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On the hyperplanes arrangements in mixed-integer techniques

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Abstract— **This paper is concerned with the improved constraints handling in mixed-integer optimization problems. The novel element is the reduction of the number of binary variables used for expressing the complement of a convex (polytopic) region. As a generalization, the problem of representing the complement of a possibly non-connected union of such convex sets is detailed. In order to illustrate the benefits of the proposed improvements, a practical implementation, the problem of obstacle avoidance using receding horizon optimization techniques is considered.**

I. Introduction

Problems like *path following with obstacle* and *collision avoidance* are relevant in many applications involving the control of cooperative systems. A popular framework for the treatment of such decision problems is *Mixed-Integer-Programming* (MIP), described in [1]. MIP has the ability to include non-convex constraints and discrete decisions in the optimization problem. For example, the evolution of a dynamical system in an environment presenting obstacle can be modeled in terms of a non-convex feasible region which can be further expressed through the use of binary decision variables, [2]. Reference like [3] detail the use of MIP for off-line trajectory design with collision avoidance constraints. In [4], the authors used the combination of MIP and *Model Predictive Control* (MPC) to stabilize general hybrid systems around equilibrium points. [5] applied MIP in a predictive control framework to plan short trajectories around nearby obstacles. The mixed-integer formulation has also proven to be useful for cooperative reconnaissance [6], path planning, [7] and air traffic management [8]. Another interesting application approach was reported in [9] where a feasible reference signal which permits set membership testing for fault detection was computed over a non-convex region leading to a MIP formulation.

MIP, despite its modeling capabilities and the availability of good solvers has serious numerical drawbacks. As stated in [10], mixed-integer techniques are NP-hard, i.e. the computational complexity will increase in an exponential relation to the number of binary variables used in the problem formulation. This highlights the importance of reducing the number of binary variables. Due to the negative influence of the increase of binary variables there are a few attempts in the literature to reduce their number. In [7] an iterative method for including the obstacles in the best path generation is provided. Other works, like [11], consider a predefined path constrained by a sequence of convex sets. In all of these papers the binary variables reduction is not tackled at the MIP level, but instead the original decision problems are reformulated in a simplified MIP form.

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In the present paper a novel approach is proposed in the context of MIP complexity reduction. We refer here specifically to problems where the binary variables are used to express a non-convex region over which a (usually quadratic) cost function has to be minimized. We formulate the problem using fewer binary variables through a more compact codification of the inequalities describing the feasible region. Thus the problem complexity will require only a polynomial number of subproblems (LPs or QPs) that have to be solved with obvious benefits for the computational effort. Additionally, the technique is extended for the treatment of non-connected non-convex regions. The method presented in this paper can be used in several fields. We choose to exemplify here with an agent control problem where it is necessary to avoid stationary obstacles in a restricted region. In this context the reduction technique is embedded within an MPC path planning for multiple obstacles avoidance.

The rest of the paper is organized as follows. In Section II the preliminaries are presented, the main idea being detailed in Section III. Further on, in Section IV the novel method is extended to non-connected non-convex regions. The improvements in the computational time for the approach are detailed in Section V. Discussions based on an example are presented in Section VI while the conclusions are drawn in Section VII.

The following notation will be used throughout the paper. The closure of a set S, $cl(S)$ is the intersection of all closed sets containing S. The collection of all possible N combinations of binary variables will be noted $\{0,1\}^N = \{(b_1,\ldots,b_N) : \, b_i \in \{0,1\}, i = 1,\ldots,N\}$. The ceiling value of $x \in \mathbb{R}$ denoted as $\lceil x \rceil$ is the smallest integer greater than x. Denote $\mathbb{B}_p^n = \{x \in \mathbb{R}^n : \|x\|_p \leq 1\}$ as the unit ball of norm p, where $\|x\|_p$ is the p-norm of vector x.

II. Preliminaries

For safety and obstacle avoidance problems (to take just a few examples) the feasible region in the space of solutions is a non-convex set. Usually this region is considered as the complement of a convex region which describes an obstacle and/or a safety region. Due to their versatility and relative low computational burden the polyhedra are the instrument of choice in characterizing these regions.

