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A Distributed Control for Multiple Photovoltaic Generators in Distribution Networks

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Abstract: **A distributed control was provided to regulate the active power of photovoltaic (PV) generators installed on a distribution power system network, making a group of PVs converge and operate at the same ratio of available power. The sufficient conditions that guarantee the convergence of the closed-loop system was discussed, including the minimal requirement of communication topology among the PVs. Simulation on a radial distribution network was provided to verify the validness of the control.**

Key words: **Photovoltaic Generator, Distribution Network, Distributed Control**

I. Introduction

In recent years, there are more and more distributed generators (DG) integrated into the modern distribution network [1], especially for the PVs due to their clean and renewable characteristics. If a distribution network with many PVs, it is necessary to control and dispatch those units to provide some ancillary services [1, 2].

Usually, there are three control modes to control the outputs of the PVs: the centralized mode, the decentralized mode and the distributed mode. The methods based on the optimal power flow strategy [3, 4] are of the centralized mode, which were employed in a distribution network that has some distributed generators [5]. This mode needs global information, so it is a difficult way for a distribution network which has numerous and geographically dispersed DGs. Another mode is the decentralized mode, such as the constant PQ of operation, the maximum photovoltaic power tracking or constant VF (Voltage and Frequency) with droop mode [6-8]. However, it is not easy to use this mode when the number of units becomes large and their outputs are intermittent.

Compared to the centralized and the decentralized control modes, the distributed control mode can use local communication networks and combine the positive features of both centralized and decentralized controls while limiting their disadvantages [9-12]. Thus, given the advances in modern communication, the distributed mode is practical to implement and also necessary to accommodate various changes of PVs. This idea was used in [13] for multiple PVs and its strategy is to make all PVs converge to an uniform output ratio autonomously. The simulation shows good feasibility for multiple PVs in a future smart grid, but the strict proof is not provided. This problem will be partly solved in this paper, in which the strict proof for the active power management will be given.

This paper provides a distributed control for many PVs in a distribution power systems network in order to control the PVs' active power outputs for some ancillary services. The requirement of the communication networks under which the control strategy is valid is presented and the stability will be proved under some trivial assumptions.

II. Problem Formulation

A. Dynamical Models

Consider a distribution power system network with n three-phase inverter based PVs, which use a decoupled d-q control method via Phase Locked Loops. The d-axis and q-axis are to be controlled for the active power and reactive power, respectively.

Ignoring the control in the q-axis, the dynamics of the distribution system can be denoted by [13]:

$$\dot{I}_{di}^{ref} = u_{1i} \tag{1}$$

$$P_i = U_i I_{di}^{ref} \tag{2}$$

$$0 = g(P_1, ..., P_n, \chi, \mathbf{X}) \tag{3}$$

where the sub-script 'i' denotes the ith PV, P_i is the active power; equation (1) denotes the dynamics in the d-loop; u_{1i} is the input to be designed in order to control the power output; χ is a vector of appropriate dimensions, which denotes the internal state variables of the system; $\mathbf{X}$ is a vector denoting the algebraic variables in the distribution network such as the voltages of buses; and (3) denotes the power flow equation of the distribution network.

B. Problems to be Solved

The control strategy is to make all PVs run at the same active power output ratios, and it is to be realized by the practical distributed control mode. The basic idea is that each PV can share its information with some others. Thus, the control of each PV should be of the general form

$$u_i = w_i(s_{i0}\mathbf{y}_0, s_{i1}\mathbf{y}_1, s_{i2}\mathbf{y}_2, ..., s_{in}\mathbf{y}_n),\ i = 1, 2, ...n \tag{4}$$

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where $\mathbf{y}_0$ denotes the output of the high level control; $\mathbf{y}_i$ $i=1,2,\dots n$ denotes the output variable of i[th] PV unit; $S=(s_{ij})$ is a time-variant matrix denoting the communication topology, defined as

$$S=\begin{bmatrix} s_{10}(t) & s_{11} & & \cdots & s_{1n}(t) \\ s_{20}(t) & s_{21}(t) & s_{22} & \cdots & s_{2n}(t) \\ \vdots & \vdots & & \ddots & \\ s_{n0}(t) & s_{n1}(t) & s_{n2}(t) & \cdots & s_{nn} \end{bmatrix} \in R^{n\times(n+1)} \qquad (5)$$

where $s_{ii}(t)\equiv 1$ is satisfied for all i; $s_{ij}=1$ if the output of the j-th PV generator is known to the i-th PV generator at time t, and $s_{ij}(t)=0$ if otherwise; $s_{i0}(t)=1$ if the i-th PV can get information from the high level control (or remote control) and $s_{i0}(t)=0$ if otherwise.

The problems to be solved are stated as follows.

Problem 1: For the system given by (1)-(3), design the controls for u_{1i} ($i=1,2,\dots,n$) such that at the equilibrium points there are

$$\frac{P_1}{P_{1\max}}=\frac{P_2}{P_{2\max}}=\dots=\frac{P_n}{P_{n\max}}=\alpha_P^0 \qquad (6)$$

where α_P^0 is the given output ratio for the active power; $P_{i\max}$ is the maximal power of the i[th] PV unit; $P_{i\max}$ is supposed to be positive without loss of any generality.

