2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

$\mathcal{L}_1$ Adaptive Controller for Nonlinear Reference Systems

Xiaofeng Wang and Naira Hovakimyan

***Abstract*— In this paper, we consider nonlinear affine-in-control systems and present the $\mathcal{L}_1$ adaptive controller for the case when the performance specifications are defined via a *nonlinear system* of similar structure. The $\mathcal{L}_1$ adaptive controller ensures that the nonlinear, affine-in-control, uncertain system follows its ideal model during the transient and steady-state, if the adaptation gain is selected sufficiently large and the bandwidth of the low-pass filter is adjusted appropriately. Simulations verify the theoretical results.**

I. INTRODUCTION

We consider a class of uncertain, nonlinear, affine-in-control systems and present the $\mathcal{L}_1$ adaptive controller for it. The control objective is to compensate for uncertainties in the system dynamics, while retaining the essential nonlinearities of the system. This problem formulation is motivated by power grids and voltage balance problems. The nominal system behavior in these applications is highly nonlinear, which motivates further development of $\mathcal{L}_1$ adaptive controller to accommodate the desired nonlinear reference behavior.

Adaptive controllers for nonlinear systems have been developed by resorting to neural networks for approximation of nonlinearities, [1], [2]. Such approximations have enabled the use of parameter update laws from adaptive control literature with local domains of attraction. The $\mathcal{L}_1$ adaptive controller in this paper helps to obtain semiglobal results, with uniform transient and steady-state performance bounds. These performance bounds are decoupled into two distinct terms, which can be adjusted independently by increasing the rate of adaptation and the bandwidth of the low-pass filter. The nonlinear nature of the desired reference system necessitates development of a new mathematical machinery, which constitutes the main contribution of this paper.

This paper is organized as follows. Section II gives the problem formulation. Section III presents the $\mathcal{L}_1$ adaptive controller for nonlinear control-affine systems. Performance bounds are analyzed in Sections IV and V. Simulation results are given in Section VI. Conclusions are in Section VII. All the proofs are in Appendix.

Notation: We denote by $\mathbb{R}^n$ the n-dimensional real vector space and by $\mathbb{R}^+$ the real positive numbers. We also use $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$. $\|\cdot\|$ is the 2-norm of a vector. $\|\cdot\|_{\mathcal{L}_1}$ is the $\mathcal{L}_1$ norm of a linear system and and $\|\cdot\|_{\mathcal{L}_\infty}$ is the $\mathcal{L}_\infty$ norm of a function. The truncated $\mathcal{L}_\infty$ norm of a function $x : \mathbb{R}_0^+ \to \mathbb{R}^n$ is defined as $\|x\|_{\mathcal{L}_\infty^{[0,\tau]}} = \sup_{0\le t\le \tau}\|x(t)\|$. Also, e is used for exponential function to distinguish it from the tracking error e. The Laplace transform of a function $x(t)$ is denoted by $x(s) = \mathfrak{L}[x(t)]$. The inverse Laplace transform of $x(s)$ is denoted as $\mathfrak{L}^{-1}[x(s)]$.

Both authors are with the Department of Mechanical Science and Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801; e-mail: wangx,nhovakim@illinois.edu. Research is supported by AFOSR under Contract No.FA9550-09-1-0265.

II. PROBLEM FORMULATION

We consider the single-input system:

$$\dot{x} = f(t,x) + g(t,x)(u + h(t,x)), \quad x(0) = x_0, \tag{1}$$

where $x : \mathbb{R}_0^+ \to \mathbb{R}^n$ is the state trajectory, $u : \mathbb{R}_0^+ \to \mathbb{R}$ is the control input, $f : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^n$ are known functions, and $h : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}$ is an unknown function. Let the performance specifications be given via the following *ideal system*

$$\dot{x}_{\rm ideal} = f_m(t, x_{\rm ideal}), \quad x_{\rm ideal}(0) = x_0\,,$$

where $f_m(t,x) = f(t,x) + g(t,x)k(t,x)$ is associated with the *ideal feedback* $u_{\rm ideal} = k(t,x_{\rm ideal}) - h(t,x_{\rm ideal})$, with $k(t,x)$ being piecewise continuous in t and Lipschitz continuous in x. The ideal controller depends upon the unknown $h(t,x)$ and is therefore not implementable. The objective is to design a state feedback controller, using only known information, to ensure that the state $x(t)$ of the real system in (1) follows the state $x_{\rm ideal}(t)$ of the ideal system in (2) with *uniform* and *quantifiable* performance bounds.

Assumption 1: $f_m(t,x)$, $\frac{\partial f_m}{\partial x}$, $g(t,x)$, and $h(t,x)$ are continuous, bounded, and Lipschitz in x, uniformly in t, for all $t \in \mathbb{R}_0^+$ and all x in any compact set.

Assumption 2: $\frac{\partial g}{\partial t}$, $\frac{\partial g}{\partial x}$, $\frac{\partial h}{\partial t}$ and $\frac{\partial h}{\partial x}$ are bounded for all $t \in \mathbb{R}_0^+$ and all x in any compact set.

Assumption 3: There exists a function $\psi : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^{1\times n}$, such that $\psi(t,x)g(t,x) \equiv 1$. Moreover, ψ, $\frac{\partial \psi}{\partial t}$, $\frac{\partial \psi}{\partial x}$ are bounded for all $t \in \mathbb{R}_0^+$ and all x in any compact set.

Assumption 4: There exist positive constants γ, c_1, c_2, c_3, c_4, and a positive definite function $V : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}_0^+$, such that for all $t \ge 0$ and all $e \in \{e \in \mathbb{R}^n \mid \|e\| \le \gamma\}$, the following inequalities hold:

$$c_1\|e\|^2 \le V(t,e) \le c_2\|e\|^2 \tag{2}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial e} f_m(t,e) \le -c_3\|e\|^2 \tag{3}$$

$$\left\|\tfrac{\partial V}{\partial e}\right\| \le c_4\|e\|, \quad \left\|\tfrac{\partial^2 V}{\partial^2 e}\right\| \le c_5, \quad \left\|\tfrac{\partial V}{\partial e \partial t}\right\| \le c_6\|e\|. \tag{4}$$

Assumption 5: There exist positive constants d_1, d_2, d_3, and a positive definite function $W : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}_0^+$, such that for all $t \ge 0$ and all $x \in \{x \in \mathbb{R}^n \mid \|x\| \le \sqrt{\frac{d_2}{d_1}}\|x_0\|\}$:

$$d_1\|x\|^2 \le W(t,x) \le d_2\|x\|^2 \tag{5}$$

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial x} f_m(t,x) \le -d_3\|x\|^2 \tag{6}$$

$$\left\|\frac{\partial W}{\partial x}\right\| \le B^{\partial W}. \tag{7}$$

Assumption 5 ensures that the ideal system in (2) is asymptotically stable. Let

$$\rho_{\mathrm{ref}} = \sqrt{\frac{d_2}{d_1}}\|x_0\| + \varepsilon \quad \text{and } \rho = \gamma + \rho_{\mathrm{ref}}, \tag{8}$$

where $\varepsilon > 0$ is an arbitrarily small constant. One can straightforwardly verify that $d_1 \rho_{\mathrm{ref}}^2 > W(0, x_0)$.