In the following we define a bounded polyhedral set, $P \subset \mathbb{R}^n$ through its implicit half-space description:

$$P = \{x \in \mathbb{R}^n : \, h_i x \leq k_i, \quad i = 1,\ldots,N\} \tag{1}$$

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with $(h_i, k_i) \in \mathbb{R}^{1\times n} \times \mathbb{R}$ and its complement, as:

$$\mathcal{C}_X(P) \triangleq cl(X \setminus P) \quad (2)$$

with the reduced notation $\mathcal{C}(P)$ whenever X is presumed known or is considered to be the entire space $\mathbb{R}^n$.

By definition every affine subspace which defines P

$$\mathcal{H}_i = \{x : \; h_i x = k_i\} \quad (3)$$

will partition the space into two disjoint[1] regions:

$$\begin{aligned} \mathcal{R}^+(\mathcal{H}_i) &= \{x : \quad h_i x \leq \quad k_i\} && (4)\\ \mathcal{R}^-(\mathcal{H}_i) &= \{x : \; -h_i x \leq -k_i\} && (5)\end{aligned}$$

with $i = 1, \ldots, N$.

The non-convex region $\mathcal{C}(P)$, denoted by (2), may be described as an union of regions that cover all space except P:

$$\mathcal{C}(P) = \bigcup_i \mathcal{R}^-(\mathcal{H}_i), \quad i = 1, \ldots, N. \quad (6)$$

Therefore, we note that the complement of a bounded polyhedra (1) is covered by an union of N overlapping regions denoted as $\mathcal{R}_i^-$ (a simplified notation for region (5) associated to the i^{th} inequality of (1)).

In order to have a tractable problem one has to use mixed integer techniques with the end result being a polyhedra in the extended space of *state + auxiliary binary variables* of the form:

$$\begin{aligned} -h_i x &\leq -k_i + M\alpha_i, \quad i = 1 \ldots N && (7)\\ \sum_{i=1}^{i=N} \alpha_i &\leq N - 1 && (8)\end{aligned}$$

with M a constant chosen appropriately (that is, significantly bigger than the rest of the variables) and $(\alpha_1, \ldots, \alpha_N) \in \{0,1\}^N$ the auxiliary binary variables.

Remark 1. A region $\mathcal{R}_i^-$ can be obtained from (7) with an adequate choice of binary variables

$$\alpha^i \triangleq (1, \ldots, 1, \underbrace{0}_{i}, 1, \ldots, 1). \quad (9)$$

However the converse is not true since no choice of binary variables will describe a region (4). Indeed if the associated binary variable is 1, the corresponding inequality degenerates such that it covers any point $x \in \mathbb{R}^n$ (this represents the limit case for $M \to \infty$). The condition (8) is then required such that at least one binary value is 0 and consequently at least one inequality is verified. ♦

As it can be seen in the representation (7)–(8) a binary variable is associated to each inequality in the description of the polytope (1). Obviously, for a big number of inequalities, the number of binary variables becomes exceedingly large. Since their number exponentially affects the resolution of any mixed integer algorithm (usually they are *branch and cut* algorithms and thus, very sensitive to the number of binary terms) the goal to reduce their number is worthwhile. A first step would be to eliminate from the half-space representation of the polytope all the redundant constraints, [12]. We suppose that this pre-treatment is performed and we are dealing with a non-redundant description of the polyhedral set.

III. Basic idea

By preserving a linear structure of the constraints, we propose in the present section a generic solution towards the binary variables reduction.

To each of the regions in (6) we associated in (7) a unique binary variable. Consequently, the total number of binary variables is N, the number of supporting hyperplanes (see (1)). However, a basic calculus shows that the minimum number of binary variables necessary to distinguish between these regions is

$$N_0 = \lceil log_2 N \rceil. \quad (10)$$

The question that arises is the following: *How to describe the regions in a linear formulation similar to* (7) *through a reduced number of binary variables?*

$$(\lambda_1, \ldots, \lambda_{N_0}) \in \Lambda \triangleq \{0,1\}^{N_0}. \quad (11)$$

The binary expression appearing in the inequalities has to remain *linear* for computational advantages related to the optimization solvers. This structural constraint is equivalent with saying that any variable α_i should be described by a linear mapping in the form:

$$\alpha_i(\lambda) = a_0^i + \sum_{k=1}^{N_0} a_k^i \lambda_k. \quad (12)$$

In the reduced space of Λ we will arbitrarely associate a tuple

$$\lambda^i \triangleq \left(\lambda_1^i \ldots \lambda_{N_0}^i\right) \quad (13)$$

to each region $\mathcal{R}_i^-$. Note that this association is not unique, and various possibilities can be considered: in the following, unless otherwise specified, the tuples will be appointed in lexicographical order.