It should be noted that in formula (5) $s_{ii}(t)\equiv 1$ will be always satisfied for each PV. Whether other PVs' information is used or not is completely determined by a nonzero entry in the communication matrix. In general, only a part of the neighboring information is necessary to ensure convergence. In addition, the communication matrix is considered to be time-varying in general, not a matrix of constants. This is a necessary to be considered since communication equipment may malfunction or some PVs could be out of service due to environmental reasons. This means that the communication matrix is piecewise continuous. Specifically, let

$$t_{\infty:0} \triangleq \{t_0,t_1,t_2,\dots\}, \quad S_{\infty:0} \triangleq \{S(t_0),S(t_1),\dots\} \qquad (7)$$

which means matrix S changes at time t_i ($t_0=0$), i.e., $S(t)=S(t_k)\in S_{\infty:0}$ for $t\in[t_k,t_{k+1})$.

Once problems 1 is solved, then those PVs can be considered to be a virtual generator with a larger capacity. Now, for each virtual generator, only the operating ratio need to be decided, resulting in a much simpler way to design a high level control for the ancillary services problems. In this paper, the ancillary service is to keep the active power consumed by loads in an area or feeder to be constant. This problem can be stated as:

Problem 2: Based on the solutions of problem 1, design an additional control for (1)-(3) such that the active power across some transmission line to be constant. Namely, at the equilibrium there is

$$P_{tran}(\cdot)=P^{ref} \qquad (8)$$

where P^{ref} is a given constant; $P_{tran}(\cdot)$ denotes the active power across a concerned line (it is a function of the PVs' output, so $(\cdot)$ is used).

C. Control Strategy

It follows from (1)-(2) that

$$\dot{P}_i=\dot{U}_i I_{di}^{ref}+U_i\dot{I}_{di}^{ref}=\frac{\dot{U}_i}{U_i}P_i+U_i u_{1i} \qquad (9)$$

Choose the control law for the i[th] PV to be

$$u_{1i}=\frac{K_0 P_{i\max}}{dz(U_i)}\left(D_{i0}\alpha_P^0+\sum_{j=1}^{n}\frac{D_{ij}P_j}{P_{j\max}}-\frac{P_i}{P_{i\max}}\right)-\frac{\dot{U}_i P_i}{\left(dz(U_i)\right)^2} \qquad (10)$$

where $dz(U_i)=\max\{U_i,\underline{U}\}$ and $\underline{U}>0$ is a given constant; $K_0>0$ is the gain;

$$D_{ij}=\frac{s_{ij}}{\sum_{j=0}^{n}s_{ij}}, \quad i=1,2,\dots n \qquad (11)$$

where $W=(w_{ij})$ denotes the weight matrix with positive elements (in this paper, those elements are set to 1 for simplicity); s_{ij} ($i,j=1,2,\dots n$) are the entries of the communication matrix defined in (5).

In addition, to satisfy problem 2, the α_P^0 is updated by a simple gradient rule shown in (12), thus the closed-loop equations can be compactly expressed by:

$$\dot{z}_0=K_P\left[P_{ref}-P_{tran}(\cdot)\right] \qquad (12)$$

$$\dot{z}_i=K_0\left[-z_i+D_{i0}z_0+\sum_{j=1,2,\dots n}D_{ij}z_j\right] \qquad (13)$$

$$g(z_1,\dots,z_n,\boldsymbol{\chi},\mathbf{X})=0 \qquad (14)$$

where $z_0=\alpha_P^0$, $z_i=P_i/P_{i\max}$ and other variables are defined previously.

The communication matrix S is designed according to the following rule.

Rule 1: the communication topology among the PVs may be intermitted, but the communication matrix S is piecewise continuous. Mathematically the sequence $S_{\infty:0}=\{S(0),S(t_1),\dots\}$ is sequentially complete.

It should be noted that the convergence rate of the closed loop dynamical system depends upon connectivity of the communication network [14].

Remark: In the proposed control strategy, we do not consider the parameter uncertainty issue, such as stochastic loads and irradiance, which is a common phenomenon in distribution power systems. If the uncertainty is considered, the distributed H-∞ control [15] may be used for this kind of dynamical models, which is one of our future work.

III. Stability Analysis for the Closed-Loop System

A. Reduced dynamical model for stability analysis

In the section IV, the communication topology and the control among all PVs are given. The remaining problem is to analyze the stability of the dynamical system (14).

It flows from formulas (12)-(14) that the equilibrium of the dynamical-algebraic equations satisfies

$$z_1^0=z_2^0=\dots=z_n^0=\alpha_P^0, \quad P_{ref}=P_{tran}^0 \qquad (15)$$

$$g(z_1^0,\dots,z_1^0,\boldsymbol{\chi}^0,\mathbf{X}'^0)=0 \qquad (16)$$

where the super-script '0' denotes the variables evaluated at the equilibrium.

