples that illustrate the effectiveness of our criteria which outperform previous criteria in the literature for nominal and uncertain delayed systems.**

I. INTRODUCTION

THE phenomena of time delays are often encountered in various practical systems, such as chemical engineering systems, biological systems, aircraft stabilization, networked control systems, etc [1]. Nonetheless, since time delays can degrade a system's performance and even cause system instability, considerable attention has been devoted to the subject of stability analysis and design of systems with time-varying delays (see, e.g., [1]–[9]).

During the last decade, the problem of time-delayed systems' stability analysis have been deeply investigated under delay-dependent criteria, for the exposure of the delay information leads to less conservative results. Various methods have been taken for deriving stability conditions using different Lyapunov–Krasovskii functionals (LKFs) [6]. Particularly, the employment of Jensen's inequality instead of the cross-terms bounding [10] is a well-established approach that leads to less conservative results. However, this still is a conservative analysis, for the time-varying delay is bounded when considering terms in the LKF derivative containing not only the delay bounds, but also the delay itself. Instead of bounding the time-varying delay, the convex optimization technique incorporated with the Jensen's inequality proved to be effective in [3]. Further impovements were obtained using similar technique with different LKFs (see, e.g., [5]–[9]). Recently, new Lyapunov functional candidates inspired on [2] have enriched the stability analysis by extending the piecewise analysis method from [2] to systems with time-varying delays, see, e.g., [7]–[9]. Particularly, [7], [9] also explore the information about the delay derivative's lower bound through the employment of delay-interval-dependent terms in the Lyapunov functional.

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Nevertheless, in practice, it is very difficult to obtain an exact mathematical model due to environmental noise or slowly varying parameters. Systems with time-varying delays almost inevitably present some uncertainties. However, most recent advances in the analysis of systems with time-varying delays aren't fully exploited in most recent works concerning robust stability of delayed systems (see, e.g., [11]–[17]). Therefore, the results from these works are usually more conservative than the results from criteria for delayed systems which do not consider the possibility of model uncertainties (see, e.g., [7], [9]).

Therefore, in this paper, we present a novel robust stability analysis for uncertain systems with delay and its derivative varying within intervals. New delay-interval-dependent LKF terms, that are ignored in previous works, are introduced to exploit all possible information about the delay derivative's lower and upper bounds. Moreover, we introduce an auxiliary delayed state in order to make further use of the delay's lower bound value. These methods considerably improve the stability analysis even for systems with no uncertainties. The resulting criteria can be applied for the case when the delay derivative is upper and lower bounded, when the lower bound is unknown, and when no restrictions are cast upon the derivative characteristics. Numerical examples illustrate the effectiveness of the proposed robust stability criteria which outperform previous criteria in the literature for time-delayed systems with and without uncertainties.

II. PRELIMINARIES

Consider the following continuous-time linear system with time-varying delay:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-d(t)), \quad && t>0 \\ x(t) &= \rho(t), && t \in [-\tau_{max},\ 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^{r_x}$ is the system's state, $\rho(t)$ is a given function which describes the state's initial condition, and the matrices A and A_d are considered not exactly known, but belonging to bounded sets: $A \in \mathscr{A} \subset \mathbb{R}^{r_x \times r_x}$ and $A_d \in \mathscr{A}_d \subset \mathbb{R}^{r_x \times r_x}$. The continuous function $d(t)$ denotes the time-varying delay that satisfies

$$\tau_{min} \leq d(t) \leq \tau_{max}, \tag{2}$$

where $0 \leq \tau_{min} \leq \tau_{max}$ are constants.

The time-varying delay is assumed to be either fast varying (with no restrictions cast upon the delay derivative) or differentiable with given bounds:

$$d_{min} \leq \dot{d}(t) \leq d_{max}, \tag{3}$$

where $d_{min} \leq d_{max}$ are constants.

Considering the parameter uncertainties, equation (1) can be rewritten as:

$$\dot{x}(t) = (A + \Delta A)x(t) + (A_d + \Delta A_d)x(t - d(t)) \tag{4}$$

The uncertainties ΔA e ΔA_d are time-varying matrices with appropriate dimensions, which are defined as follows:

$$\begin{bmatrix}\Delta A & \Delta A_d\end{bmatrix} = DF(t)\begin{bmatrix}E_A & E_{Ad}\end{bmatrix} \tag{5}$$

where D, E_A, and E_{Ad} are known real constant matrices with appropriate dimensions and $F(t)$ represents an unknown time-varying matrix, which is Lebesque measurable in t and satisfies $F(t)^T F(t) \leq I$.

Throughout this paper, the following results will be useful to derive conditions for the establishment of new delay-depedent stability criteria for the uncertain system with time-varying delay (4).

Lemma 1 ([18]) *For given scalars r_1, r_2 and matrix $M \in \mathbb{R}^{m\times m}$ such that $(r_2 - r_1) \geq 0$ and $M > 0$, and any vectorial function $x : [r_1, r_2] \longrightarrow \mathbb{R}^m$, we have:*

$$(r_2 - r_1)\int_{r_1}^{r_2} x^T(s)Mx(s)ds \geq \left(\int_{r_1}^{r_2} x(s)ds\right)^T M\left(\int_{r_1}^{r_2} x(s)ds\right).$$

Lemma 2 ([19]) *Given matrices $M = M^T \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{r\times m}$, the following statement*

$$x^T Mx > 0 \;\Leftrightarrow M + FB + B^T F^T > 0,$$

holds for some $F \in \mathbb{R}^{m\times r}$ and any $x \in \mathbb{R}^m \backslash \{0\}$ such that $Bx = 0$.

III. STABILITY ANALYSIS

This section presents the main results of this paper. Firstly, we shall, similarly to [7]–[9], divide the delay range $[\tau_{min}, \tau_{max}]$. Here we will consider two equally spaced subintervals: $[\tau_1, \tau_2]$ and $[\tau_2, \tau_3]$, where $\tau_1 = \tau_{min}$, $\tau_3 = \tau_{max}$, and $\tau_2 = \dfrac{\tau_{max} + \tau_{min}}{2}$. Therefore, the linear uncertain delayed system (4) can be rewritten as

$$\begin{aligned}\dot{x}(t) = {} & (A + \Delta A)x(t) + \chi_{[\tau_1,\tau_2]}(d(t))(A_d + \Delta A_d)x(t - d(t)) \\ & + \left(1 - \chi_{[\tau_1,\tau_2]}(d(t))\right)(A_d + \Delta A_d)x(t - d(t))\end{aligned} \tag{6}$$

where $\chi_{[\tau_1,\tau_2]}: \mathbb{R} \to \{0, 1\}$ is the characteristic function of $[\tau_1, \tau_2]$:

$$\chi_{[\tau_1,\tau_2]}(s) = \begin{cases} 1, & \text{if } s \in [\tau_1, \tau_2] \\ 0, & \text{otherwise.}\end{cases}$$