The problem of finding a mapping in Λ which describes region $\mathcal{R}_i^-$ reduces then to finding the coefficients $\left(a_0^i, a_1^i, \ldots, a_{N_0}^i\right)$ for which $\alpha_i = 0$ for the associated tuple and $\alpha_i = 1$ everywhere else. This translates into the following conditions for any $\lambda^i, \lambda^j \in \{0,1\}^{N_0}$:

$$\begin{cases} a_0^i + \sum\limits_{k=1}^{N_0} a_k^i \lambda_k^i = 0 \\ a_0^i + \sum\limits_{k=1}^{N_0} a_k^i \lambda_k^j \geq 1, \quad \forall j \neq i \end{cases} \quad (14)$$

with λ_k^i the k^{th} component of the tuple associated to $\mathcal{R}_i^-$. Note that, in (14) the equality constraints for $j \neq i$ were relaxed to inequalities since the value of $M\alpha_i$ needs only to be sufficiently large (any $\alpha_i \geq 1$ being a feasible choice).

Nothing is said a priori about the non-emptiness of the set described by (14). We need at least a point in the coefficients

[1] The relative interiors of these regions do not intersect but their closures have as a common bounday the affine subspace $\mathcal{H}_i$.

space $(a_0, a_1, \ldots, a_{N_0})$ which verifies conditions (14) in order to prove the non-emptiness. To this end, we present the following proposition:

Proposition 1. *A mapping $\alpha_i(\lambda) : \{0,1\}^{N_0} \to \{0\} \cup [1, \infty)$ which verifies (14) is given by:*

$$\alpha_i(\lambda) = \sum_{k=1}^{N_0} t_k, \quad \text{where } t_k = \begin{cases} \lambda_k, & \text{if } \lambda_k^i = 0 \\ 1-\lambda_k, & \text{if } \lambda_k^i = 1 \end{cases} \tag{15}$$

where λ_k denotes the k^{th} variable and λ_k^i its value for the tuple associated to region $\mathcal{R}_i^-$.

The coefficients $\left(a_0^i, \ldots, a_{N_0}^i\right)$ of an equivalent linear mapping (12) can be then obtained as:

$$a_0^i = \sum_{k=1}^{N_0} \lambda_k^i, \quad a_k^i = \begin{cases} 1, & \text{if } \lambda_k^i = 0 \\ -1, & \text{if } \lambda_k^i = 1 \end{cases}, \quad k = 1, \ldots, N_0 \tag{16}$$

Proof: The claim is constructive, by introducing mapping (16) in (14) it can be seen by simple inspection that the conditions are verified. □

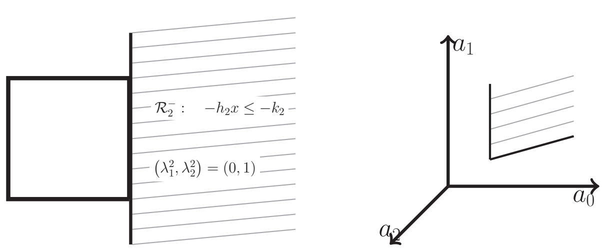
For exemplification the case of a square will be described:

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{17}$$

As stated in this section the number of binary variables (similar to the formulation (7)) is $N = 4$, equal with the number of half-spaces described in (17). The reduced number of variables will be $N_0 = \lceil log_2 4 \rceil = 2$, according to (10). Following the problem formulation (15) the variables α_i can be expressed as in (12) by

$$\alpha_i = a_0^i + a_1^i \lambda_1^i + a_2^i \lambda_2^i.$$

We associate to each region a tuple of two values (λ_1, λ_2) in lexicographical order.