With Assumptions 1 – 3, we have that for all $t \geq 0$ and all $x_1, x_2, x \in \{x \in \mathbb{R}^n \mid \|x\| \leq \rho\}$:

$$\begin{aligned} \|\Xi(t, x_1) - \Xi(t, x_2)\| &\leq L_\rho^{\Xi}\|x_1 - x_2\| && (9)\\ \|\Psi(t, x)\| &\leq B_\rho^{\Psi}, && (10) \end{aligned}$$

where $\Psi \in \left\{f_m, g, h, \psi, \frac{\partial h}{\partial t}, \frac{\partial h}{\partial x}, \frac{\partial g}{\partial t}, \frac{\partial g}{\partial x}, \frac{\partial \psi}{\partial t}, \frac{\partial \psi}{\partial x}\right\}$ and $\Xi \in \left\{f_m, \frac{\partial f_m}{\partial x}, g, h\right\}$. We assume that for the uncertainty h, the parameters L_ρ^h, B_ρ^h, $B_\rho^{\frac{\partial h}{\partial t}}$, $B_\rho^{\frac{\partial h}{\partial x}}$ are known.

Assumption 6: The constant γ from Assumption 4 also verifies $2L_\rho^{\frac{\partial f_m}{\partial x}}\gamma < \frac{c_1 c_3}{c_2 c_4}$.

Remark 1: Assumption 6 restricts the relationship between the achievable performance bound, given by γ, and the rate of change of the nonlinearity of the desired ideal system, given by $f_m(x)$. For linear systems, this condition is trivially satisfied, since $L_\rho^{\frac{\partial f_m}{\partial x}} \equiv 0$. For nonlinear systems, Assumption 6 can be ensured by choosing γ small enough.

III. $\mathcal{L}_1$ Adaptive Controller

A. Definitions

The design of $\mathcal{L}_1$ controller involves a strictly proper and stable low-pass filter $C(s)$, subject to $C(0) = 1$. Let the bandwidth of it be ω. For simplicity, let $C(s) = \frac{\omega}{s+\omega}$. Further, let

$$\begin{aligned} &B_\rho^{\dot{x}} = B_\rho^{f_m} + (1 + \|C(s)\|_{\mathcal{L}_1}) B_\rho^g B_\rho^h\\ &B_\rho^{\dot{g}} = B_\rho^{\frac{\partial g}{\partial t}} + B_\rho^{\frac{\partial g}{\partial x}} B_\rho^{\dot{x}},\\ &B^a = \tfrac{c_3}{c_2} - (2L_\rho^{\frac{\partial f_m}{\partial x}}\gamma + L_\rho^{\frac{\partial f_m}{\partial x}}\rho_{\mathrm{ref}})\tfrac{c_4}{c_1}\\ &M = c_5 B_\rho^g (B_\rho^g L_\rho^h + L_\rho^g B_\rho^h)\|1 - C(s)\|_{\mathcal{L}_1} + c_6 B_\rho^g\\ &\qquad + c_5 B_\rho^g \left(L_\gamma^{f_m} + L_\rho^{\frac{\partial f_m}{\partial x}}(2\gamma + \rho_{\mathrm{ref}})\right) + c_4 B_\rho^{\dot{g}}\\ &B_{\rho_{\mathrm{ref}}}^{\dot{h}_{\mathrm{ref}}} = B_\rho^{\frac{\partial h}{\partial t}} + B_\rho^{\frac{\partial h}{\partial x}}(B_{\rho_{\mathrm{ref}}}^{f_m} + \|1 - C(s)\|_{\mathcal{L}_1} B_{\rho_{\mathrm{ref}}}^g B_\rho^h), \end{aligned} \tag{11}$$

where $B_\rho^{f_m}$, B_ρ^g, B_ρ^h, $B_\rho^{\frac{\partial g}{\partial t}}$, $B_\rho^{\frac{\partial g}{\partial x}}$, $B_\rho^{\frac{\partial h}{\partial t}}$, $B_\rho^{\frac{\partial h}{\partial x}}$, $L_\rho^{\frac{\partial f_m}{\partial x}}$ are defined in (9) and (10).

Suppose that Assumptions 1- 6 hold. Given any initial condition x_0, let ω verify the following inequalities

$$\rho_{\mathrm{ref}}^2 \geq \tfrac{W(0, x_0)}{d_1} + \tfrac{B^{\partial W} B_{\rho_{\mathrm{ref}}}^g d_2}{d_1}\left(\tfrac{\|h(0, x_0)\|}{|d_3 - d_2\omega|} + \tfrac{B_{\rho_{\mathrm{ref}}}^{\dot{h}_{\mathrm{ref}}}}{d_3\omega}\right) \tag{12}$$

$$\mu = 2L_\rho^{\frac{\partial f_m}{\partial x}}\gamma + L_\rho^{\frac{\partial f_m}{\partial x}}\epsilon < \frac{c_1 c_3}{c_2 c_4} \tag{13}$$

$$\delta_1(\omega) + \delta_2(\omega) < c_1, \tag{14}$$

where T is the convergence time for the reference system to its ultimate bound (yet to be defined), and

$$\epsilon = \sqrt{\tfrac{e^{-\frac{d_3}{d_2}T} W(0, x_0)}{d_1} + \tfrac{B^{\partial W} B_{\rho_{\mathrm{ref}}}^g d_2}{d_1}\left(\tfrac{\|h(0, x_0)\|}{|d_3 - d_2\omega|} + \tfrac{B_{\rho_{\mathrm{ref}}}^{\dot{h}_{\mathrm{ref}}}}{d_3\omega}\right)} \tag{15}$$

$$\delta_1(\omega) = L_\rho^h \frac{c_4 B_\rho^g + \frac{\varrho}{\hat{\alpha}}(B^a c_4 B_\rho^g + M)}{\omega} \tag{16}$$

$$\delta_2(\omega) = \frac{\|h(0, x_0)\| c_4 L_\rho^g \varrho}{|\hat{\alpha} - \omega|} + \frac{B_{\rho_{\mathrm{ref}}}^{\dot{h}_{\mathrm{ref}}} c_4 L_\rho^g \varrho}{\hat{\alpha}\omega} \tag{17}$$

$$\hat{\alpha} = \frac{c_3}{c_2} - \mu\frac{c_4}{c_1} > 0, \quad \varrho = e^{\frac{c_4 L_\rho^{\frac{\partial f_m}{\partial x}} \rho_{\mathrm{ref}} T}{c_1}} \geq 1. \tag{18}$$

Note that these conditions can always be satisfied by choosing ω and T large enough. The physical meaning of these parameters will be further discussed later.

Remark 2: As seen in (15), the larger ω (bandwidth) is, the smaller T (convergence time) is required for the same ϵ (ultimate bound). It agrees with the intuition that faster bandwidth leads to faster convergence to the ultimate bound.

B. $\mathcal{L}_1$ Control Architecture

This section introduces the structure of $\mathcal{L}_1$ adaptive controller for the nonlinear system in (1). We consider the following feedback structure:

$$u(t) = k(t, x) + u_2(t), \tag{19}$$

where $u_2 : \mathbb{R}_0^+ \to \mathbb{R}$ is the adaptive signal.