The proposed stability analysis for systems with time-varying delay and model uncertainties is based on the Lyapunov–Krasovskii functional candidate

$$V(t) = \sum\nolimits_{i=1}^{6} V_i(t), \tag{7}$$

where

$$\begin{aligned}V_1(t) = {} & \chi_{[\tau_1,\tau_2]}(d(t))x^T(t)\left[\frac{d(t) - \tau_1}{\tau_2 - \tau_1}P_1 + \frac{\tau_2 - d(t)}{\tau_2 - \tau_1}P_2\right]x(t) \\ & + \left(1 - \chi_{[\tau_1,\tau_2]}(d(t))\right)x^T(t)\left[\frac{d(t) - \tau_2}{\tau_3 - \tau_2}P_3 + \frac{\tau_3 - d(t)}{\tau_3 - \tau_2}P_1\right]x(t),\end{aligned}$$

$$V_2(t) = \int_{t-d(t)}^{t-\tau_1} x^T(s)Q_1x(s)ds$$

$$V_3(t) = \int_{t-\tau_2}^{t-\tau_1}\begin{bmatrix}x(s)\\ x(s - \tau_2 + \tau_1)\end{bmatrix}^T\begin{bmatrix}N_{11} & N_{12}\\ N_{12}^T & N_{22}\end{bmatrix}\begin{bmatrix}x(s)\\ x(s - \tau_2 + \tau_1)\end{bmatrix}ds,$$

$$V_4(t) = \int_{t-\frac{1}{2}\tau_1}^{t}\begin{bmatrix}x(s)\\ x(s - \frac{\tau_1}{2})\end{bmatrix}^T\begin{bmatrix}M_{11} & M_{12}\\ M_{12}^T & M_{22}\end{bmatrix}\begin{bmatrix}x(s)\\ x(s - \frac{\tau_1}{2})\end{bmatrix}ds,$$

$$\begin{aligned}V_5(t) = {} & \frac{\tau_1}{2}\int_{-\frac{1}{2}\tau_1}^{0}\int_{t+\beta}^{t}\dot{x}^T(s)Z_1\dot{x}(s)dsd\beta + \frac{\tau_1}{2}\int_{-\tau_1}^{-\frac{1}{2}\tau_1}\int_{t+\beta}^{t}\dot{x}^T(s) \\ & \times Z_2\dot{x}(s)dsd\beta + (\tau_2 - \tau_1)\int_{-\tau_2}^{-\tau_1}\int_{t+\beta}^{t}\dot{x}^T(s)Z_3\dot{x}(s)dsd\beta \\ & + (\tau_3 - \tau_2)\int_{-\tau_3}^{-\tau_2}\int_{t+\beta}^{t}\dot{x}^T(s)Z_4\dot{x}(s)dsd\beta,\end{aligned}$$

$$\begin{aligned}V_6(t) = {} & \chi_{[\tau_1,\tau_2]}(d(t))\left[\int_{-\tau_2}^{-d(t)}\int_{t+\beta}^{t}\dot{x}^T(s)(R_1 - R_3)\dot{x}(s)dsd\beta\right] \\ & + \left(1 - \chi_{[\tau_1,\tau_2]}(d(t))\right)\left[\int_{-d(t)}^{-\tau_2}\int_{t+\beta}^{t}\dot{x}^T(s)(R_3 - R_1)\dot{x}(s)dsd\beta\right] \\ & + \int_{-d(t)}^{0}\int_{t+\beta}^{t}\dot{x}^T(s)(R_1 + R_2)\dot{x}(s)dsd\beta \\ & + \int_{-\tau_3}^{-d(t)}\int_{t+\beta}^{t}\dot{x}^T(s)(R_3 + R_4)\dot{x}(s)dsd\beta.\end{aligned}$$

One can note that the if the conditions

$$\begin{aligned}& P_1 = \frac{P_3 + P_2}{2},\quad P_2 > 0,\quad P_3 > 0,\quad Q_1 \geq 0,\quad Z_j > 0,\ j \in \{1,2,3,4\}, \\ & ((\tau_2 - \tau_1)Z_3 + R_1 - R_3) > 0,\ ((\tau_3 - \tau_2)Z_4 + R_3 - R_1) > 0,\ (R_1 + R_2) > 0, \\ & (R_3 + R_4) > 0,\ N = \begin{bmatrix}N_{11} & N_{12}\\ N_{12}^T & N_{22}\end{bmatrix} \geq 0, \text{ and } M = \begin{bmatrix}M_{11} & M_{12}\\ M_{12}^T & M_{22}\end{bmatrix} \geq 0,\end{aligned} \tag{8}$$

are satisfied, we guarantee the positiveness of (7). Also, it can be seen that $V(t)$ in (7) is continuous in t, since

$$\lim_{d(t)\to\tau_2} V_1(t) = x^T(t)P_1x(t),$$

$$\begin{aligned}\lim_{d(t)\to\tau_2} V_6(t) = {} & \int_{-\tau_2}^{0}\int_{t+\beta}^{t}\dot{x}^T(s)(R_1 + R_2)\dot{x}(t)dsd\beta \\ & + \int_{-\tau_3}^{-\tau_2}\int_{t+\beta}^{t}\dot{x}^T(s)(R_3 + R_4)\dot{x}(t)dsd\beta.\end{aligned}$$

In the following, we propose novel robust stability criteria for linear systems with model uncertainties and delay and its derivative varying within intervals.

Theorem 1 *For given scalars τ_{min}, τ_{max}, d_{min}, and d_{max} such that $0 < \tau_{min} \leq \tau_{max}$ and $d_{min} < d_{max}$, the system (4) with time-varying delay $d(t)$ satisfying (2)-(3), and uncertainties described by (5) is robust asymptotically stable if there exist scalars $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, and matrices P_i, $i \in \{1,2,3\}$, Q_1, Z_j, R_j, $j \in \{1,2,3,4\}$, N, and M, with appropriate dimensions, satisfying (8) and*

$$Z_1 + U_1|_{\dot{d}(t)\to d_{max}} > 0,\qquad Z_2 + U_1|_{\dot{d}(t)\to d_{max}} > 0, \tag{9}$$

$$Z_1 + U_1|_{\dot{d}(t)\to d_{min}} > 0,\qquad Z_2 + U_1|_{\dot{d}(t)\to d_{min}} > 0, \tag{10}$$

and free-weighting matrices $H_1 \in \mathbb{R}^{7r_x\times 3r_x}$ and $H_2 \in \mathbb{R}^{7r_x\times 3r_x}$, such that the following LMIs hold:

$$\begin{aligned}&\Omega_{11}|_{\dot{d}(t)\to d_{min}} < 0; && \Omega_{11}|_{\dot{d}(t)\to d_{max}} < 0; \\ &\Omega_{12}|_{\dot{d}(t)\to d_{min}} < 0; && \Omega_{12}|_{\dot{d}(t)\to d_{max}} < 0; \\ &\Omega_{21}|_{\dot{d}(t)\to d_{min}} < 0; && \Omega_{21}|_{\dot{d}(t)\to d_{max}} < 0; \\ &\Omega_{22}|_{\dot{d}(t)\to d_{min}} < 0; && \Omega_{22}|_{\dot{d}(t)\to d_{max}} < 0,\end{aligned} \tag{11}$$