(A) Feasible region for coefficients (B) $R^-(\mathcal{H}_2)$ and its tuple

FIG. 1: Outer regions and their associated tuples

The case of the 2^{nd} half-space, associated to tuple $(\lambda_1^2, \lambda_2^2) = (0, 1)$, is detailed in Figure 1(a). Using (14) we obtain, as depicted in Figure 1(b), the feasible set of the coefficients described by

$$a_0^2 + a_2^2 = 0, \quad a_0^2 \ge 1, \quad a_1^2 \ge 1.$$

This represents a polytopic region in the coefficients space $(a_0, a_1, a_2) \in \mathbb{R}^3$ and, according to (15), the non-emptiness is assured by the existence of at least a feasible combination of coefficients leading to the mapping $\alpha_2 = 1 + \lambda_1 - \lambda_2$. This means that the region R_2^- is associated with

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le -1 + M(1 + \lambda_1 - \lambda_2)$$

Further, the same computations will be performed for the rest of the regions, resulting in an extended system of linear inequalities over mixed decision variables:

$$\begin{bmatrix} 0 & -1 \\ 0 & 1 \\ -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} -1 + M(\quad \lambda_1 + \lambda_2) \\ -1 + M(1 - \lambda_1 + \lambda_2) \\ -1 + M(1 + \lambda_1 - \lambda_2) \\ -1 + M(2 - \lambda_1 - \lambda_2) \end{bmatrix}$$

A. *Interdicted tuples*

By the choice of the cardinal N_0 as in (10), the number of tuples allowed by the reduced set of binary variables (11) may be greater than the actual number of regions. For further use we define Λ^{alloc} as the set of N allocated tuples and Λ^{int} as the set of $N_{int} = 2^{\lceil log_2 N \rceil} - N$ unallocated tuples which evidently verify $\Lambda = \Lambda^{alloc} \cup \Lambda^{int}$ and $\Lambda^{alloc} \cap \Lambda^{int} = \emptyset$.

The tuples left unallocated will be labeled as *interdicted* and additional inequalities will have to be added to the extended set of constraints (7). These restrictions are justified by the fact that, under construction (16), an unallocated tuple will not enforce the verification of any of the constraints of (7) (see Remark 1). It then becomes evident that the single constraint of (8) has to be substituted by a set of constraints that implicitly make all the points Λ^{int}, defining the interdicted combinations, infeasible. This raises the question of how many such unallocated tuples may exist. An upper bound is given by:

$$0 \le N_{int} \le 2^{\lceil log_2 N \rceil} - 2^{\lceil log_2 N \rceil - 1} - 1 = 2^{\lceil log_2 N \rceil - 1} - 1 \tag{18}$$

where the bound is reached for the most unfavorable case of $N = 2^{\lceil log_2 N \rceil - 1} + 1$.

From the above relation it can be seen that the number of unallocated tuples can be important. If we associate to each of them an inequality intended to discard the combination from the set of feasible points, we negatively influence the speed of the associated optimization algorithm. This can be alleviated by noting (as previously mentioned) that the association between regions and tuples is arbitrary. One could then chose favorable associations which will permit more than one tuple to be removed through a single inequality.

Geometrically, the tuples are extreme points on the hypercube $\mathbb{B}_\infty^{N_0}$ and the inequalities we require are then half-spaces which separate the points of the hypercube into allocated and unallocated tuples. As a first step one should note that there always exists a half-space that separates the points of a facet from the rest of the hypercube.

An intuitive method is then to label as unallocated the extreme points which compose entire facets on the hypercube $\mathbb{B}_\infty^{N_0}$. By writing N_{int} as a sum of consecutive powers of 2 $(N_{int} = \sum_{i=0}^{\lceil log_2 N_{int} \rceil - 1} b_i 2^i)$ and using (18), an upper bound for the number of inequalities can be computed:

$$N_{hyp} = \sum_{i=0}^{\lceil log_2 N_{int} \rceil - 1} b_i \leq \lceil log_2 N \rceil - 1 \quad (19)$$

where $b_i \in \{0, 1\}$.

IV. Refinements for the complement of a union of convex sets

In the previous section the basic reduction method was applied for treatment of the complement of a convex set. A generic cases will be detailed in the following by considering the complement of a union of convex (bounded polyhedral) sets $\mathbb{P} = \bigcup_l P_l$:

$$\mathcal{C}_X(\mathbb{P}) = cl(X \setminus \mathbb{P}) \quad (20)$$

with[2] $P_l = \bigcap_{k_l=1}^{\mathcal{K}_l} R^+(\mathcal{H}_{k_l})$ and $N \triangleq \sum_l \mathcal{K}_l$.