To define the adaptive signal, consider the state predictor:

$$\dot{\hat{x}} = f_m(t, x) + g(t, x)(u_2 + \hat{\sigma}) + A_m\tilde{x} \tag{20}$$

with $\hat{x}(0) = x_0$, where $\hat{x} : \mathbb{R}_0^+ \to \mathbb{R}^n$ is the state of the state predictor, $\tilde{x} = \hat{x} - x$ is the prediction error, and $A_m \in \mathbb{R}^{n\times n}$ is any $n \times n$ Hurwitz matrix. Then, given arbitrary symmetric matrix $Q > 0$, there exists a symmetric matrix $P > 0$, such that

$$PA_m + A_m^\top P = -Q\ . \tag{21}$$

Let $H(s) = (s\mathbb{I} - A_m)^{-1}$. In equation (20), $\hat{\sigma} : \mathbb{R}_0^+ \to \mathbb{R}$ is updated according to the adaptive law

$$\dot{\hat{\sigma}}(t) = \Gamma \mathbf{Proj}_\Omega(\hat{\sigma}(t), -g(t, x(t))^\top P\tilde{x}(t)) \tag{22}$$

where $\Gamma \in \mathbb{R}^+$ is the adaptation gain, $\hat{\sigma}(0) \in \Omega$, $\Omega = \{\sigma \in \mathbb{R} \mid |\sigma| \leq B_\rho^h\}$, and B_ρ^h is in (10). The projection-based operator ensures $|\hat{\sigma}(t)| \leq B_\rho^h$ for $\forall t \geq 0$ and given $\hat{\sigma} \in \mathbb{R}$ and $\sigma \in \Omega$, $(\hat{\sigma} - \sigma)^\top(\mathbf{Proj}_\Omega(\hat{\sigma}, \sigma) - \sigma) \leq 0$ holds [3].

The adaptive feedback $u_2(s)$ is defined as follows

$$u_2(s) = -C(s)\hat{\sigma}(s). \tag{23}$$

The $\mathcal{L}_1$ controller is defined by (19) – (23), subject to conditions (12) – (14).

IV. Performance Analysis

Consider the following reference system:

$$\begin{aligned} \dot{x}_{\mathrm{ref}}(t) &= f_m(t, x_{\mathrm{ref}}) + g(t, x_{\mathrm{ref}})(-\eta_{\mathrm{ref}}(t) + h(t, x_{\mathrm{ref}}))\\ \eta_{\mathrm{ref}}(s) &= C(s)\mathfrak{L}[h(t, x_{\mathrm{ref}})] && (24)\\ u_{\mathrm{ref}}(t) &= k(t, x_{\mathrm{ref}}) - \eta_{\mathrm{ref}}(t), \quad x_{\mathrm{ref}}(0) = x_0. \end{aligned}$$

Similar to other architectures of $\mathcal{L}_1$ adaptive control theory, this reference system assumes only partial cancellation of uncertainties within the bandwidth of $C(s)$. The next lemma proves its stability.

Lemma 1: Let Assumptions 1, 2, 4, 5 hold. If the inequality in (12) holds, then $\|x_{\rm ref}\|_{\mathcal{L}_\infty} \le \rho_{\rm ref}$, and the system is uniformly ultimately bounded, i.e. $\|x_{\rm ref}(t)\| \le \epsilon, \quad \forall\, t \ge T$, where ϵ is defined in (15), and T satisfies (13) – (14).

The next lemma establishes the bound on the derivative of the system state.

Lemma 2: Consider the system in (1) with $\mathcal{L}_1$ adaptive controller, given by (19) – (23). Suppose that Assumptions 1 – 2 hold. If there exists $\rho \in \mathbb{R}^+$, such that $\|x(t)\| \le \rho$ holds for all $t \in [0, \tau]$, then

$$\|\dot{x}(t)\| \le B_\rho^{\dot{x}} \ \text{ and } \|\dot{h}(t, x(t))\| \le B_\rho^{\dot{h}} \qquad (25)$$

hold for all $t \in [0, \tau]$, where $B_\rho^{\dot{x}}$ is defined in (11) and $B_\rho^{\dot{h}} = B_\rho^{\frac{\partial h}{\partial t}} + B_\rho^{\frac{\partial h}{\partial x}} B_\rho^{\dot{x}}$.

Next we derive the bound for the prediction error. Let $\tilde{x}(t) = \hat{x}(t) - x(t)$. By equations (1) and (20), the error dynamics between the system and the predictor are given as follows:

$$\dot{\tilde{x}}(t) = A_m \tilde{x}(t) + g(t, x(t))\tilde{\sigma}(t), \quad \tilde{x}(0) = 0,$$

where $\tilde{\sigma}(t) = \hat{\sigma}(t) - h(t, x(t))$. With the bounds obtained in Lemma 2, we are able to derive the bound on $\|\tilde{x}\|_{\mathcal{L}_\infty^{[0,\tau]}}$. More importantly, with this bound, we can obtain the bound on $\|\tilde{\eta}\|_{\mathcal{L}_\infty^{[0,\tau]}}$, where $\tilde{\eta}(t)$ is the inverse Laplace transform of $C(s)\tilde{\sigma}(s)$.

Lemma 3: Consider the system (1) with the $\mathcal{L}_1$ adaptive controller in (19) – (23). Suppose that Assumptions 1 – 2 hold. If there exists $\rho \in \mathbb{R}^+$, such that $\|x(t)\| \le \rho$ holds for all $t \in [0, \tau]$, then $\|\tilde{x}\|_{\mathcal{L}_\infty^{[0,\tau]}} \le \frac{\alpha}{\sqrt{\Gamma}}$ and $\|\tilde{\eta}\|_{\mathcal{L}_\infty^{[0,\tau]}} \le \frac{\beta}{\sqrt{\Gamma}}$, where

$$\begin{aligned} \alpha &= \sqrt{\tfrac{4 {B_\rho^h}^2}{\lambda_{\min}(P)} + \tfrac{4\lambda_{\max}(P) B_\rho^h B_\rho^{\dot{h}}}{\lambda_{\min}(Q)\lambda_{\min}(P)}}, \\ \beta &= \left(\|C(s)s\|_{\mathcal{L}_1} B_\rho^\psi + \|C(s)\|_{\mathcal{L}_1} B_\rho^\psi \|A_m\|\right)\alpha \\ &\quad + (B_\rho^{\frac{\partial \psi}{\partial t}} + B_\rho^{\frac{\partial \psi}{\partial x}} B_\rho^{\dot{x}})\alpha. \end{aligned} \qquad (26)$$

Let $e(t) = x(t) - x_{\rm ref}(t)$. Consider the error dynamics between the real system and the reference system. By equations (1) and (24), we have

$$\dot{e} \ = \ f_m(t, e) + \Delta(t, e) + \Phi(t, x), \qquad (27)$$

where $\Delta(t, e) = f_m(t, x_{\rm ref} + e) - f_m(t, x_{\rm ref}) - f_m(t, e)$ and $\Phi(t, x) = g(t, x)\left(h(t, x) - \hat{\eta}(t)\right) - g(t, x_{\rm ref})\left(h(t, x_{\rm ref}) - \eta_{\rm ref}(t)\right)$.

The next lemma states the lower bound on the adaptation gain for obtaining a bound on the error $e(t)$.