If the equilibrium is a nonsingular point of the power flow equation, it follows from the inversed function theorem that around the equilibrium there exist smooth functions of appropriate dimensions, say $\varphi_1'(\cdot)$ and $\varphi_2'(\cdot)$, satisfying the power flow equation, i.e.,

$$g(z_1,...,z_n,\varphi_1'(\cdot),\varphi_2'(\cdot))=0$$

Thus, there exists a smooth function $\varphi_1(\cdot)$ such that P_{tran} can be expressed as:

$$P_{tran}=\varphi_1(z_1,...,z_n) \qquad (17)$$

Substituting (17) into (12)-(14), the reduced dynamical equations can be expressed as

$$\dot{z}_0=K_P\left[P_{ref}-\varphi_1(z_1,...,z_n)\right] \qquad (18)$$

$$\dot{z}_i=K_0\left[-z_i+D_{i0}z_0+\sum_{j=1,2,...n}D_{ij}z_j\right], i=1,2,...,n \qquad (19)$$

Thus, the stability is completely determined by system (18)-(19). In the next subsection, the stability will be proved under some additional assumptions that are usually satisfied in power systems.

B. Basic properties for the stability analysis

For a general distribution network, the following facts can be assumed to be satisfied:

***Fact 1**:* The positive direction of P_{tran} is chosen to be an increasing function of α_P^0, so $P_{sum}=\varphi_1(\cdot)$ is an increasing smooth function of P_i, i.e.,

$$\partial\varphi_1(\cdot)/\partial P_i>0 \qquad (20)$$

Fact 2: the angles at both sides of the transmission line of concern satisfy that

$$\left|\sin(\delta_1-\delta_2)\right| \ll \left|\cos(\delta_1-\delta_2)\right| \qquad (21)$$

where δ_i ($i=1,2$) denotes the angle of the concerned transmission line.

The linearization system of (18)-(19) at equilibrium (denoted by E_0) can be expressed as:

$$\begin{cases}\dot{z}_0=-K_P\sum_{j=1}^{n}c_{1j}(z_j-\alpha_P^0)\\ K_0^{-1}\dot{z}_i=-z_i+D_{i0}z_0+\sum_{j=1,2,...n}D_{ij}z_j\end{cases} \qquad (22)$$

where $c_{1j}\triangleq\partial\varphi_1/\partial z_j\big|_{E_0}$.

Changing the equilibrium of (22) to the origin results in a system with the same stability property as that of the following system:

$$\begin{cases}\dot{z}_0=-K_0^{-1}K_Pc_1^{\mathrm{T}}\mathbf{z}\\ \dot{\mathbf{z}}=D_{*0}z_0+(-\mathbf{I}+\mathbf{D})\mathbf{z}\end{cases} \qquad (23)$$

where $D_{*0}=[D_{10},D_{20},...,D_{n0}]^{\mathrm{T}}$, $\mathbf{D}$ is the $n\times n$ matrix whose entries are time-varying D_{ij}.

Thus, the stability of system (18)-(19) is locally determined by system (23).

Lemma 1. Suppose facts 1-2 are satisfied, then in (23), $M=-K_Pc_1^T\mathbf{1}<0$ is satisfied.

Proof: facts 1-2 implies c_{1j} is positive, thus the conclusion is correct obviously.

It follows from lemma 1 and lemma 3 (in appendix) that the stability of system (23) can be stated as:

Theorem 1. Consider system (23). Suppose that the following conditions are satisfied:

1) K_P is small enough;

2) Facts 1-2 are satisfied;

3) The communication among the PVs satisfies the sequentially complete (rule 1 in section III).

Then system (23) is asymptotically stable.

***Proof**:* Clearly, system (23) can be rewritten as the system considered in lemma 3. It follows from the given conditions 1-2 and Lemma 1 that the condition 1 in lemma 3 is satisfied. The given condition 3 implies that the condition 2 in lemma 3 is also satisfied. Thus, it follows from lemma 3 that system (23) is asymptotically stable. □

It follows from this theorem that the linearized system is asymptotically stable, so if the initial states lie in the neighborhood of the equilibrium, the active outputs of PVs satisfies $[z_0,z_1,...,z_n]^{\mathrm{T}}\to\alpha_P^0\mathbf{1}$, where α_P^0 satisfies (8). Consequently, the theorem 1 guarantees that the proposed control strategy is the solution to problems 1-2.

IV. Simulation

A 50hz radial network is considered in this section. The main voltage in this network is 10KV, and the topology is shown in Figure 1, where 5 PVs and 6 loads are considered to be connected to the low voltage network.