This type of regions arises naturally in the context of obstacle avoidance when there is more than a single object to be taken into account.

In order to deal with the complement of a non-convex region in the context of mixed-integer techniques several additional theoretical tools will be introduced in the following.

Definition 1 (Hyperplane arrangements – [13])**.** A collection of hyperplanes $\mathbb{H} = \{\mathcal{H}_i\}_{i=1:N}$ will partition the space in an union of disjoint[3] cells defined as follows:

$$\mathcal{A}(\mathbb{H}) = \bigcup_{l=1,\ldots,\gamma(N)} \underbrace{\left(\bigcap_{i=1}^{N} R^{\sigma_l(i)}(\mathcal{H}_i)\right)}_{A_l} \quad (21)$$

where $\sigma_l \in \{-,+\}^N$ denotes feasible combinations of regions (4)–(5) obtained for the hyperplanes in $\mathbb{H}$.

Several computational aspects are of interest. The number of feasible cells, $\gamma(N)$, (in relation with the space dimension – d and the number of hyperplanes – N) is bounded by ([14]):

$$\gamma(N) \leq \sum_{i=0}^{d} \binom{N}{i} \quad (22)$$

Efficient (as computation time and storage requirements) algorithms for cell enumerating were presented in [15].

In (7) a single binary variable was associated to a single inequality but the mechanism can be applied similarly to more inequalities. Thus, one can describe (21) in an extended space of *state + auxiliary binary variables* as follows:

$$\begin{array}{c} \vdots \\ \left.\begin{array}{c} \sigma_l(1)h_1 x \leq \sigma_l(1)k_1 + M\alpha_l \\ \vdots \\ \sigma_l(N)h_N x \leq \sigma_l(N)k_N + M\alpha_l \end{array}\right\} A_l \\ \vdots \end{array} \quad (23)$$

with condition

$$\sum_{l=1}^{l=\gamma(N)} \alpha_l \leq \gamma(N) - 1 \quad (24)$$

imposing that at least a set of constraints will be verified.

Construction (23)–(24) will permit, through projection along the binary variables α_l (see (9)), to obtain any of the cells of hyperplane arrangement (21).

Analogously to Section III we propose in the following the reduction of the number of binary variables by associating to each of the cells an unique tuple. The binary part will be computed following the constructive result in Proposition 1 and used accordingly in (23). Additional inequalities, that render infeasible the unallocated tuples or describe interdicted regions are introduced analogously to the case presented in Subsection III-A.

Remark 2. Note that if we discard the linear structure and allow a nonlinear formulation involving products of binary variables, the hyperplane arrangements (21) can be represented as:

$$\begin{array}{rl} & \vdots \\ -h_i x \leq & -k_i + M \cdot \displaystyle\prod_{\substack{l=1,\ldots,\gamma(N) \\ \sigma_l(i)='-'}} \alpha_l \\ h_i x \leq & \ \ k_i + M \cdot \displaystyle\prod_{\substack{l=1,\ldots,\gamma(N) \\ \sigma_l(i)='+'}} \alpha_l \\ & \vdots \end{array} \quad (25)$$

where we used the fact that the cells of (21) use the same half-spaces and thus they can be concatenated. The method presented in [16] transforms an inequality with nonlinear binary components into a set of inequalities with linear binary components. However, this can be made only at the expense of introducing additional binary variables. ♦

In the following an illustrative example is provided in terms of a region composed by two triangles considered as obstacles. The hyperplane arrangement resulting from the 6 hyperplanes in the definition of triangles will lead, according to (22), to $\gamma(6) = 22$ cells. In order to allocate a tuple to each of these cells we require 5 binary variables.

As it can be seen in Figure 2 the interdicted regions are described by 4 cells and there are 10 unallocated tuples remaining. For a minimum number of separating hyperplanes we position them, as discussed in Subsection III-A, on 3

[2]The "+" superscript was chosen for the homogeneity of notation, equivalently one could have chosen any combination of signs in the half-space representation (4)–(5).