where

$$\Omega_{1k}=\begin{bmatrix}\left(\Psi^{(1)}|_{d(t)\to\tau_k}+H_1B_1+(H_1B_1)^T\right) & (\tau_2-\tau_1)H_1\Gamma_k & H_1\Gamma_3D & \varepsilon_1T_E^T\\ * & -(\tau_2-\tau_1)\Lambda_{1k} & 0 & 0\\ * & * & -\varepsilon_1I & 0\\ * & * & * & -\varepsilon_1I\end{bmatrix}$$

$$\Omega_{2k}=\begin{bmatrix}\left(\Psi^{(2)}|_{d(t)\to\tau_{(k+1)}}+H_2B_2+(H_2B_2)^T\right) & (\tau_3-\tau_2)H_2\Gamma_k & H_2\Gamma_3D & \varepsilon_2T_E^T\\ * & -(\tau_3-\tau_2)\Lambda_{2k} & 0 & 0\\ * & * & -\varepsilon_2I & 0\\ * & * & * & -\varepsilon_2I\end{bmatrix}$$

with $k\in\{1,2\}$*, and*

$$\Gamma_1=\begin{bmatrix}0 & I & 0\end{bmatrix}^T,\qquad \Gamma_2=\begin{bmatrix}I & 0 & 0\end{bmatrix}^T,\qquad \Gamma_3=\begin{bmatrix}0 & 0 & I\end{bmatrix}^T,$$

$$\Lambda_{11}=((\tau_2-\tau_1)Z_3+R_1+R_4),\quad \Lambda_{12}=\left((\tau_2-\tau_1)Z_3+R_1+\left(1-\dot{d}(t)\right)R_2+\dot{d}(t)R_4\right),$$

$$\Lambda_{21}=((\tau_3-\tau_2)Z_4+R_3+R_4),\quad \Lambda_{22}=\left((\tau_3-\tau_2)Z_4+R_3+\left(1-\dot{d}(t)\right)R_2+\dot{d}(t)R_4\right),$$

$$B_1=\begin{bmatrix}0 & I & 0 & 0 & -I & 0 & 0\\ 0 & -I & 0 & 0 & 0 & I & 0\\ A & A_d & -I & 0 & 0 & 0 & 0\end{bmatrix},\quad B_2=\begin{bmatrix}0 & I & 0 & 0 & 0 & -I & 0\\ 0 & -I & 0 & 0 & 0 & 0 & I\\ A & A_d & -I & 0 & 0 & 0 & 0\end{bmatrix},$$

$$T_E=\begin{bmatrix}E_A & E_{A_d} & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

$$\Psi^{(1)}=\begin{bmatrix}\Psi_{11} & 0 & \frac{d(t)-\tau_1}{\tau_2-\tau_1}P_1+\frac{\tau_2-d(t)}{\tau_2-\tau_1}P_2 & \Psi_{14} & 0 & 0 & 0\\ * & \Psi_{22} & 0 & 0 & 0 & 0 & 0\\ * & * & \Psi_{33}^{(1)}(d(t))+\Psi_{33} & 0 & 0 & 0 & 0\\ * & * & * & \Psi_{44} & \Psi_{45} & 0 & 0\\ * & * & * & 0 & \Psi_{55} & N_{12} & 0\\ * & * & * & 0 & * & N_{22}-N_{11}-U_2 & U_2-N_{12}\\ * & * & * & 0 & * & * & -U_2-N_{12}\end{bmatrix},$$

$$\Psi^{(2)}=\begin{bmatrix}\Psi_{11} & 0 & \frac{d(t)-\tau_2}{\tau_3-\tau_2}P_3+\frac{\tau_3-d(t)}{\tau_3-\tau_2}P_1 & \Psi_{14} & 0 & 0 & 0\\ * & \Psi_{22} & 0 & 0 & 0 & 0 & 0\\ * & * & \Psi_{33}^{(2)}(d(t))+\Psi_{33} & 0 & 0 & 0 & 0\\ * & * & * & \Psi_{44} & \Psi_{45} & 0 & 0\\ * & * & * & * & \Psi_{55}-U_3 & N_{12}+U_3 & 0\\ * & * & * & * & * & N_{22}-N_{11}-U_3 & -N_{12}\\ * & * & * & * & * & * & -N_{22}\end{bmatrix},$$

with

$$U_1=\frac{2}{\tau_1}\left(R_1+\left(1-\dot{d}(t)\right)R_2+\dot{d}(t)R_4\right),$$

$$U_2=\frac{1}{\tau_3-\tau_2}((\tau_3-\tau_2)Z_4+R_3+R_4),$$

$$U_3=\frac{1}{\tau_2-\tau_1}\left((\tau_2-\tau_1)Z_3+R_1+\left(1-\dot{d}(t)\right)R_2+\dot{d}(t)R_4\right),$$

$$\Psi_{11}=\frac{\dot{d}(t)}{\tau_2-\tau_1}(P_1-P_2)+M_{11}-Z_1-U_1,$$

$$\Psi_{22}=-\left(1-\dot{d}(t)\right)Q_1,$$

$$\Psi_{33}=\left(\frac{\tau_1}{2}\right)^2(Z_1+Z_2)+(\tau_2-\tau_1)^2Z_3+(\tau_3-\tau_2)^2Z_4+\tau_2R_1+(\tau_3-\tau_2)R_3,$$

$$\Psi_{33}^{(1)}(d(t))=(\tau_3-\tau_2)R_4+(\tau_2-d(t))R_4+\tau_2\frac{(d(t)-\tau_1)}{\tau_2-\tau_1}R_2+\tau_1\frac{(\tau_2-d(t))}{\tau_2-\tau_1}R_2,$$

$$\Psi_{33}^{(2)}(d(t))=(\tau_3-d(t))R_4+\tau_3\frac{(d(t)-\tau_2)}{\tau_3-\tau_2}R_2+\tau_2\frac{(\tau_3-d(t))}{\tau_3-\tau_2}R_2,$$

$$\Psi_{44}=M_{22}-M_{11}-Z_1-Z_2-2U_1,$$

$$\Psi_{55}=Q_1+N_{11}-M_{22}-Z_2-U_1,$$

$$\Psi_{14}=Z_1+M_{12}+U_1,$$

$$\Psi_{45}=Z_2-M_{12}+U_1. \tag{12}$$

■

It is also interesting to consider two special cases of the previous result. The case when the lower bound of the time-varying delay derivative is unknown and the case when no restrictions are cast upon delay derivative. For the first case, by fullfilling the restrictions

$$P_3>P_2 \text{ and } R_2>R_4, \tag{13}$$

the following corollary arises directly from Theorem 1.