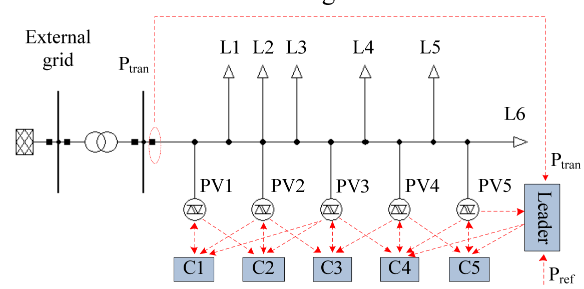

Figure 1 A radial system with multiple PVs (The dash arrows among controllers represent the information flow)

The detailed parameters of the grid are:

1) Every segment of the transmission line (from one bus to its neighboring bus) is 0.85km; the impedances is 0.443+j0.3ohm/km; the external grid is considered to be an infinite bus whose voltage is 1.05 p.u.; the short-circuit voltage of the transformer is 5% and its capacity and copper loss are 1MVA and 5kW.
2) Spot loads (balanced) are shown in Table 1 and the constant impedance model is considered.
3) The communication topology is shown by the dash arrows in Figure 1, $K_0=20$ and $K_p=1$; the concerned line is chosen to be the objects which are measured by the leader control.
4) The maximum of every PV is 0.2MW+j0.04MVAR and the initial output is 0.15MW+j0.0MVAR.

Table 1 Load information

Load	Active power (KW)	Reactive power (KVAR)
L1	346.28	92.34
L2	364.50	58.32
L3	473.85	97.20

L4	394.88	63.18
L5	413.10	121.5
L6	273.38	77.76

It follows from the information flow in Figure 1 that the communication topology can be represented by the following matrix:

$$S = \left[\begin{array}{c|ccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad (24)$$

Clearly, if the communication network is kept to be constant, then its sequence is sequentially complete.

Suppose that the expected disturbance is that: all loads decrease 10% active power and reactive power on their normal basis at 0s, and increase 20% on their normal basis at 3.5s. Figure 2 plots the dynamical responses of the proposed distributed control and the PVs' outputs. It follows from Figure 2 that the distributed control guarantees that the active power outputs of PVs converge to the uniform ratio. Thus, the requirement of problem 1 is satisfied.

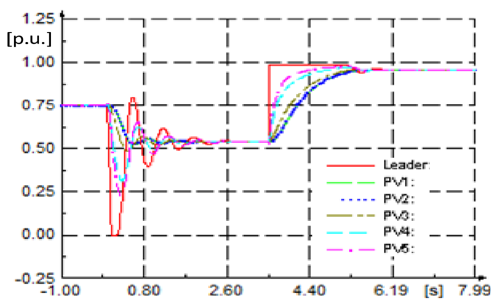

Figure 2 Active power output ratios

V. Conclusion

A distributed control scheme is provided for the power output control for a group of PVs in the distribution network. This method has the advantage of robust communication topology. Simulations based on a radial distribution network show the validity of the proposed method. The proposed design methodology is also applicable to distribution networks with different types of DGs including solar-, wind- and ocean-energy power generators.

Further research should be done on the uncertain parameter issue in the dynamical models. How to use the distributed H-∞ control to solve this problem is our future work.

Reference

[1] F. Katiraei,M. Iravani, "Power Management Strategies for a Microgrid with Multiple Distributed Generation Units," *IEEE Transactions on Power Systems,* vol. 21, no. 4, pp. 1821-1831, Nov 2006.

[2] A. Woyte, V. Van Thong, R. Belmans,J. Nijs, "Voltage Fluctuations on Distribution Level Introduced by Photovoltaic Systems," *IEEE Transactions on Energy Conversion,* vol. 21, no. 1, pp. 202-209, Mar 2006.

[3] D. Gan, R. Thomas,R. Zimmerman, "Stability-Constrained Optimal Power Flow," *IEEE Transactions on Power Systems,* vol. 15, no. 2, pp. 535-540, May 2000.

[4] H. Xin, D. Gan, Z. Huang, K. Zhang, and L. Cao, "Applications of Stability-Constrained Optimal Power Flow in the East China System," *IEEE Transaction on Power Systems,* vol. 25, no. 3, pp. 1423-1433, 2010.

[5] A. Tsikalakis,N. Hatziargyriou, "Centralized Control for Optimizing Microgrids Operation," *IEEE Transactions on Energy Conversion,* vol. 23, no. 1, pp. 241-248, Mar 2008.

[6] G. Diaz, C. Gonzaalez-Moran, J. Gomez-Aleixandre,A. Diez, "Complex-Valued State Matrices for Simple Representation of Large Autonomous Microgrids Supplied by Pq and V F Generation," *IEEE Transactions on Power Systems,* vol. 24, no. 4, pp. 1720-1730, Nov 2009.

[7] R. Majumder, B. Chaudhuri, A. Ghosh, R. Majumder, G. Ledwich, and F. Zare, "Improvement of Stability and Load Sharing in an Autonomous Microgrid Using Supplementary Droop Control Loop," *IEEE Transactions on Power Systems,* vol. 25, no. 2, pp. 796-808, May 2010.

[8] E. Barklund, N. Pogaku, M. Prodanovic, C. Hernandez-Aramburo, and T. C. Green, "Energy Management in Autonomous Microgrid Using Stability-Constrained Droop Control of Inverters," *IEEE Transactions on Power Electronics,* vol. 23, no. 5, pp. 2346-2352, Sep 2008.