Lemma 4: Consider the system (1) with the $\mathcal{L}_1$ adaptive controller in (19) – (23). Suppose that Assumptions 1 – 6 hold, and $\|x(t)\| \le \rho$ for $t \in [0, \tau]$, where ρ is defined in (8). Then, assuming that inequalities (12)–(14) hold, and the adaptation gain is selected large enough to verify

$$\frac{\varrho B_\rho^g \beta\left(\omega c_4 + L_\rho^h B_\rho^g c_5\right)}{\omega \hat{\alpha} \gamma (c_1 - \delta_1(\omega) - \delta_2(\omega))} < \sqrt{\Gamma}, \qquad (28)$$

where ϱ, β, $\hat{\alpha}$, δ_1, δ_2 are defined in (26), (18), (16), (17), respectively, we have $\|x_{\rm ref} - x\|_{\mathcal{L}_\infty^{[0,\tau]}} < \gamma$.

The next theorem summarizes the performance bounds.

Theorem 1: Consider the system (1) with the $\mathcal{L}_1$ adaptive controller in (19) – (23). Suppose that Assumptions 1 – 6 hold. Then, provided that inequalities (12)–(14) and (28) hold, the inequality $\|x_{\rm ref} - x\|_{\mathcal{L}_\infty} < \gamma$ holds. Moreover, if $k(t, x)$ is locally Lipschitz in x, uniformly in t, then $\|u_{\rm ref} - u\|_{\mathcal{L}_\infty} \le \gamma_u$, where $\gamma_u = \left(L_\rho^k + \|C(s)\|_{\mathcal{L}_1} L_\rho^h\right)\gamma + \frac{\beta}{\sqrt{\Gamma}}$, β is defined in (26), and L_ρ^k is the Lipschitz constant of $k(t, x)$ over the compact set $\{x \in \mathbb{R}^n \mid \|x\| \le \rho\}$.

Remark 3: The filter's bandwidth ω satisfies three conditions (12)-(14). Condition (12) is sufficient to establish the stability of the reference system. For the given performance bound γ the condition in (13) restricts the size of the ultimate bound ϵ on the reference system. This condition is used, together with the condition in (14), to establish the boundedness on the difference between the closed-loop system and the reference system.

Remark 4: We note that the conditions in (12) – (14) do not involve the $\mathcal{L}_1$ norm bound, similar to the ones used in [4]. Instead, the right hand side in (12) represents an upper bound for the $\mathcal{L}_1$ norm-like constraint that cannot be explicitly expressed due to the nonlinear nature of the desired ideal system behavior. For linear reference systems, the $\mathcal{L}_1$ norm conditions can be recovered by applying the analysis from this paper.

Remark 5: Note that γ and γ_u can be rendered arbitrarily small by increasing the adaptation gain Γ. If $C(s) = 1$, β in (26) will be unbounded because the term $\|C(s)s\|_{\mathcal{L}_1}$ is then equal to $\|s\|_{\mathcal{L}_1}$, which is unbounded. Consequently, the lower bound on the adaptation gain in (28) will be unbounded, thus eliminating the uniform bound for the control signal.

V. Design Analysis

Note that the closed-loop reference system in (24) depends upon the unknown nonlinearity h, which prevents prediction of its performance. Next, we consider the ideal system in (2). According to Assumption 5, it is asymptotically stable. The following theorem establishes the relation between the reference system and the ideal system.

Theorem 2: Consider the reference system in (24) and the ideal system in (2). Suppose that Assumptions 1 – 6 hold. If inequalities (12)–(14) hold, then there exist decreasing functions $\chi_1, \chi_2, \chi_3 : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and a positive constant a, independent of t, such that

$$\begin{aligned} \|x_{\rm ref} - x_{\rm ideal}\|_{\mathcal{L}_\infty} &\le \chi_1(\omega) \\ \|u_{\rm ref}(t) - u_{\rm ideal}(t)\| &\le \chi_2(\omega) + a\chi_3(\omega t), \quad \forall\, t \ge 0\,, \end{aligned}$$

where $\lim_{\omega \to \infty} \chi_i(\omega) = 0$ for $i = 1, 2, 3$ and $\chi_3(0) = 1$.

Proof: The proof is similar to the proof of Theorem 1 and is omitted. ■

Remark 6: The introduction of the reference system decouples adaptation from robustness. Fast adaptation ensures that the real system can be arbitrarily close to the reference system, while the selection of $C(s)$ provides the trade-off between performance and robustness [5]. Further, we notice that minimization of $\|x_{\mathrm{ref}} - x_{\mathrm{ideal}}\|_{\mathcal{L}_\infty}$ must be done with the consideration of the robustness requirements.

VI. An Illustrative Example

This section uses an example to illustrate how $\mathcal{L}_1$ adaptive controller works. We consider the Lorenz attractor:

$$\begin{aligned}\dot{x}_1 &= \sigma(x_2 - x_1)\\ \dot{x}_2 &= rx_1 - x_2 - x_1x_3 + (x_1^2 + x_2^2 + 0.1)(u + h(t,x))\\ \dot{x}_3 &= x_1x_2 - bx_3\,,\end{aligned}$$

where $\sigma = 1$, $r = 0.5$, $b = 1$. The ideal system corresponds to $u = -h(t,x)$. We choose the initial condition satisfying $\|x(0)\| \leq 2$. The adaptation gain is $\Gamma = 10^7$. The matrix A_m in the state predictor is $A_m = -\mathbb{I}$.

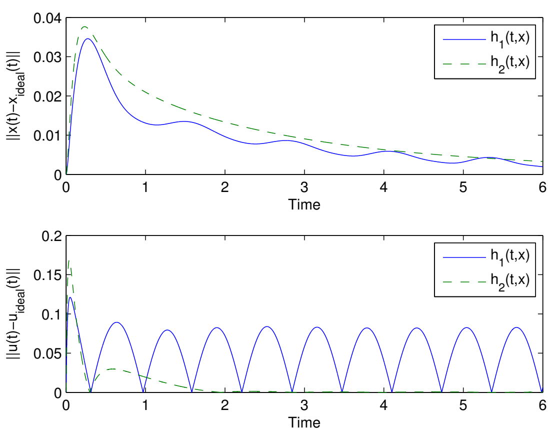

Fig. 1. Time history of the errors in the states and inputs with different uncertainties using the same $\mathcal{L}_1$ controller

We examine different uncertainties using the same $\mathcal{L}_1$ controller. Two different uncertainties are considered: $h_1(t,x) = 0.5\sin(3t)(x_1^2 + x_2^2 + x_3^2) + \sin(2t)$ and $h_2(t,x) = 3\sin(2t)\frac{x_1^3 + x_3}{\sqrt{x_2^2+1}}$. The controller remains the same with $\omega = 60$. The time histories of the errors in the states and the inputs of the real and the ideal systems are plotted in Figure 1. The control signal changes accordingly to ensure uniform transient response. We observe that the $\mathcal{L}_1$ adaptive controller guarantees smooth and uniform transient response without any retuning of the controller in the presence of different types of uncertainties and disturbances.

VII. Conclusion

This paper presents the $\mathcal{L}_1$ adaptive controller for stabilization of uncertain nonlinear systems. The performance specifications are defined via a different nonlinear system of similar structure. The $\mathcal{L}_1$ adaptive controller achieves guaranteed transient behavior and quantifiable performance bounds w.r.t the ideal system. Uniform bounds on the difference in the states and the inputs of the real, the reference system and the ideal systems are derived. Future papers will address unmodeled dynamics and output feedback.

References

[1] F. Lewis, "Neural network control of robot manipulators," *IEEE Expert*, vol. 11, no. 3, pp. 64–75, 1996.