Corollary 1 *For given scalars τ_{min}, τ_{max}, and d_{max} such that $0<\tau_{min}\le\tau_{max}$, the system (4) with the delay $d(t)$ satisfying (2) and $\dot{d}(t)\le d_{max}$, and uncertainties described by (5) is robust asymptotically stable if there exist scalars $\varepsilon_1>0$, $\varepsilon_2>0$, and matrices P_i, $i\in\{1,2,3\}$, Q_1, Z_j, R_j, $j\in\{1,2,3,4\}$, N, and M, with appropriate dimensions, satisfying (8), (9), and (13), and free-weighting matrices $H_1\in\mathbb{R}^{7r_x\times3r_x}$ and $H_2\in\mathbb{R}^{7r_x\times3r_x}$ such that the following LMIs, with notations given in (12), hold:*

$$\Omega_{11}|_{\dot{d}(t)\to d_{max}}<0;\qquad \Omega_{12}|_{\dot{d}(t)\to d_{max}}<0;$$
$$\Omega_{21}|_{\dot{d}(t)\to d_{max}}<0;\qquad \Omega_{22}|_{\dot{d}(t)\to d_{max}}<0.$$

□

We shall now consider the second case, i.e. fast-varying delays. In this case, as we have no information about the delay derivative, by assuming

$$P_1=P_2=P_3,\quad Q_1=0,\quad \text{and}\quad R_2=R_4, \tag{14}$$

we can eliminate the terms with $\dot{d}(t)$ from (12). Then it is straightforward to obtain the following corollary.

Corollary 2 *For given scalars τ_{min} and τ_{max} such that $0<\tau_{min}\le\tau_{max}$, the system (4) with time-varying delay $d(t)$ satisfying (2), and uncertainties described by (5) is robust asymptotically stable if there exist scalars $\varepsilon_1>0$ and $\varepsilon_2>0$, and matrices P_i, $i\in\{1,2,3\}$, Q_1, Z_j, R_j, $j\in\{1,2,3,4\}$, N, and M, with appropriate dimensions, satisfying (8) and (14), and free-weighting matrices $H_1\in\mathbb{R}^{7r_x\times3r_x}$ and $H_2\in\mathbb{R}^{7r_x\times3r_x}$ such that the following LMIs, with notations given in (12), hold:*

$$\Omega_{11}<0;\qquad \Omega_{12}<0;\qquad \Omega_{21}<0;\qquad \Omega_{22}<0.$$

□

Remark 1 Because of the term U_1, the results are valid only for minimum delay strictly greater than zero. However it is straightforward to extend these results to the case where $\tau_{min}=0$ by considering $U_1=0$.

Theorem 1, Corollaries 1 and 2 provide stability conditions for linear systems liable to model uncertainties and time-varying delays, and are the main results of the paper. Compared with previous criteria, the conservativeness of the stability analysis is considerably reduced. To improve the results, we have introduced a new auxiliary delayed state $x\left(t-\frac{\tau_1}{2}\right)$ in the Lyapunov functional, which allows further exploitation of the delay's lower bound value. Moreover, further improvements were obtained through the introduction of new delay-interval-dependent terms in (7). The employment of these terms yields different expression in the derivative of the Lyapunov functional when $d(t)<\tau_2$ and when $\tau_2<d(t)$.

The examples in the next section illustrate the effectiveness of our criteria. It is important to emphasize that the stability results are less conservative than previous published criteria not only for systems with uncertainties, but also for nominal time-delayed systems.

IV. NUMERICAL EXAMPLES

Example 1 Consider the system (4) with no uncertainties and

$$A=\begin{bmatrix}-2 & 0\\ 0 & -0.9\end{bmatrix},\quad A_d=\begin{bmatrix}-1 & 0\\ -1 & -1\end{bmatrix},\quad \Delta A=0,\ \Delta A_d=0.$$

TABLE I

ALLOWABLE τ_{max} VALUE FOR d_{max}=0.1 AND d_{min}=−0.1 (EX. 1)

Method $\setminus \tau_{min}$		0	1	2	3	4	5
He et al. [4]		3.605	–	–	3.612	4.064	–
Sun et al. [20]		3.918	–	–	3.918	4.178	5.038
Fridman et al. [7]	thm 1	4.260	4.571	4.622	4.216	4.090	–
	thm 2	3.663	4.203	4.456	4.425	4.429	5.097
Theorem 1		**4.363**	**4.604**	**4.711**	**4.698**	**4.577**	**5.098**

TABLE II

ADMISSIBLE τ_{max} VALUE FOR τ_{min}=1 AND GIVEN d_{min} AND d_{max} (EX. 2)

Method		unknown d_{min}, (d_{max}=0.3)	(unkn d_{max})	$d_{min}=-0.1$, (d_{max}=0.3)	(d_{max}=1)
He et al. [4]		2.2125	1.5187	–	–
Shao [6]		2.247	1.617	–	–
Orihuela et al. [8]		2.353	1.792	–	–
Fridman et al. [7]	Thm 2	2.41	1.76	2.57	1.77
	Thm 1	2.42	1.79	2.60	1.85
Theorem 1		**2.454**	**1.797**	**2.770**	**1.895**

Assuming slow-varying delays ($-0.1 \leq \dot{d}(t) \leq 0.1$), the maximum values of τ_{max} which maintain the system's asymptotical stability for various τ_{min} are listed in Table I. It is clear that the obtained results are less conservative than those in [4], [7], [20]

Example 2 Consider the following delayed system described by

$$A=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix},\ A_d=\begin{bmatrix}0 & 0\\ -1 & 1\end{bmatrix},\quad \Delta A=0,\ \ \Delta A_d=0.$$

For τ_{min}=1, and various d_{min} and d_{max}, the results from various criteria in the literature are listed in Table II. For unknown d_{min} and for fast-varying delays the results are obtained using Corollaries 1 and 2, respectively. From the table, it can be seen that our criteria present superior results when compared to previous methods. Moreover, one can note that τ_{max} grows for $d_{min}\to 0$ and for $d_{max}\to 0$.

Example 3 Consider now the uncertain system (4) with

$$A=\begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix},\quad A_d=\begin{bmatrix}-1 & 0\\ -1 & -1\end{bmatrix},$$
$$D=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad E_A=\begin{bmatrix}1.6 & 0\\ 0 & 0.05\end{bmatrix},\quad E_{A_d}=\begin{bmatrix}0.1 & 0\\ 0 & 0.3\end{bmatrix}.$$

From Corollary 2, we find that the uncertain delayed system is stable for τ_{min}=0 and various values for d_{max} with admissible τ_{max} given in Table III. The obtained result represents an important improvement over those from previous robust criteria.