[9] Z. Qu, J. Wang,R. Hull, "Cooperative Control of Dynamical Systems with Application to Autonomous Vehicles," *IEEE Transactions on Automatic Control,* vol. 53, no. 4, pp. 894-911, May 2008.

[10] Z. Qu, *Cooperative Control of Dynamical Systems.* London: Springer, 2009.

[11] J. Choi, S. Oh,R. Horowitz, "Distributed Learning and Cooperative Control for Multi-Agent Systems," *Automatica,* vol. 45, no. 12, pp. 2802-2814, Dec 2009.

[12] T. Kim,T. Sugie, "Cooperative Control for Target-Capturing Task Based on a Cyclic Pursuit Strategy," *Automatica,* vol. 43, no. 8, pp. 1426-1431, Aug 2007.

[13] H. Xin, Z. Qu, J. Seuss,A. Maknouninejad, "A Self Organizing Strategy for Power Flow Control of Photovoltaic Generators in a Distribution Network," *IEEE Transactions on Power systems,* vol. Early Access, 2011.

[14] C. Li, Z. Qu, A. Das,F. Lewis, "Cooperative Control with Improved Network Connectivity," presented at the American Control Conference, Baltimore, 2010.

[15] B. Shen, Z. D. Wang,Y. S. Hung, "Distributed H-Infinity-Consensus Filtering in Sensor Networks with Multiple Missing Measurements: The Finite-Horizon Case," *Automatica,* vol. 46, no. 10, pp. 1682-1688, Oct 2010.

[16] H. En, "On Some New Discrete Inequalities of the Bellman-Bihari Type," *Nonlinear Analysis-Theory Methods & Applications,* vol. 7, no. 11, pp. 1237-1246, 1983.

Appendix

Lemma 2. Suppose that $D(t_k)$ is row-stochastic matrix for every k. The sequence $\{D(t_0), D(t_1), D(t_2),...\}$ is sequentially complete, then there exist a constant $\mu \in (0,1)$ and an integer $\kappa > 0$ such that:

$$\lambda\left(P_{k+\kappa:k}\right) = \lambda\left(\prod_{i=k}^{k+\kappa} P_i\right) \le \mu\ ,\quad \forall k \ge 0 \tag{25}$$

$$\lambda(E) \le 1\ ,\quad \delta(E) \le 1 \tag{26}$$

$$\delta(EF) \le \lambda(E)\delta(F)\ ,\quad \lambda(EF) \le \lambda(E)\lambda(F) \tag{27}$$

are satisfied uniformly for all $k > 0$, where

$$P_i \triangleq e^{-(I+D_i)(t_{i+1}-t_i)} \tag{28}$$

$$\lambda(E) = 1 - \min_{1\le i_1, i_2 \le n} \sum_{j=1}^{n} \min(e_{i_1 j}, e_{i_2 j}) \tag{29}$$

$$\delta(E) = \max_{1\le j\le n} \max_{1\le i_1, i_2 \le n} \left|e_{i_1 j} - e_{i_2 j}\right| \tag{30}$$

where $E = (e_{i_1 j})$, both E and F are row-stochastic matrices with appropriate dimensions.

Moreover, there exists a $\Delta T_0 > 0$ such that

$$t_{k+\kappa} - t_k \ge \Delta T_0 \tag{31}$$

is uniformly satisfied for every $k \ge 0$.

Proof: The results (25)-(27) can be found in [10] (lemma 4.41). The (25) implies that (31) is satisfied, otherwise, there is $t_{(k+1)\kappa} - t_{k\kappa} \to 0$, so $t_{i+1} - t_i \to 0$ and $P_i \to I$ for all i. The definition in (29) implies that $\lambda\left(P_{k+\kappa:k}\right) \to 1 > \mu$ is satisfied. It is contradictory to (25). □

Lemma 3. Consider a time-varying system as follows:

$$\begin{cases} \dot{\mathbf{x}} = \varepsilon \mathbf{A}_{11}\mathbf{x} + \varepsilon \mathbf{A}_{12}\mathbf{z} \\ \dot{\mathbf{z}} = D_{*0}(t)\mathbf{c}^{\mathrm{T}}\mathbf{x} + (-\mathbf{I} + \mathbf{D}(t))\mathbf{z} \end{cases} \tag{32}$$

where $x \in R^{n_1}$, $z \in R^{n_2}$ are the states; $\mathbf{c} \in R^{n_1\times 1}$, $A_{11} \in R^{n_1\times n_1}$ and $A_{12} \in R^{n_1\times n_2}$ are constant; $\varepsilon > 0$ is a small constant; $\mathbf{I} \in R^{n_2\times n_2}$ is the identity; $D_{*0}(t) \in R^{n_2\times 1}$ and $\mathbf{D}(t) \in R^{n_2\times n_2}$ are non-negative piecewise continuous.