[2] A. Calise, N. Hovakimyan, and M. Idan, "Adaptive output feedback control of nonlinear systems using neural networks* 1," *Automatica*, vol. 37, no. 8, pp. 1201–1211, 2001.

[3] J. Pomet and L. Praly, "Adaptive nonlinear regulation: estimation from the Lyapunovequation," *IEEE Transactions on Automatic Control*, vol. 37, no. 6, pp. 729–740, 1992.

[4] N. Hovakimyan and C. Cao, *$\mathcal{L}_1$ Adaptive Control Theory*. SIAM, 2010.

[5] C. Cao and N. Hovakimyan, "Stability margin of $\mathcal{L}_1$ adaptive control architecture," *IEEE Transactions on Automatic Control*, vol. 55, no. 2, pp. 480–487, 2010.

[6] W. Rugh, *Linear system theory*. Prentice Hall Upper Saddle River, NJ, 1996.

Appendix

Lemma 5: Consider a system

$$\begin{aligned}\dot{z}(t) &= a(t)z(t) + b(t)\upsilon(t)\\ \upsilon(s) &= (1 - C(s))\sigma(s), \quad z(0) = 0,\end{aligned} \tag{29}$$

where $z, \upsilon : \mathbb{R}_0^+ \to \mathbb{R}$ are the state and the input, $a : \mathbb{R}_0^+ \to \mathbb{R}$ is continuous and $b : \mathbb{R}_0^+ \to \mathbb{R}$ is differentiable, $C(s) = \frac{\omega}{s+\omega}$ is a low-pass filter, and $\sigma : \mathbb{R}_0^+ \to \mathbb{R}$ is a bounded input signal. Assume that there exist positive constants p_1, p_2, p_3, such that $|a(t)| \leq p_1$, $|b(t)| \leq p_2$, $|\dot{b}(t)| \leq p_3$ for all $t \in [0,\tau]$. Then the following inequality holds:

$$\begin{aligned}\|z\|_{\mathcal{L}_\infty^{[0,\tau]}} \leq \|\sigma\|_{\mathcal{L}_\infty^{[0,\tau]}} \cdot \\ \int_0^\tau \left(p_2 e^{-\omega t} + (p_1p_2 + p_3)\int_0^t e^{-\omega\lambda}\varphi(t,\lambda)d\lambda\right)dt,\end{aligned} \tag{30}$$

where $\varphi(t,\tau) \geq 0$ is the transition function of the system $\dot{z}(t) = a(t)z(t) + b(t)\upsilon(t)$. Moreover, if $\|\dot{\sigma}\|_{\mathcal{L}_\infty^{[0,\tau]}}$ is also bounded, then

$$\begin{aligned}\|z\|_{\mathcal{L}_\infty^{[0,\tau]}} \leq \|\sigma(0)\|p_2 \cdot \\ \int_0^\tau e^{-\omega t}\varphi(\tau,t)dt + \|\dot{\sigma}\|_{\mathcal{L}_\infty^{[0,\tau]}} p_2 \int_0^\tau \int_0^t e^{-\omega\lambda}\varphi(t,\lambda)d\lambda dt.\end{aligned} \tag{31}$$

Proof: Note that the system in (29) is a linear time-varying system. Therefore,

$$\|z\|_{\mathcal{L}_\infty^{[0,\tau]}} \leq \|\mathcal{H}\|_{\mathcal{L}_1^{[0,\tau]}} \|\sigma\|_{\mathcal{L}_\infty^{[0,\tau]}}, \tag{32}$$

where $\mathcal{H}$ is the map from the input σ to z.

We next examine the impulse response of $\mathcal{H}$ [6], denoted by $q(t)$. Let $\delta(t)$ be the impulse function. Then

$$\begin{aligned}q(t) &= \varphi(t,t)b(t)e^{-\omega t}\\ &\quad - \int_0^t e^{-\omega\tau}\left(\frac{\partial}{\partial\tau}\varphi(t,\tau)b(\tau) + \varphi(t,\tau)\dot{b}(\tau)\right)d\tau.\end{aligned}$$

Note that $\varphi(t,t) = 1$ and $\frac{\partial}{\partial\tau}\varphi(t,\tau) = -a(\tau)\varphi(t,\tau)$. Therefore, for any $t \in [0,\tau]$

$$\begin{aligned}|q(t)| = \Big|b(t)e^{-\omega t} \\ - \int_0^t e^{-\omega\tau}\left(-a(\tau)\varphi(t,\tau)b(\tau) + \varphi(t,\tau)\dot{b}(\tau)\right)d\tau\Big| \\ \leq p_2 e^{-\omega t} + \sup_{t\in[0,\tau]}\{|a(t)b(t)| + p_3\}\int_0^t e^{-\omega\tau}\varphi(t,\tau)d\tau\end{aligned}$$

holds. With this inequality, we obtain (30), since $\|\mathcal{H}\|_{\mathcal{L}_1^{[0,\tau]}} = \int_0^\tau |q(t)|dt$. Notice that the system can be rewritten as

$$\begin{aligned}\dot{z}(t) &= a(t)z(t) + b(t)\upsilon(t)\\ \upsilon(s) &= \frac{1-C(s)}{s}s\sigma(s), \quad z(0) = 0,\end{aligned}$$

which is equal to

$$\begin{aligned}\dot{z}(t) &= a(t)z(t) + b(t)\upsilon(t)\\ \dot{\upsilon}(t) &= -\omega\upsilon(t) + \dot{\sigma}(t), \quad \upsilon(0) = \eta(0), \quad z(0) = 0\,.\end{aligned}$$

Next, consider the impulse response of the map $\hat{\mathcal{H}}$ from the input $\dot{\sigma}$ to z: $\hat{q}(t) = \int_0^t \varphi(t,\tau)b(\tau)\mathrm{e}^{-\omega\tau}d\tau$. We can upper bound like $|\hat{q}(t)| \le p_2 \int_0^t \mathrm{e}^{-\omega\tau}\varphi(t,\tau)d\tau$, which leads to the inequality in (31). ■

A. Proof of Lemma 1

Proof: We first show that $\|x_{\mathrm{ref}}\|_{\mathcal{L}_\infty^{[0,\tau]}}$ is bounded by ρ_{ref} using a contradictive argument. Suppose the statement is not true. Notice that $\|x_{\mathrm{ref}}(0)\| < \rho_{\mathrm{ref}}$. Since $x_{\mathrm{ref}}(t)$ is continuous, there must exist a time instant τ^*, such that $\|x_{\mathrm{ref}}(\tau^*)\| = \rho_{\mathrm{ref}}$ an $\|x_{\mathrm{ref}}(t)\| < \rho_{\mathrm{ref}}$ for any $t \in [0,\tau^*)$. Consider $x_{\mathrm{ref}}(t)$ over $[0,\tau^*]$. By Assumption 5, there exist a positive definite function $W(t, x_{\mathrm{ref}})$ and constants d_1, d_2, d_3, such that

$$\dot{W} \le -\frac{d_3}{d_2}W + \tfrac{\partial W}{\partial x_{\mathrm{ref}}} g(t,x_{\mathrm{ref}})(-\eta_{\mathrm{ref}}(t) + h(t,x_{\mathrm{ref}})),$$

where $\eta_{\mathrm{ref}}(t)$ is defined in (24). Solving this inequality with the initial condition $W(0,x_0)$, we have

$$\begin{aligned}&W(t,x_{\mathrm{ref}}(t)) \le \mathrm{e}^{-\frac{d_3}{d_2}t}W(0,x_0)\\ &\quad + \textstyle\int_0^t \mathrm{e}^{-\frac{d_3}{d_2}(t-\tau)} \tfrac{\partial W}{\partial x_{\mathrm{ref}}} g(\tau,x_{\mathrm{ref}})(-\eta_{\mathrm{ref}}(\tau) + h(\tau,x_{\mathrm{ref}}))d\tau.\end{aligned} \tag{33}$$