[3]By disjoint cells we refer to their relative interior's intersection since their closures have one of the hyperplanes $\mathcal{H}_i$ as a common boundary.

faces of the dimension 5 hypercube, with 8, 4 and respectively 2 points.

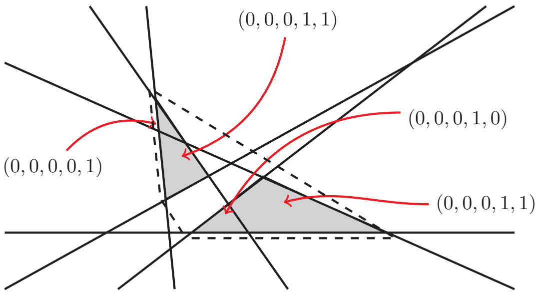


FIG. 2: Exemplification of hyperplane arrangement

A. Practical implementation for hyperplane arrangements

In the above, we presented a modality of describing the cell arrangement (21) through MIP techniques. This allows, by the interdiction of the tuples associated with the cells describing the set $\mathbb{P}$, to describe the non-connected and non-convex region (20). However the resulting linear representation is not unique or necessarily minimal. In order to offer a complete procedure for the complexity reduction scheme in the case of union of convex sets, the techniques detailed in Section III and Section IV can be used jointly to achieve a better solution. To this end we provide Algorithm 1:

Algorithm 1: Hybrid scheme for representing $\mathcal{C}(\mathbb{P})$

1. obtain the cell arrangement as in (21) for $\mathbb{P}$;
2. compute the convex hull of $\mathbb{P}$ as $\mathbb{P}^\circ = ConvexHull\{\mathbb{P}\}$;
3. select the cells of (21) which intersect $\mathbb{P}^\circ$;
4. obtain the regions describing $\mathcal{C}(\mathbb{P}^\circ)$;
5. associate tuples to the selected cells and the regions computed à priori;

As described in Algorithm 1 this hybrid scheme permits to express (20) as an union of the cells of (21) which intersect $\mathbb{P}^\circ$ and of the regions (in the sense of (5)) which describe $\mathcal{C}(\mathbb{P}^\circ)$. As long as the number of cells in (21) is significant we observe a sensible reduction in the required number of binary variables as a result of using this hybrid technique. Additionally, the number of total inequalities in (23) decreases since the regions (5) are described by a single inequality.

Figure 2 illustrates $ConvexHull\{\mathbb{P}\}$ (dashed contour) and by using the proposed algorithm, the number of regions that have to be described reduces to 14, 10 cells and 4 half-spaces, which reduces the number of binary variables to 4.

V. Numerical considerations

In this Section we will test the computation time improvements for our approach versus the standard technique encountered in the literature. As previously mentioned, a MIP problem is NP-hard in the number of binary variables. Therefore, a small reduction will render sensible improvements.

The complexity of the MIP algorithm with constraints in the classical form (7)–(8) will be of the order of $\mathcal{O}(2^N \cdot l_{QP}(N+1))$ where l_{QP} denotes the cost of solving a QP. Using the alternative formulation proposed in Section III we obtain the complexity as

$$\mathcal{O}(2^{\lceil log_2 N\rceil} \cdot l_{QP}(N+\lceil log_2 N\rceil-1)) = \mathcal{O}(N \cdot l_{QP}(N)). \quad (26)$$

In fact, one can see that the MIP problem is now P-hard in the number of QP subproblems.

In Section IV a method for describing in the MIP formalism of the complement of a possibly non-connected union of polytopes was presented. The main drawback is that in both classical and reduced formulation the problem depends on the number of cells. Supposing the hyperplanes from the hyperplane arrangement (21) are in random position we obtain for formulation (23)–(24) a complexity of order

$$\mathcal{O}(2^\gamma(N) \cdot l_{QP}(N \cdot N^d + 1)) \quad (27)$$

and employing the techniques from Section III the complexity reduces to

$$\mathcal{O}(2^{\lceil log_2 \gamma(N)\rceil} \cdot l_{QP}(N \cdot N^d + 1)) = \mathcal{O}(\gamma(N) \cdot l_{QP}(N^{d+1})). \quad (28)$$

Again, we observe that the MIP problem becomes P-hard in the number QP subproblems.