Suppose that the following conditions are satisfied:

1) $\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{1}\mathbf{c}^{\mathrm{T}}$ are Hurwitz;

2) The expanded matrix $\mathbf{D}_e = [D_{*0}, \mathbf{D}]$ is row-stochastic, piecewise continuous, which satisfies

$$\mathbf{D}_{e,k} \triangleq \mathbf{D}_e(t_k) = [D_{*0}(t_k), \mathbf{D}(t_k)]\ ,\quad t \in [t_k, t_{k+1})\ ,\quad k = 0,1,2...$$

and sequence $\{\mathbf{D}_{e,0}, \mathbf{D}_{e,1},...\}$ is sequentially complete.

Then there exists an $\varepsilon_0 > 0$ such that system (32) is asymptotically stable for every $\varepsilon \in (0, \varepsilon_0)$.

Proof: Perform a coordinate transformation as follows:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{z}_f \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{1}\mathbf{c}^{\mathrm{T}} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix}$$

Since the matrix $\mathbf{D}_e = [D_{*0}, \mathbf{D}]$ is row-stochastic, there is

$$D_{*0}\mathbf{c}^{\mathrm{T}}\mathbf{x} + (-\mathbf{I} + \mathbf{D})\mathbf{1}\mathbf{c}^{\mathrm{T}}\mathbf{x} = \left(\mathbf{D}_e\mathbf{1} - \mathbf{1}\right)\mathbf{c}^{\mathrm{T}}\mathbf{x} = \mathbf{0}$$

Thus, in the coordinate $[\mathbf{x}, \mathbf{z}_f]$, (32) can be rewritten as

$$\dot{\mathbf{x}} = \varepsilon(\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{1}\mathbf{c}^{\mathrm{T}})\mathbf{x} + \varepsilon \mathbf{A}_{12}\mathbf{z}_f \tag{33}$$

$$\dot{\mathbf{z}}_f = (-\mathbf{I} + \mathbf{D})\mathbf{z}_f + \varepsilon \mathbf{w} \tag{34}$$

where $\mathbf{w} = -\mathbf{1}\mathbf{c}^{\mathrm{T}}\left[(\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{1}\mathbf{c}^{\mathrm{T}})\mathbf{x} + \mathbf{A}_{21}\mathbf{z}_f\right]$.

Let

$$\tilde{\mathbf{D}}_k \triangleq \tilde{\mathbf{D}}(t_k) = \begin{bmatrix} 1 & 0 \\ \mathbf{D}_{*0}(t_k) & \mathbf{D}(t_k) \end{bmatrix},\quad \tilde{\mathbf{I}} \triangleq \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{I} \end{bmatrix}$$

$$\tilde{\mathbf{w}} \triangleq \begin{bmatrix} 0 \\ \mathbf{w} \end{bmatrix},\quad \tilde{\mathbf{z}}_f \triangleq \begin{bmatrix} z_{f,0} \\ \mathbf{z}_f \end{bmatrix}$$

The solution of system (34) can be calculated by:

$$\begin{cases} \dot{\tilde{\mathbf{z}}}_f = (-\tilde{\mathbf{I}} + \tilde{\mathbf{D}})\tilde{\mathbf{z}}_f + \varepsilon\tilde{\mathbf{w}} \\ \tilde{\mathbf{z}}_{f,0}(0) = 0 \end{cases} \tag{35}$$

The sequential completeness of $\{\mathbf{D}_{e,0}, \mathbf{D}_{e,1},...\}$ implies that $\{\tilde{\mathbf{D}}_0, \tilde{\mathbf{D}}_1,...\}$ is also sequentially complete. Thus by lemma 2 there exists a $\mu \in (0,1)$ and an integer $\kappa > 0$ such that:

$$\lambda\left(\prod_{i=k}^{k+\kappa} e^{(-\tilde{\mathbf{I}}+\tilde{\mathbf{D}}_i)(t_{i+1}-t_i)}\right) \le \mu\ ,\quad \forall k \ge 0 \tag{36}$$

where λ is defined in lemma 2.

Let

$$N = \lfloor 1/\varepsilon \rfloor \tag{37}$$

$$y(m) \triangleq \max_{t\in[t_{mN\kappa}, t_{(m+1)N\kappa})} \max_{1\le i_1, i_2 \le n} \{\tilde{z}_{f,i_1}(t) - \tilde{z}_{f,i_2}(t)\}\ ,\quad m \ge 0 \tag{38}$$

$$y'(m) \triangleq \max_{t\in[t_{mN\kappa}, t_{(m+1)N\kappa})} \| x(t) \|_\infty\ ,\quad m \ge 0 \tag{39}$$

where $\lfloor 1/\varepsilon \rfloor$ is the maximum integer less than $1/\varepsilon$.