Example 4 Consider the following uncertain system (4) with

$$A=\begin{bmatrix}-0.5 & -2\\ 1 & -1\end{bmatrix},\ A_d=\begin{bmatrix}-0.5 & -1\\ 0 & 0.6\end{bmatrix},\ D=I,\ \ E_A=E_{A_d}=\begin{bmatrix}0.2 & 0\\ 0 & 0.2\end{bmatrix}.$$

In Table IV, we compare the results from Corollaries 1 and 2 with those in [11], [12], [15], [17] for τ_{min}=0, unknown d_{min}, and various d_{max}. From the table, it is clear

TABLE III

ALLOWABLE UPPER BOUND VALUE OF τ_{max} FOR τ_{min}=0, UNKNOWN d_{min} AND VARIOUS d_{max} (EX. 3)

Methods $\setminus d_{max}$	0.2	0.4	0.6	0.8
Wu et al. [11]	1.063	0.973	0.873	0.760
Lien [13]	1.063	0.973	0.873	0.760
Yue & Han [21]	1.063	0.973	0.873	0.760
Qian et al. [17]	1.083	1.023	0.986	0.964
Park & Ko [3]	1.099	1.077	1.070	1.068
Corollary 1	**1.219**	**1.104**	**1.089**	**1.089**

TABLE IV

MAX. τ_{max} VALUE FOR τ_{min}=0 AND UNKNOWN d_{min} (EX. 4)

Methods $\setminus d_{max}$	0,5	0,9	Unknown
Wu et al. [11]	0.243	0.242	0.242
Jing et al. [12]	0.243	0.242	0.242
He et al. [15]	0.342	0.338	0.336
Qian et al. [17]	0.379	0.379	0.379
Corollary 1	**0.4471**	**0.4461**	–
Corollary 2	–	–	**0.4461**

that our results are considerably less conservative than those in previous criteria in the literature.

V. CONCLUSIONS

This work's main result concern the establishment of new stability criteria for time-delayed systems liable to model uncertainties and with delay and its derivative varying within bounded intervals. The case when the derivative's lower bound is unknown is also considered, as the case when no restrictions are cast upon the delay derivative. The conservativeness of the stability analysis is considerably reduced with the introduction of new delay-interval-dependent terms and a new auxiliary delayed state in the LKF. Although this paper deals mainly with uncertain delayed systems, our criteria, when applied to nominal systems, also yields less conservative results than previous criteria in the literature. These analyses are ratified with numerical examples that illustrate the effectiveness of the proposed criteria.

APPENDIX
PROOF OF THEOREM 1

Firstly, we shall consider the case where $d(t) < \tau_2$. Taking the time derivative of the Lyapunov functional candidate (7) with $\chi = 1$ yields

$$\dot{V}_1(t)|_{d(t)<\tau_2}=\dot{d}(t)x^T(t)\frac{P_1-P_2}{\tau_2-\tau_1}x(t)+2\dot{x}^T(t)\left[\frac{d(t)-\tau_1}{\tau_2-\tau_1}P_1+\frac{\tau_2-d(t)}{\tau_2-\tau_1}P_2\right]x(t)$$

$$\dot{V}_2(t)=x^T(t-\tau_1)Q_1x(t-\tau_1)-\left(1-\dot{d}(t)\right)x^T(t-d(t))Q_1x(t-d(t)),$$

$$\dot{V}_3(t)=\begin{bmatrix}x(t-\tau_1)\\ x(t-\tau_2)\end{bmatrix}^T\begin{bmatrix}N_{11} & N_{12}\\ N_{12}^T & N_{22}\end{bmatrix}\begin{bmatrix}x(t-\tau_1)\\ x(t-\tau_2)\end{bmatrix}-\begin{bmatrix}x(t-\tau_2)\\ x(t-\tau_3)\end{bmatrix}^T\begin{bmatrix}N_{11} & N_{12}\\ N_{12}^T & N_{22}\end{bmatrix}\begin{bmatrix}x(t-\tau_2)\\ x(t-\tau_3)\end{bmatrix},$$

$$\dot{V}_4(t)=\begin{bmatrix}x(t)\\ x(t-\frac{\tau_1}{2})\end{bmatrix}^T\begin{bmatrix}M_{11} & M_{12}\\ M_{12}^T & M_{22}\end{bmatrix}\begin{bmatrix}x(t)\\ x(t-\frac{\tau_1}{2})\end{bmatrix}-\begin{bmatrix}x(t-\frac{\tau_1}{2})\\ x(t-\tau_1)\end{bmatrix}^T\begin{bmatrix}M_{11} & M_{12}\\ M_{12}^T & M_{22}\end{bmatrix}\begin{bmatrix}x(t-\frac{\tau_1}{2})\\ x(t-\tau_1)\end{bmatrix},$$

$$\dot{V}_5(t)=\dot{x}^T(t)\left[\left(\frac{\tau_1}{2}\right)^2(Z_1+Z_2)+(\tau_2-\tau_1)^2Z_3+(\tau_3-\tau_2)^2Z_4\right]\dot{x}(t)-\frac{\tau_1}{2}$$
$$\times\int_{t-\frac{1}{2}\tau_1}^{t}\dot{x}^T(s)Z_1\dot{x}(s)ds-\frac{\tau_1}{2}\int_{t-\tau_1}^{t-\frac{1}{2}\tau_1}\dot{x}^T(s)Z_2\dot{x}(s)ds-(\tau_2-\tau_1)$$
$$\times\int_{t-\tau_2}^{t-\tau_1}\dot{x}^T(s)Z_3\dot{x}(s)ds-(\tau_3-\tau_2)\int_{t-\tau_3}^{t-\tau_2}\dot{x}^T(s)Z_4\dot{x}(s)ds,$$

$$\dot{V}_6(t)|_{d(t)<\tau_2}=\dot{x}^T(t)[(\tau_2-d(t))(R_1-R_3)+d(t)(R_1+R_2)+(\tau_3-d(t))(R_3+R_4)]\dot{x}(t)$$

$$-\int_{t-d(t)}^{t}\dot{x}^T(s)\left(R_1+(1-\dot{d}(t))R_2+\dot{d}(t)R_4\right)\dot{x}(s)ds$$
$$-\int_{t-\tau_2}^{t-d(t)}\dot{x}^T(s)(R_1-R_3)\dot{x}(s)ds-\int_{t-\tau_3}^{t-d(t)}\dot{x}^T(s)(R_3+R_4)\dot{x}(s)ds. \qquad (15)$$