Consider the following system:

$$\begin{aligned}\dot{z}(t) &= -\frac{d_3}{d_2}z(t) + \frac{\partial W}{\partial x_{\mathrm{ref}}} g(t,x_{\mathrm{ref}})\zeta(t).\\ \zeta(s) &= (1-C(s))\mathfrak{L}[h(t,x_{\mathrm{ref}})], \quad z(0) = 0.\end{aligned}$$

It is easy to verify that for all $t \in [0,\tau^*]$, we have

$$\left\|\tfrac{\partial W}{\partial x_{\mathrm{ref}}} g(t,x_{\mathrm{ref}})\right\| \le B^{\partial W} B^g_{\rho_{\mathrm{ref}}} \quad \text{and} \quad \left\|\dot{h}(t,x_{\mathrm{ref}})\right\| \le B^{\dot{h}_{\mathrm{ref}}}_{\rho_{\mathrm{ref}}}.$$

By Lemma 5, we have the following bound for all $t \in [0,\tau^*]$:

$$\begin{aligned}&\|z(t)\| \le B^{\partial W} B^g_{\rho_{\mathrm{ref}}} \|h(0,x_0)\| \textstyle\int_0^t \mathrm{e}^{-\omega\tau}\mathrm{e}^{-\frac{d_3}{d_2}(t-\tau)}d\tau\\ &\quad + B^{\partial W} B^g_{\rho_{\mathrm{ref}}} B^{\dot{h}_{\mathrm{ref}}}_{\rho_{\mathrm{ref}}} \textstyle\int_0^t\int_0^\tau \mathrm{e}^{-\omega\lambda}\mathrm{e}^{-\frac{d_3}{d_2}(\tau-\lambda)}d\lambda d\tau\\ &\quad \le B^{\partial W} B^g_{\rho_{\mathrm{ref}}} d_2 \left(\tfrac{\|h(0,x_0)\|}{|d_3-d_2\omega|} + \tfrac{B^{\dot{h}_{\mathrm{ref}}}_{\rho_{\mathrm{ref}}}}{d_3\omega}\right) \triangleq r.\end{aligned}$$

Applying this inequality to (33) yields $W(t,x_{\mathrm{ref}}(t)) \le \mathrm{e}^{-\frac{d_3}{d_2}t}W(0,x_0) + r$, which means that for all $t \in [0,\tau^*]$, $\|x_{\mathrm{ref}}(t)\|^2 \le \frac{\mathrm{e}^{-\frac{d_3}{d_2}t}W(0,x_0)}{d_1} + \frac{r}{d_1}$. Therefore $\rho^2_{\mathrm{ref}} = \|x_{\mathrm{ref}}(\tau^*)\|^2 < \frac{W(0,x_0)}{d_1} + \frac{r}{d_1}$, which contradicts the inequality (12). Thus, $\|x_{\mathrm{ref}}\|_{\mathcal{L}_\infty^{[0,\tau]}} \le \rho_{\mathrm{ref}}$. Also note that it implies that the reference system is uniformly ultimately bounded. ■

B. Proof of Lemma 2

Proof: We first consider $\|\dot{x}(t)\|$. For the system in (1) the controller in (19) – (23) leads to the following dynamics: $\dot{x}(t) = f_m(t,x) + g(t,x)(u_2(t) + h(t,x))$, which gives the following upper bound $\|\dot{x}(t)\| \le \|f_m(t,x)\| + \|g(t,x)u_2(t)\| + \|g(t,x)\|\|h(t,x))\|$.

Since $\|x(t)\| \le \rho$ for all $t \in [0,\tau]$, the inequalities in (10) hold. Therefore, for all $t \in [0,\tau]$ the following bound

$$\|\dot{x}(t)\| \le B^{f_m}_\rho + B^g_\rho B^h_\rho + \|g(t,x)u_2(t)\| \tag{34}$$

holds. By the definition of $u_2(t)$ in (23), for $t \in [0,\tau]$, $\|g(t,x)u_2(t)\| \le B^g_\rho \|C(s)\|_{\mathcal{L}_1} B^h_\rho$, where $\|\hat{\sigma}(t)\| \le B^h_\rho$ is ensured by the projection operator. Applying this inequality to (34) implies that we have

$$\|\dot{x}(t)\| \le B^{f_m}_\rho + B^g_\rho B^h_\rho + \|C(s)\|_{\mathcal{L}_1} B^g_\rho B^h_\rho = B^{\dot{x}}_\rho \tag{35}$$

for $t \in [0,\tau]$. Notice that $\dot{h}(t,x(t)) = \frac{\partial h}{\partial t}(t,x(t)) + \frac{\partial h}{\partial x}(t,x(t))\dot{x}$, which leads to (25) for all $t \in [0,\tau]$. ■

C. Proof of Lemma 3

Proof: Let $U(\tilde{x},\tilde{\sigma}) = \tilde{x}^\top P\tilde{x} + \tilde{\sigma}^\top\Gamma^{-1}\tilde{\sigma}$. Compute the time derivative of U:

$$\dot{U} = -\tilde{x}^\top Q\tilde{x} + 2\tilde{x}^\top Pg(x)\tilde{\sigma} + 2\tilde{\sigma}^\top\Gamma^{-1}\dot{\hat{\sigma}} - 2\tilde{\sigma}^\top\Gamma^{-1}\dot{h}.$$

The adaptive law from (22) leads to $\dot{U} \le -\tilde{x}^\top Q\tilde{x} - 2\tilde{\sigma}^\top\Gamma^{-1}\dot{h}(t,x)$. Using the upper bound in (25), along with $\|\tilde{\sigma}(t)\| \le \|\hat{\sigma}(t)\| + \|h(t,x(t))\| \le 2B^h_\rho$, leads to $\dot{U} \le -\lambda_{\min}(Q)\|\tilde{x}\|^2 + 4B^h_\rho\Gamma^{-1}B^{\dot{h}}_\rho$. It implies that for all $t \ge 0$, either $\dot{U} \le 0$ or $\|\tilde{x}(t)\|^2 \le \frac{4B^h_\rho B^{\dot{h}}_\rho}{\Gamma\lambda_{\min}(Q)}$ holds. Since $U(0) \le \|\hat{\sigma}(0) - h(0,x_0)\|^2\Gamma^{-1} \le \frac{4{B^h_\rho}^2}{\Gamma}$, we have $\|\tilde{x}(t)\|^2 \le \frac{4{B^h_\rho}^2}{\lambda_{\min}(P)\Gamma} + \frac{4\lambda_{\max}(P)B^h_\rho B^{\dot{h}}_\rho}{\Gamma\lambda_{\min}(Q)\lambda_{\min}(P)}$ for $\forall t \in [0,\tau]$.