Clearly, there exists two positive constants β_1 and β_2 such that

$$\| \varepsilon\tilde{\mathbf{w}} \|_\infty \le \varepsilon\beta_1 \| \mathbf{x} \|_\infty + \varepsilon\beta_2 \| \mathbf{z}_f \|_\infty \le \varepsilon\beta_1 y'(l) + \varepsilon\beta_2 y(l) \tag{40}$$

Consider $t \in [t_{lN\kappa}, t_{(l+1)N\kappa})$. The solution of (35) satisfies

$$\begin{aligned} \tilde{\mathbf{z}}_f(t) &= \varphi(t,0)\tilde{\mathbf{z}}_f(0) + \int_0^t \varphi(t,\tau)\tilde{\mathbf{w}}(\tau)d\tau \\ &= \varphi(t, t_{lN\kappa}) \prod_{i=0}^{l-1}\prod_{j=0}^{N-1} \varphi(t_{(iN+j+1)\kappa}, t_{(iN+j)\kappa})\mathbf{z}_f(0) \\ &\quad + \int_{t_{lN\kappa}}^{t} \varphi(t,\tau)\tilde{\mathbf{w}}(\tau)d\tau + \sum_{i=0}^{l-1}\sum_{j=0}^{N-1} \int_{t_{(iN+j)\kappa}}^{t_{(iN+j+1)\kappa}} \varphi(t,\tau)\tilde{\mathbf{w}}(\tau)d\tau \end{aligned} \tag{41}$$

where $\varphi(t,0) = e^{(-\tilde{\mathbf{I}}+\tilde{D})t}$.

Substituting (40) into (41) and considering (26)-(27) and (36), we have

$$\begin{aligned} &\max_{i,j}\left\{\left|\tilde{z}_{f,i}(t) - \tilde{z}_{f,j}(t)\right|\right\} \\ &\le \mu^{lN}\left\|\tilde{z}_f(0)\right\|_\infty + \int_{t_{lN\kappa}}^{t} \lambda(\varphi(t,\tau))\left\|\tilde{w}(\tau)\right\|_\infty d\tau \\ &\quad + \sum_{i=0}^{l-1}\sum_{i=0}^{N-1}\int_{t_{(iN+i)\kappa}}^{t_{(iN+j+1)\kappa}} \lambda(\varphi(t_{lN\kappa},\tau))\left\|\tilde{w}(\tau)\right\|_\infty d\tau \\ &\le \mu^{Nl}\left\|\tilde{z}_f(0)\right\|_\infty + \varepsilon\left(\beta_1 y'(l) + \beta_2 y(l)\right) \\ &\quad + \frac{\varepsilon\kappa\mu}{1-\mu}\sum_{i=0}^{l-1}\left[\mu^{N(l-i-1)}\left(\beta_1 y'(i) + \beta_2 y(i)\right)\right] \end{aligned} \tag{42}$$

Since expression (42) is satisfied for all $t \in [t_{lN\kappa}, t_{(l+1)N\kappa})$, it follows that

$$\begin{aligned} y(l) &\le \mu^{Nl}\left\|\tilde{z}_f(0)\right\|_\infty + \varepsilon\left(\beta_1 y'(l) + \beta_2 y(l)\right) \\ &\quad + \frac{\varepsilon\kappa\mu}{1-\mu}\sum_{i=0}^{l-1}\left[\mu^{N(l-i-1)}\left(\beta_1 y'(i) + \beta_2 y(i)\right)\right] \end{aligned}$$

$$= a_0\bar{\mu}^l + \varepsilon\left(\beta_1 y'(l) + \beta_2 y(l)\right) + \varepsilon\alpha_3\sum_{i=0}^{l-1}\left[\bar{\mu}^{(l-i-1)}\left(\beta_1 y'(i) + \beta_2 y(i)\right)\right] \quad (43)$$

where $\bar{\mu} = \mu^N$, $a_0 = \|\tilde{z}_f(0)\|_\infty$, $\alpha_3 = \frac{\kappa\mu}{1-\mu}$.

Next we perform the similar skills for the solution of (33). When $t \in [t_{lN\kappa}, t_{(l+1)N\kappa})$, the solution of (33) is:

$$\mathbf{x}(t) = e^{\varepsilon(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{1c}^{\mathrm{T}})t}\mathbf{x}(0) + \varepsilon\int_0^t e^{\varepsilon(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{1c}^{\mathrm{T}})(t-\tau)}\mathbf{A}_{12}\mathbf{z}_f(\tau)d\tau$$

$$= e^{\varepsilon(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{1c}^{\mathrm{T}})t}\mathbf{x}(0) + \varepsilon\int_{t_{lNk}}^t e^{\varepsilon(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{1c}^{\mathrm{T}})(t-\tau)}\mathbf{A}_{12}\mathbf{z}_f(\tau)d\tau + \sum_{i=0}^{l-1}\sum_{j=0}^{N-1}\int_{t_{(iN+j)\kappa}}^{t_{(iN+j+1)\kappa}} e^{\varepsilon(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{1c}^{\mathrm{T}})(t-\tau)}\mathbf{A}_{12}\mathbf{z}_f(\tau)d\tau \quad (44)$$

Since $\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{1c}^{\mathrm{T}}$ are Hurwitz, there exist constants $\beta_3 > 0$ and $\beta_4 > 0$ such that

$$\left\|e^{(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{1c}^{\mathrm{T}})t}\right\|_\infty \le \beta_3 e^{-\beta_4 t} \quad (45)$$