Suppose now we take $\dot{V}_5(t)$ and $\dot{V}_6(t)|_{d(t)<\tau_2}$ in (15) and expand the integral terms using the fact that $\tau_1 \le d(t) \le \tau_2$. Then, defining

$$\gamma_{1d}:=\frac{1}{d(t)-\tau_1}\int_{t-d(t)}^{t-\tau_1}\dot{x}(s)ds \quad \text{and} \quad \gamma_{d2}:=\frac{1}{\tau_2-d(t)}\int_{t-\tau_2}^{t-d(t)}\dot{x}(s)ds,$$

where $\lim_{d(t)\to\tau_1}\gamma_{1d}=\dot{x}(t-\tau_1)$, and $\lim_{d(t)\to\tau_2}\gamma_{d2}=\dot{x}(t-\tau_2)$, and applying Jensen's inequality (Lemma 1), we have the following inequalities

$$\dot{V}_5(t)+\dot{V}_6(t)|_{d(t)<\tau_2} \le \dot{x}^T(t)\Big[\Big(\frac{\tau_1}{2}\Big)^2(Z_1+Z_2)+(\tau_2-\tau_1)^2Z_3+(\tau_3-\tau_2)^2Z_4+\tau_2R_1$$
$$+(\tau_3-\tau_2)R_3+(\tau_3-\tau_2)R_4+(\tau_2-d(t))R_4+\tau_2\frac{d(t)-\tau_1}{\tau_2-\tau_1}R_2+\tau_1\frac{\tau_2-d(t)}{\tau_2-\tau_1}R_2\Big]\dot{x}(t)$$
$$-\Big[x(t)-x\Big(t-\frac{\tau_1}{2}\Big)\Big]^T\Big(Z_1+\frac{2}{\tau_1}\left(R_1+(1-\dot{d}(t))R_2+\dot{d}(t)R_4\right)\Big)\Big[x(t)-x\Big(t-\frac{\tau_1}{2}\Big)\Big]$$
$$-\Big[x\Big(t-\frac{\tau_1}{2}\Big)-x(t-\tau_1)\Big]^T\Big(Z_2+\frac{2}{\tau_1}\left(R_1+(1-\dot{d}(t))R_2+\dot{d}(t)R_4\right)\Big)\Big[x\Big(t-\frac{\tau_1}{2}\Big)-x(t-\tau_1)\Big]$$
$$-[x(t-\tau_2)-x(t-\tau_3)]^T\Big(Z_4+\frac{1}{\tau_3-\tau_2}(R_3+R_4)\Big)[x(t-\tau_2)-x(t-\tau_3)]$$
$$-\gamma_{1d}^T\left((d(t)-\tau_1)\left((\tau_2-\tau_1)Z_3+R_1+(1-\dot{d}(t))R_2+\dot{d}(t)R_4\right)\right)\gamma_{1d}$$
$$-\gamma_{d2}^T((\tau_2-d(t))((\tau_2-\tau_1)Z_3+R_1+R_4))\gamma_{d2}. \qquad (16)$$

Then, from (15) and (16), after some manipulation, one can conclude that

$$\dot{V}(t)|_{d(t)<\tau_2} \le \zeta_1^T(t)\left(\Omega|_{d(t)<\tau_2}\right)\zeta_1(t), \qquad (17)$$

where

$$\Omega|_{d(t)<\tau_2}=\begin{bmatrix}\Psi^{(1)} & 0\\ * & -\Lambda^{(1)}\end{bmatrix},\ \Lambda^{(1)}=\begin{bmatrix}(d(t)-\tau_1)\Lambda_{12} & 0\\ 0 & (\tau_2-d(t))\Lambda_{11}\end{bmatrix},$$

and $\Psi^{(1)}$, Λ_{11}, and Λ_{12} are defined in (12). Also, we have defined $\zeta_1^T(t):=\begin{bmatrix}\zeta_x^T & \gamma_{1d}^T & \gamma_{d2}^T\end{bmatrix}\in\mathbb{R}^{9r_x}$, where

$$\zeta_x^T(t):=\begin{bmatrix}x^T(t) & x^T(t-d(t)) & \dot{x}^T(t) & x^T(t-\frac{\tau_1}{2}) \\ x^T(t-\tau_1) & x^T(t-\tau_2) & x^T(t-\tau_3)\end{bmatrix}, \qquad (18)$$

Suppose now we introduce $\widetilde{B}_1=\begin{bmatrix}\widetilde{B}_{11} & \widetilde{B}_{12}\end{bmatrix}\in\mathbb{R}^{3r_x\times 9r_x}$ and $\widetilde{H}_1=\begin{bmatrix}H_1^T & 0\end{bmatrix}^T\in\mathbb{R}^{9r_x\times 3r_x}$, where H_1 is a $7r\times 3r$ free-weighting matrix, and

$$\widetilde{B}_{11}=\begin{bmatrix}0 & I & 0 & 0 & -I & 0 & 0\\ 0 & -I & 0 & 0 & 0 & I & 0\\ (A+\Delta A) & (A_d+\Delta A_d) & -I & 0 & 0 & 0 & 0\end{bmatrix},\ \widetilde{B}_{12}=\begin{bmatrix}(d(t)-\tau_1)I & 0\\ 0 & (\tau_2-d(t))I\\ 0 & 0\end{bmatrix}.$$

It is interesting to note that $\widetilde{B}_1\zeta_1(t)=0$. Then a straightforward consequence of applying Finsler's lemma (Lemma 2) is that the right side of (17) is negative definite if $\Xi_1<0$ holds, where

$$\Xi_1=\Omega|_{d(t)<\tau_2}+\widetilde{H}_1\widetilde{B}_1+\widetilde{B}_1^T\widetilde{H}_1^T=\begin{bmatrix}\left(\Psi^{(1)}+H_1\widetilde{B}_{11}+\widetilde{B}_{11}^TH_1^T\right) & H_1\widetilde{B}_{12}\\ * & -\Lambda^{(1)}\end{bmatrix}.$$

Here we shall consider the terms Ξ_{11} and Ξ_{12} that arise from Ξ_1 when $d(t)\to\tau_1$ and $d(t)\to\tau_2$, respectively

$$\Xi_{1k}=\begin{bmatrix}\left(\Psi^{(1)}|_{d(t)\to\tau_k}+H_1\widetilde{B}_{11}+\widetilde{B}_{11}^TH_1^T\right) & (\tau_2-\tau_1)H_1\Gamma_k\\ * & -(\tau_2-\tau_1)\Lambda_{1k}\end{bmatrix} \qquad (19)$$

where Γ_1 and Γ_2 are defined in (12), and $k\in\{1,2\}$.

Note that we have deleted the zero row and column from Ξ_{11} and Ξ_{12}. Now, it can be seen that

$$\zeta_1^T(t)\Xi_1\zeta_1(t)=\frac{\tau_2-d(t)}{\tau_2-\tau_1}\zeta_{11}^T(t)\Xi_{11}\zeta_{11}(t)+\frac{d(t)-\tau_1}{\tau_2-\tau_1}\zeta_{12}^T(t)\Xi_{12}\zeta_{12}(t),$$

where $\zeta_{11}^T(t):=\begin{bmatrix}\zeta_x^T & \gamma_{d2}^T\end{bmatrix}$, $\zeta_{12}^T(t):=\begin{bmatrix}\zeta_x^T & \gamma_{1d}^T\end{bmatrix}$, and ζ_x is defined in (18). Thus, $\zeta_1^T(t)\Xi_1\zeta_1(t)$ is convex in $d(t)$ and is negative definite only if the vertices (Ξ_{11} and Ξ_{12}) are.