VI. Collision avoidance example

Collision avoidance has been shown to be crucial in many applications involving the control of multi-agent systems. The goal of this example is to present a control approach for an agent operating in an environment filled with stationary obstacles. The agent evolves from an initial position to a target position while avoiding the randomly distributed obstacles.

We consider the dynamics of the agent described by a LTI system as follows:

$$\xi_{k+1} = A\xi_k + Bu_k. \quad (29)$$

The agent model is used in a predictive control [17] context which permits the use of non-convex state constraints for obstacle avoidance behavior.

An optimal control action u^* is obtained from the control sequence $\mathbf{u} \triangleq \{u_{k|k}, u_{k+1|k}, \cdots, u_{k+N-1|k}\}$ as a result of the optimization problem:

$$\begin{aligned} u^* \;=\; & \arg\min_{\mathbf{u}} (\xi^T_{k+N|k} P \xi_{k+N|k} + \sum_{l=1}^{N-1} \xi^T_{k+l|k} Q \xi_{k+l|k} + \\ & + \sum_{l=0}^{N-1} u^T_{k+l|k} R u_{k+l|k}) \\ & \text{subject to: } \begin{cases} \xi_{k+l|k} = A\xi_{k+l-1|k} + Bu_{k+l-1|k} \\ \xi_{k+l|k} \in \mathcal{C}(\mathbb{P}), \quad l = 1, \ldots, N \end{cases} \end{aligned} \quad (30)$$

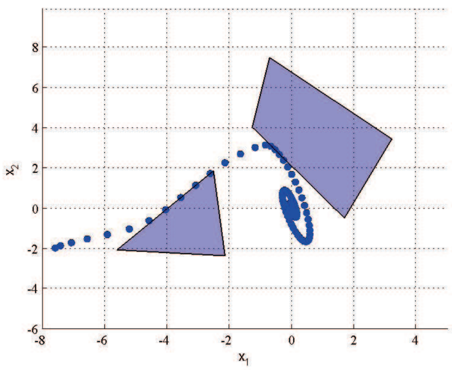


FIG. 3: The trajectory of the agent with collision free behavior

Here $Q \geq 0$, $R > 0$ are the weighting matrices, $P \geq 0$ defines the terminal cost and $\mathbb{P}$ is an union of polytopes describing the obstacles.

As previously stated this requires the use of MIQP techniques which are NP-hard in the number of binary variables. The methods presented in this paper would certainly fit very nicely with this type of problem. The non-convex and non-compact feasible region can be described as in Sections III or IV and, ultimately, the number of binary variables will be reduced accordingly.

As a practical application we consider a linear system (vehicle, pedestrian or agent in general form) whose dynamics are described by:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{\mu}{m} & 0 \\ 0 & 0 & 0 & -\frac{\mu}{m} \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{m} \end{bmatrix} \tag{31}$$

where $\xi = [x \ y \ v_x \ v_y]^T$, $u = [u_x \ u_y]^T$ are the state and the input of the system. With the components of the state being (x, y), the position, and (v_x, v_y) the velocities of the agent, m is the mass of the agent and μ its damping factor.

We consider the position component of the agent state to be constrained by two obstacles defined by 4 and 4 hyperplanes. These 8 hyperplanes will result in 29 cells as detailed in (21). This would require $\lceil log_2 29 \rceil = 5$ binary variables for representing the feasible region. If the enhancements presented in Subsection IV-A are used we will find 5 cells inside $ConvexHull(\mathbb{P})$ and its exterior is described by 5 regions of form (5). This will result in a reduced number of binary variables $\lceil log_2 10 \rceil = 4$ and less inequality constraints.

We apply the predictive control strategy for horizon $N = 2$ and cost matrices $Q = 10^5 \cdot I_4$, $R = I_2$ and $P = 10^5 \cdot I_4$ and obtain the trajectory depicted in Figure 3. Note that increasing the horizon length will enhance the accuracy of the trajectory.

VII. CONCLUSIONS

In this paper we present several remarks leading to computational improvements of the MIP techniques of solving optimization problems with real and binary variables. We introduce a novel *linear* constraints expression for reducing the number of binary variables necessary in describing the exterior of convex sets. An additional method for describing in the same framework a collection of non-connected convex sets (or their complement) was also provided. The numerical improvements over previous techniques were presented and tested in an obstacle avoidance control problem.

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