It follows from lemma 2 that there exists a $\Delta T_0 > 0$ such that $t_{l\kappa} - t_{i\kappa} \ge (l-i)\Delta T_0$ is satisfied. In addition, it follows from the definition in (37) that $\varepsilon N \ge 1$ is satisfied. Thus there exists a $\mu' \triangleq e^{-\beta_4\Delta T_0} \in (0,1)$ such that

$$\left\|e^{\varepsilon(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{1c}^{\mathrm{T}})(t_{lN\kappa}-t_{iN\kappa})}\right\|_\infty \le \beta_3 e^{-\varepsilon\beta_4(t_{lN\kappa}-t_{iN\kappa})} \le \beta_3 e^{-\varepsilon N\beta_4(l-i)\Delta T_0} \le \beta_3\mu'^{l-i} \quad (46)$$

is satisfied for every pair of $\{i,l\}$ ($l \ge i$).

It follows from (44) and (46) that

$$\|\mathbf{x}(t)\|_\infty \le \left\|e^{\varepsilon(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{1c}^{\mathrm{T}})t}\right\|_\infty \|\mathbf{x}(0)\|_\infty + \varepsilon\int_{t_{lNk}}^t \left\|e^{\varepsilon(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{1c}^{\mathrm{T}})(t-\tau)}\right\|_\infty \left\|\mathbf{A}_{12}\mathbf{z}_f(\tau)\right\|_\infty d\tau + \varepsilon\sum_{i=0}^{l-1}\sum_{j=0}^{N-1}\int_{t_{(iN+j)\kappa}}^{t_{(iN+j+1)\kappa}} \left\|e^{\varepsilon(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{1c}^{\mathrm{T}})(t-\tau)}\right\|_\infty \left\|\mathbf{A}_{12}\mathbf{z}_f(\tau)\right\|_\infty d\tau$$

$$\le \beta_3\mu'^l\|\mathbf{x}(0)\|_\infty + \varepsilon y(l)\beta_3\|\mathbf{A}_{12}\|_\infty + \frac{\varepsilon\mu'\beta_3\kappa\|\mathbf{A}_{12}\|_\infty}{1-\mu'}\sum_{i=0}^{l-1}\left[y(i)\mu'^{\varepsilon[(l-i)N-1]}\mu'^{\varepsilon(-N+1)}\right]$$

$$\triangleq a'_0\mu'^l + \alpha'_2 y(l) + \varepsilon\alpha'_3\sum_{i=0}^{l-1}\mu'^{l-i-1}y(i) \quad (47)$$

where $a'_0 = \beta_3\|\mathbf{x}(0)\|_\infty$, $\alpha'_2 = \beta_3\|\mathbf{A}_{12}\|_\infty$, $\alpha'_3 = \frac{\mu'\beta_3\kappa\|\mathbf{A}_{12}\|_\infty}{1-\mu'}$.

Since (47) is satisfied for all $t \in [t_{lN\kappa}, t_{(l+1)N\kappa})$, we have

$$y'(l) \le a'_0\mu'^l + \varepsilon\alpha'_2 y(l) + \varepsilon\alpha'_3\sum_{i=0}^{l-1}\mu'^{l-i-1}y(i) \quad (48)$$

Since we consider that $\varepsilon > 0$ is small, N is a large integer and $\bar{\mu} < \mu'$. Thus, simultaneously considering (43) and (48), we have

$$\max\{y(l), y'(l)\} \le \max\{a_0, a'_0\}\mu'^l + \varepsilon\max\{\beta_1+\beta_2, \alpha'_2\}\max\{y(l), y'(l)\} + \varepsilon\max\{\alpha_3\beta_1+\alpha_3\beta_2, \alpha'_3\}\sum_{i=0}^{l-1}\mu'^{l-i-1}\max\{y(i), y'(i)\}$$

i.e.,

$$\mu'^{-l}\bar{y}(l) \le \bar{a}_0 + \varepsilon\mu'^{-1}\bar{M}_2\sum_{i=0}^{l-1}\mu'^{-i}\bar{y}(i) \quad (49)$$

where $\bar{M}_1 = \max\{\beta_1+\beta_2, \alpha'_2\}$, $\bar{a}_0 = \left(1-\varepsilon\bar{M}_1\right)^{-1}\max\{a_0, a'_0\}$, $\bar{y} = \max\{y, y'\}$, $\bar{M}_2 = \left(1-\varepsilon\bar{M}_1\right)^{-1}\max\{\alpha_3\beta_1+\alpha_3\beta_2, \alpha'_3\}$.

Applying the Gronwall-Bellman [16] inequality into (49), we have

$$\bar{y}(l) \le \bar{a}_0\mu'^l\left(\frac{1+\varepsilon\mu'^{-1}\bar{M}_2}{1-\varepsilon\bar{M}_1}\right)^l \quad (50)$$

Clearly, it follows from (50) that if $\varepsilon > 0$ is small enough then $\bar{y}(l) \to 0$ is satisfied as $l \to \infty$. Therefore, system (32) is asymptotically stable in turn. □