We now consider $\|\tilde{\eta}\|_{\mathcal{L}_\infty^{[0,\tau]}}$. Since Assumption 3 holds, based on equation (26), we know that $\psi(t,x)\dot{\tilde{x}}(t) = \psi(t,x)A_m\tilde{x}(t) + \tilde{\sigma}(t)$, which implies

$$\frac{d}{dt}[\psi(t,x)\tilde{x}(t)] = \psi(t,x)A_m\tilde{x}(t) + \tilde{x}(t)\frac{d}{dt}\psi(t,x) + \tilde{\sigma}(t).$$

Therefore, with the equation above,

$$\begin{aligned}&\|\tilde{\eta}\|_{\mathcal{L}_\infty^{[0,\tau]}} = \|C(s)\tilde{\sigma}(s)\|_{\mathcal{L}_\infty^{[0,\tau]}}\\ &\le (\|C(s)s\|_{\mathcal{L}_1}B^\psi_\rho + \|C(s)\|_{\mathcal{L}_1}B^\psi_\rho\|A_m\| + B^{\dot{\psi}}_\rho)\frac{\alpha}{\sqrt{\Gamma}},\end{aligned}$$

where $B^{\dot{\psi}}_\rho = B^{\frac{\partial\psi}{\partial t}}_\rho + B^{\frac{\partial\psi}{\partial x}}_\rho B^{\dot{x}}_\rho$. ■

D. Proof of Lemma 4

Proof: We prove the statement by contradiction. Suppose that the statement is not true. Since $e(t)$ is continuous and $\|e(0)\| = 0 < \gamma$, there must exist a time constant $\tau^* > 0$, such that

$$\|e(\tau^*)\| = \gamma \tag{36}$$

$$\|e(t)\| < \gamma, \ \forall t \in [0,\tau^*). \tag{37}$$

Then we consider the system in (27) over $[0,\tau^*)$. By the mean value theorem, the i^{th} entry of Δ can be rewritten as

$$\Delta_i(t,e) = \left[\frac{\partial f^i_m}{\partial x}(t,\lambda_1 e + x_{\mathrm{ref}}) - \frac{\partial f^i_m}{\partial x}(t,\lambda_2 e)\right]e, \tag{38}$$

where $0 < \lambda_i < 1$. Since $\frac{\partial f_m}{\partial x}$ is locally Lipschitz, we have

$$\|\Delta(t,e)\| \le 2L^{\frac{\partial f_m}{\partial x}}_\rho \|e\|^2 + L^{\frac{\partial f_m}{\partial x}}_\rho \|x_{\mathrm{ref}}\|\|e\|. \tag{39}$$

Recall that $\eta(s) = \mathfrak{L}[h(t,x)]$, $\hat{\eta} = C(s)\hat{\sigma}(s)$, and $\eta_{\text{ref}} = \mathfrak{L}[h(t,x_{\text{ref}})]$. The term $\Phi(t,x)$ is equal to $\Phi(t,x) = \Phi_1(t,x) + \Phi_2(t,x)$, where

$$\begin{aligned}\Phi_1(t,x) &= g(t,x)(\eta(t) - \hat{\eta}(t))\\ \Phi_2(t,x) &= g(t,x)\left(h(t,x) - \eta(t) - h(t,x_{\text{ref}}) + \eta_{\text{ref}}(t)\right)\\ &\quad + \left(g(t,x) - g(t,x_{\text{ref}})\right)\left(h(t,x_{\text{ref}}) - \eta_{\text{ref}}(t)\right).\end{aligned} \tag{40}$$

From Lemma 3 we have $\|\Phi_1(t,x)\| \le \frac{B_\rho^g \beta}{\sqrt{\Gamma}}$. Since $\|e(t)\| \le \gamma$ over $[0,\tau^*]$, by Assumption 4 and simple calculation, there exist c_1, c_2, c_3, c_4, and a positive definite function $V(t,e)$ for the system in (27), such that for all $t \in [0,\tau^*]$, we have

$$\dot{V}(t,e) \le -\left(\tfrac{c_3}{c_2} - \tfrac{c_4}{c_1}\kappa(t)\right)V + \tfrac{c_4 B_\rho^g \beta\gamma}{\sqrt{\Gamma}} + \tfrac{\partial V}{\partial e}\Phi_2(t,x), \tag{41}$$

where

$$\kappa(t) \;=\; 2L_\rho^{\frac{\partial f_m}{\partial x}}\gamma + L_\rho^{\frac{\partial f_m}{\partial x}}\|x_{\text{ref}}(t)\|. \tag{42}$$

Since $e(0) = 0$, $V(0) = 0$, we can upper bound

$$V(t) \le \int_0^t \varphi(t,\tau)\left(\tfrac{c_4 B_\rho^g \beta\gamma}{\sqrt{\Gamma}} + \tfrac{\partial V}{\partial e}(\tau, e(\tau))\Phi_2(\tau, x(\tau))\right)d\tau, \tag{43}$$

where $t \in [0,\tau^*]$, and

$$\varphi(t,\tau) = \mathrm{e}^{-\frac{c_3}{c_2}(t-\tau) + \frac{c_4}{c_1}\int_\tau^t \kappa(\lambda)d\lambda}. \tag{44}$$

Consider $\kappa(t)$ in (42). From Lemma 1, there exists a positive constant ϵ and a time instant $T > 0$, such that

$$\begin{aligned}\|x_{\text{ref}}(t)\| &\le \epsilon, \quad \forall\, t \ge T\\ 2L_\rho^{\frac{\partial f_m}{\partial x}}\gamma + L_\rho^{\frac{\partial f_m}{\partial x}}\epsilon &< \frac{c_1 c_3}{c_2 c_4}\\ \|x_{\text{ref}}(t)\| &\le \rho_{\text{ref}}, \quad \forall\, t \ge 0.\end{aligned}$$

Therefore, for all $t \ge 0$, the following upper bound $\int_0^t \|x_{\text{ref}}(\tau)\|d\tau \le \epsilon t + \rho_{\text{ref}}T$ holds. Further, it implies that for all $t \in [0,\tau^*]$, $\kappa(t)$ satisfies

$$\int_0^t \kappa(\tau)d\tau \le \mu t + L_\rho^{\frac{\partial f_m}{\partial x}}\rho_{\text{ref}}T, \tag{45}$$

where μ is defined in (13).

From (44), (45) and (13), we know

$$\varphi(t,\tau) \le \mathrm{e}^{-\hat{\alpha}(t-\tau)}\varrho, \tag{46}$$

where $\hat{\alpha}$ and ϱ are defined in (18). Using this inequality in (43), we obtain

$$V(t,e(t)) \le \tfrac{\varrho}{\hat{\alpha}}\tfrac{c_4 B_\rho^g \beta\gamma}{\sqrt{\Gamma}} + \int_0^t \varphi(t,\tau)\tfrac{\partial V}{\partial e}(\tau, e(\tau))\Phi_2(\tau, x)d\tau. \tag{47}$$