Furthermore, to eliminate the time-varying matrix $F(t)$ from (19), we use the definition of ΔA and ΔA_d from (5) and rewrite $\widetilde{B}_{11}$ as

$$\widetilde{B}_{11}=B_1+\Gamma_3\begin{bmatrix}\Delta A & \Delta A_d & 0 & 0 & 0 & 0 & 0\end{bmatrix}=B_1+\Gamma_3DF(t)T_E, \qquad (20)$$

where B_1, Γ_3, and T_E are defined in (12). Then, according to (20), Ξ_{1k} in (19) is rewritten as

$$\Xi_{1k}=\begin{bmatrix}\left(\Psi^{(1)}|_{d(t)\to\tau_k}+H_1B_1+B_1^TH_1^T\right) & (\tau_2-\tau_1)H_1\Gamma_k\\ * & -(\tau_2-\tau_1)\Lambda_{1k}\end{bmatrix}+\alpha F(t)\beta+\beta^TF(t)^T\alpha^T,$$

where $\alpha=\begin{bmatrix}(H_1\Gamma_3D)^T & 0\end{bmatrix}^T$ and $\beta=\begin{bmatrix}T_E & 0\end{bmatrix}$.

Then it follows from applying Lemma 3 in [22] that $\Xi_{1k}<0$ holds if and only if there exists a scalar $\varepsilon_1>0$ such that

$$\begin{bmatrix}\left(\Psi^{(1)}|_{d(t)\to\tau_k}+H_1B_1+B_1^TH_1^T\right) & (\tau_2-\tau_1)H_1\Gamma_k\\ * & -(\tau_2-\tau_1)\Lambda_{1k}\end{bmatrix}+\frac{1}{\varepsilon_1}\alpha\alpha^T+\varepsilon_1\beta^T\beta<0$$

holds for $k\in\{1,2\}$. Moreover, taking the Schur's complement, we have Ω_{1k} as described in (12). Therefore, Ξ_1 is negative definite if and only if Ω_{11} and Ω_{12} are.

Furthermore, given (3), the following expressions hold

$$\Omega_{11}=\frac{d_{max}-\dot{d}(t)}{d_{max}-d_{min}}\Omega_{11}|_{\dot{d}(t)\to d_{min}}+\frac{\dot{d}(t)-d_{min}}{d_{max}-d_{min}}\Omega_{11}|_{\dot{d}(t)\to d_{max}},$$
$$\Omega_{12}=\frac{d_{max}-\dot{d}(t)}{d_{max}-d_{min}}\Omega_{12}|_{\dot{d}(t)\to d_{min}}+\frac{\dot{d}(t)-d_{min}}{d_{max}-d_{min}}\Omega_{12}|_{\dot{d}(t)\to d_{max}}.$$

Therefore, Ω_{11} and Ω_{12} are convex in $\dot{d}(t)\in[d_{min},d_{max}]$.

We will now consider the case where $\tau_2<d(t)\le\tau_3$. We shall prove that analogous results can be derived using exactly the same arguments of the former case. Taking the time derivative of the Lyapunov functional candidate (7) with $\chi=0$ yields

$$\dot{V}_1(t)|_{d(t)>\tau_2}=x^T(t)\dot{d}(t)\frac{P_3-P_1}{\tau_3-\tau_2}x(t)+2\dot{x}^T(t)\left[\frac{d(t)-\tau_2}{\tau_3-\tau_2}P_3+\frac{\tau_3-d(t)}{\tau_3-\tau_2}P_1\right]x(t),$$
$$\dot{V}_6(t)|_{d(t)>\tau_2}=\dot{x}^T(t)[(d(t)-\tau_2)(R_3-R_1)+d(t)(R_1+R_2)+(\tau_3-d(t))(R_3+R_4)]\dot{x}(t)$$
$$-\int_{t-d(t)}^{t}\dot{x}^T(s)\left(R_1+(1-\dot{d}(t))R_2+\dot{d}(t)R_4\right)\dot{x}(s)ds$$
$$-\int_{t-d(t)}^{t-\tau_2}\dot{x}^T(s)(R_3-R_1)\dot{x}(s)ds-\int_{t-\tau_3}^{t-d(t)}\dot{x}^T(s)(R_3+R_4)\dot{x}(s)ds. \qquad (21)$$

and $\dot{V}_2(t)$ to $\dot{V}_5(t)$ are defined in (15). Then, similarly to (16), we apply Jensen's inequality (Lemma 1) to $\dot{V}_5(t)$ and $\dot{V}_6(t)|_{d(t)>\tau_2}$:

$$\dot{V}_5(t)+\dot{V}_6(t)|_{d(t)<\tau_2}\le\dot{x}^T(t)\Big[\Big(\frac{\tau_1}{2}\Big)^2(Z_1+Z_2)+(\tau_2-\tau_1)^2Z_3+(\tau_3-\tau_2)^2Z_4$$
$$+\tau_2R_1+(\tau_3-\tau_2)R_3+(\tau_3-d(t))R_4+\tau_3\frac{(d(t)-\tau_2)}{\tau_3-\tau_2}R_2+\tau_2\frac{(\tau_3-d(t))}{\tau_3-\tau_2}R_2\Big]\dot{x}(t)$$
$$-\Big[x(t)-x\Big(t-\frac{\tau_1}{2}\Big)\Big]^T\Big(Z_1+\frac{2}{\tau_1}\left(R_1+(1-\dot{d}(t))R_2+\dot{d}(t)R_4\right)\Big)\Big[x(t)-x\Big(t-\frac{\tau_1}{2}\Big)\Big]$$
$$-\Big[x\Big(t-\frac{\tau_1}{2}\Big)-x(t-\tau_1)\Big]^T\Big(Z_2+\frac{2}{\tau_1}\left(R_1+(1-\dot{d}(t))R_2+\dot{d}(t)R_4\right)\Big)\Big[x\Big(t-\frac{\tau_1}{2}\Big)-x(t-\tau_1)\Big]$$
$$-[x(t-\tau_1)-x(t-\tau_2)]^T\Big(Z_3+\frac{1}{\tau_2-\tau_1}\left(R_1+(1-\dot{d}(t))R_2+\dot{d}(t)R_4\right)\Big)[x(t-\tau_1)-x(t-\tau_2)]$$
$$-\gamma_{2d}^T(d(t)-\tau_2)\left((\tau_3-\tau_2)Z_4+(1-\dot{d}(t))R_2+R_3+\dot{d}(t)R_4\right)\gamma_{2d}$$
$$-\gamma_{d3}^T(\tau_3-d(t))((\tau_3-\tau_2)Z_4+R_3+R_4)\gamma_{d3}, \qquad (22)$$

where γ_{2d} and γ_{d3} are defined by

$$\gamma_{2d}:=\frac{1}{d(t)-\tau_2}\int_{t-d(t)}^{t-\tau_2}\dot{x}(s)ds \quad \text{and} \quad \gamma_{d3}:=\frac{1}{\tau_3-d(t)}\int_{t-\tau_3}^{t-d(t)}\dot{x}(s)ds,$$

with $\lim_{d(t)\to\tau_2}\gamma_{2d}=\dot{x}(t-\tau_2)$ and $\lim_{d(t)\to\tau_3}\gamma_{d3}=\dot{x}(t-\tau_3)$.

We denote $\zeta_2^T(t):=\begin{bmatrix}\zeta_x^T & \gamma_{2d}^T & \gamma_{d3}^T\end{bmatrix}\in\mathbb{R}^{9r_x}$ where ζ_x is defined in (18). Then taking $\dot{V}_2(t)$, $\dot{V}_3(t)$ and $\dot{V}_4(t)$ from (15),

$\dot{V}_1(t)|_{d(t)>\tau_2}$ (21), and (22) one concludes that

$$\dot{V}(t)|_{d(t)>\tau_2} \leq \zeta_2^T(t)\left(\Omega|_{d(t)>\tau_2}\right)\zeta_2(t), \tag{23}$$

where

$$\Omega|_{d(t)>\tau_2}=\begin{bmatrix}\Psi^{(2)} & 0\\ * & -\Lambda^{(2)}\end{bmatrix},\ \Lambda^{(2)}=\begin{bmatrix}(d(t)-\tau_2)\Lambda_{22} & 0\\ 0 & (\tau_3-d(t))\Lambda_{21}\end{bmatrix},$$

and $\Psi^{(2)}$, Λ_{21}, and Λ_{22} are defined in (12).

Suppose now that analogously to the case where $\chi=1$, we define a matrix $\widetilde{B}_2=\left(\begin{bmatrix}B_2 & \widetilde{B}_{22}\end{bmatrix}+\begin{bmatrix}\Gamma_3 DF(t)T_E & 0\end{bmatrix}\right)$ such that $\widetilde{B}_2\zeta_2(t)=0$, where B_2 is defined in (12), and $\widetilde{B}_{22}$ is defined in an analogous fashion to $\widetilde{B}_{12}$.

Then the condition that arises from applying Finsler's lemma (Lemma 2) to the right side of (23) is that $\zeta_2^T(t)\left(\Omega|_{d(t)>\tau_2}\right)\zeta_2(t)$ is negative definite if there exists a matrix $\widetilde{H}_2=\begin{bmatrix}H_2^T & 0\end{bmatrix}^T\in\mathbb{R}^{9r_x\times 3r_x}$ such that $\Xi_2<0$ holds, where $H_2\in\mathbb{R}^{7r_x\times 3r_x}$ is a free-weighting matrix and

$$\Xi_2=\Omega|_{d(t)>\tau_2}+\widetilde{H}_2\widetilde{B}_2+\widetilde{B}_2^T\widetilde{H}_2^T. \tag{24}$$

Similarly to (19), we consider the terms Ξ_{21} and Ξ_{22} that arise from Ξ_2 when $d(t)\to\tau_2$ and $d(t)\to\tau_3$, respectively, After some manipulation, it can be seen that

$$\zeta_2^T(t)\Xi_2\zeta_2(t)=\frac{\tau_3-d(t)}{\tau_3-\tau_2}\zeta_{21}^T(t)\Xi_{21}\zeta_{21}(t)+\frac{d(t)-\tau_2}{\tau_3-\tau_2}\zeta_{22}^T(t)\Xi_{22}\zeta_{22}(t),$$

where $\zeta_{21}^T(t):=\begin{bmatrix}\zeta_x^T & \gamma_{d3}^T\end{bmatrix}$, $\zeta_{22}^T(t):=\begin{bmatrix}\zeta_x^T & \gamma_{2d}^T\end{bmatrix}$, and ζ_x is defined in (18). Then, from the convexity of $\zeta_2^T(t)\Xi_2\zeta_2(t)$, it is sufficient to verify the feasibility for Ξ_{21} and for Ξ_{22}.

Then it follows from applying Lemma 3 in [22] that $\Xi_{1k}<0$ holds if and only if there exists a scalar $\varepsilon_2>0$ such that

$$\begin{bmatrix}\left(\Psi^{(2)}|_{d(t)\to\tau_{k+1}}+H_2B_2+B_2^TH_2^T\right) & (\tau_3-\tau_2)H_2\Gamma_k\\ * & -(\tau_3-\tau_2)\Lambda_{2k}\end{bmatrix}+\frac{1}{\varepsilon_2}\alpha\alpha^T+\varepsilon_2\beta^T\beta<0$$

holds for $k\in\{1,2\}$. Moreover, taking the Schur's complement, we have Ω_{2k} as described in (12). Therefore, Ξ_1 is negative definite if and only if Ω_{21} and Ω_{22} are.

Moreover, given (3), the expressions

$$\Omega_{21}=\frac{d_{max}-\dot{d}(t)}{d_{max}-d_{min}}\Omega_{21}|_{\dot{d}(t)\to d_{min}}+\frac{\dot{d}(t)-d_{min}}{d_{max}-d_{min}}\Omega_{21}|_{\dot{d}(t)\to d_{max}},$$

$$\Omega_{22}=\frac{d_{max}-\dot{d}(t)}{d_{max}-d_{min}}\Omega_{22}|_{\dot{d}(t)\to d_{min}}+\frac{\dot{d}(t)-d_{min}}{d_{max}-d_{min}}\Omega_{22}|_{\dot{d}(t)\to d_{max}}$$

hold. Thus, Ω_{21} and Ω_{22} are convex in $\dot{d}(t)\in[d_{min},d_{max}]$.

We are now ready to complete the proof by establishing conditions that guarantee the negativeness of the Lyapunov functional's derivative. For the first case where $d(t)\neq\tau_2$, it is easy to check that

$$\dot{V}(t)|_{d(t)\neq\tau_2}\leq\chi_{[\tau_1,\tau_2]}(d(t))\zeta_1^T(t)\Omega_1\zeta_1(t)+\left(1-\chi_{[\tau_1,\tau_2]}(d(t))\right)\zeta_2^T(t)\Omega_2\zeta_2(t).$$

For the second case where $d(t)=\tau_2$, using exactly the same arguments of [7] and [9], we conclude that

$$\dot{V}(t)|_{d(t)=\tau_2}\leq\max\{\zeta_1^T(t)\Omega_1\zeta_1(t),\ \zeta_2^T(t)\Omega_2\zeta_2(t)\}.$$

Therefore, it is straightforward to conclude that if the conditions in (11) are fullfilled, then we guarantee that $\dot{V}(t)<0$, which concludes the proof.

ACKNOWLEDGMENTS

The authors are supported by research grants funded by the National Council of Technological and Scientific Development - CNPq, Brazil. The authors would also like to thank the support of the Scientific and Technological Developments Foundation – FINATEC, Brazil, and the Dean of Research and Graduate Studies – DPP, University of Brasilia – UnB, Brazil.

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