Consider the last term in the preceding inequality. With the definition of $\Phi_2(t,x)$ in (40), this integral term is in fact equivalent to the sum of the states of two scalar systems that share the same transition function $\varphi(t,\tau)$, given by:

$$\begin{aligned}\dot{z}_1(t) &= \left(\tfrac{c_4\kappa(t)}{c_1} - \tfrac{c_3}{c_2}\right)z_1(t) + \tfrac{\partial V}{\partial e}g(t,x)\zeta_1(t)\\ \zeta_1(s) &= (1 - C(s))\mathfrak{L}[h(t,x) - h(t,x_{\text{ref}})], \;\; z_1(0) = 0\end{aligned} \tag{48}$$

and

$$\begin{aligned}\dot{z}_2(t) &= \left(\tfrac{c_4\kappa(t)}{c_1} - \tfrac{c_3}{c_2}\right)z_2(t) + \tfrac{\partial V}{\partial e}(g(t,x) - g(t,x_{\text{ref}}))\zeta_2(t)\\ \zeta_2(s) &= (1 - C(s))\mathfrak{L}[h(t,x_{\text{ref}})], \quad z_2(0) = 0.\end{aligned} \tag{49}$$

It is easy to verify, based on the previous analysis and Assumption 4, that for all $t \in [0,\tau^*]$, we have

$$\begin{aligned}\left\|\tfrac{c_3}{c_2} - \kappa(t)\tfrac{c_4}{c_1}\right\| &\le B^a\\ \left\|\tfrac{\partial V}{\partial e}g(t,x)\right\| &\le c_4 B_\rho^g\gamma\\ \left\|\tfrac{d}{dt}\left(\tfrac{\partial V}{\partial e}g(t,x)\right)\right\| &\le M\gamma + \tfrac{c_5 {B_\rho^g}^2\beta}{\sqrt{\Gamma}}\\ \|h(t,x) - h(t,x_{\text{ref}})\| &\le L_\rho^h\gamma\\ \left\|\tfrac{\partial V}{\partial e}(g(t,x) - g(t,x_{\text{ref}}))\right\| &\le c_4 L_\rho^g\gamma^2\\ \left\|\dot{h}(t,x_{\text{ref}})\right\| &\le B_{\rho_{\text{ref}}}^{\dot{h}_{\text{ref}}},\end{aligned}$$

where the parameters on the right hand sides of the inequalities are defined in (11). Then we can apply Lemma 5 to the system in (48) to obtain

$$\begin{aligned}|z_1(t)| \le L_\rho^h\gamma\Big(&\int_0^t c_4 B_\rho^g\gamma\mathrm{e}^{-\omega\tau}d\tau +\\ &\left[\left(B^a c_4 B_\rho^g + M\right)\gamma + \tfrac{c_5 {B_\rho^g}^2\beta}{\sqrt{\Gamma}}\right]\int_0^t\int_0^\tau \mathrm{e}^{-\omega\lambda}\varphi(\tau,\lambda)d\lambda d\tau\Big).\end{aligned}$$

for all $t \in [0,\tau^*]$. Using (46) in the preceding inequality leads to the bound $|z_1(t)| \le \gamma^2\delta_1(\omega) + \gamma\frac{\delta_3}{\sqrt{\Gamma}}$, for all $t \in [0,\tau^*]$, where $\delta_3 = \frac{L_\rho^h c_5 {B_\rho^g}^2\beta\varrho}{\hat{\alpha}\omega}$. Similarly, applying Lemma 5 to the system in (49) yields $\|z_2(t)\|_{\mathcal{L}_\infty^{[0,\tau]}} \le \gamma^2\delta_2(\omega)$. Combining the preceding two inequalities, we have

$$\begin{aligned}&\left|\int_0^t \varphi(t,\tau)\tfrac{\partial V}{\partial e}(\tau,e(\tau))\Phi_2(\tau,x(\tau))d\tau\right|\\ &= |z_1(t) + z_2(t)| \le (\delta_1(\omega) + \delta_2(\omega))\gamma^2 + \gamma\tfrac{\delta_3}{\sqrt{\Gamma}}.\end{aligned}$$

Let $\delta_4 = \frac{\varrho c_4 B_\rho^g\beta}{\hat{\alpha}}$. Substituting it into (47), we obtain the bound $V(t) \le \frac{\gamma(\delta_3+\delta_4)}{\sqrt{\Gamma}} + (\delta_1(\omega) + \delta_2(\omega))\gamma^2$ for all $t \in [0,\tau^*]$, which, along with (2), implies that for all $t \in [0,\tau^*]$, $c_1\|e(t)\|^2 \le \frac{\gamma(\delta_3+\delta_4)}{\sqrt{\Gamma}} + (\delta_1(\omega) + \delta_2(\omega))\gamma^2$. Since $\sup_{t\in[0,\tau^*]}\|e(t)\| = \gamma$, we have $c_1\gamma^2 \le \frac{\gamma(\delta_3+\delta_4)}{\sqrt{\Gamma}} + (\delta_1(\omega)+\delta_2(\omega))\gamma^2$. Since $\frac{\delta_1(\omega)+\delta_2(\omega)}{c_1} < 1$, by inequality (14), the inequality above implies $\sqrt{\Gamma} \le \frac{\delta_3+\delta_4}{\gamma(c_1-\delta_1(\omega)-\delta_2(\omega))}$. This contradicts the assumption in (28). Hence, $\|e\|_{\mathcal{L}_\infty^{[0,\tau]}} < \gamma$. ■

E. Proof of Theorem 1

Proof: We first show that $\|x\|_{\mathcal{L}_\infty} < \rho$ using a contradiction argument. Suppose that $\|x\|_{\mathcal{L}_\infty}$ is not bounded by ρ. Since $\|x(0)\| < \rho$ and $x(t)$ is continuous, there must exist a time instant τ^*, such that

$$\|x(\tau^*)\| = \rho \text{ and } \|x(t)\| < \rho, \quad \forall\, t \in [0,\tau^*). \tag{50}$$

According to Lemma 4, the inequality $\|x_{\text{ref}}(t) - x(t)\| < \gamma$ holds for all $t \in [0,\tau^*]$. Since $\|x_{\text{ref}}\|_{\mathcal{L}_\infty} \le \rho_{\text{ref}}$, it implies $\|x(t)\| < \gamma + \rho_{\text{ref}} = \rho$ for all $t \in [0,\tau^*]$, which contradicts (50). Therefore, $\|x\|_{\mathcal{L}_\infty} < \rho$. As a result, according to Lemma 4, $\|x_{\text{ref}} - x\|_{\mathcal{L}_\infty} < \gamma$.

We next prove the bound for $\|u_{\text{ref}}(t) - u(t)\|$. According to the definitions of $u_{\text{ref}}(t)$ in (24) and $u(t)$ in (19) – (23), we have $\|u_{\text{ref}}(t) - u(t)\| \le L_\rho^k\|x_{\text{ref}} - x\| + \|\eta_{\text{ref}}(t) - \hat{\eta}(t)\|$. Consider the last term on the right hand side of this inequality. Recall that $\hat{\eta}(s) = C(s)\hat{\sigma}(s)$, $\eta_{\text{ref}}(s) = \mathfrak{L}[h(t,x_{\text{ref}})]$, and $\eta(s) = \mathfrak{L}[h(t,x)]$. Then $\|\eta_{\text{ref}}(t) - \hat{\eta}(t)\| \le \|\eta_{\text{ref}}(s) - \eta(s)\|_{\mathcal{L}_\infty} + \|\eta(t) - \hat{\eta}(t)\|_{\mathcal{L}_\infty} \le \|C(s)\|_{\mathcal{L}_1}L_\rho^h\gamma + \frac{\beta}{\sqrt{\Gamma}}$, where the last inequality is obtained by applying Lemma 3. The proof is complete. ■