2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Multiplayer Nash Solution for Noncooperative Cost Density-Shaping Games

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***Abstract*— Finite-horizon, LQ cost density-shaping has been achieved through several different control paradigms that are based on the Multiple-Cumulant Cost Density-Shaping (MCCDS) theory. With these cost density-shaping control methods, the shape of a target cost density can be transformed into a linear control law. However, the existing MCCDS theory does not permit control design that accounts for competing objectives among multiple noncooperative agents. The aim of this work is to derive the Nash equilibrium solution to an N-Player MCCDS game posed for the LQG framework. Simulation results are provided to support the new theory.**

***Index Terms*— cost density-shaping games, stochastic optimal control, cost cumulant control, structural control**

I. Introduction

The "k Cost Cumulant" (kCC) control theory [1] provides the linear control input to a system that minimizes an arbitrarily-weighted linear combination of k cumulants for an integral-quadratic, random cost. As the first cost cumulant is the expectation of the cost, kCC control can naturally be thought of as a true generalization to the classical LQG theory that permits additional terms. The flexibility of choosing weights in the kCC performance index has motivated the development of the Multiple-Cumulant Cost Density-Shaping (MCCDS) paradigm [2]. In particular, the weights do not correspond directly to the general "shape" of the cost density achieved under kCC control. Since the shape of the cost density is intimately related to the performance and stability properties of the control input underlying the cost cumulants [3], it has been beneficial to develop MCCDS controls, which are capable of minimizing probability distance functions between multi-cumulant approximations to the cost density and to the target density. In essence, the MCCDS translates a shape of a target density into a linear control law.

In this way, MCCDS control can obtain certain target statistical characterizations for the cost functional. However, MCCDS currently cannot accommodate competing objectives in the control design. Indeed, it would be ideal for MCCDS if competing agents could formulate a strategy to deliberately shape the density function of that agent's random cost with respect to how other agents influence the process evolution. Given this limitation, an N-Player MCCDS Nash game is formulated and solved in this paper.

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Cost cumulant control theory has been successfully adapted to noncooperative stochastic games in recent years. For the case of linear controls and integral-quadratic cost functionals, kCC control theory is the foundation for the study of kCC zero-sum and Nash games [1]. For the more general case of nonlinear controls and integral non-quadratic cost functionals, the cost cumulant games theory has been developed, again in the non-cooperative setting [4]. This work has in fact generalized the classical H_2/H_∞ theory and strengthened the connection between cost cumulant control and robust control theories.

This paper is organized in four main sections. In the first, the problem class, basic definitions, and notation are given. The second section goes into the problem formulation, and then the third section contains the solution of the N-Player MCCDS Nash game. This development is followed by simulation results in the fourth section that illustrate the derived control solution. Proofs have been omitted due to space limitations.

II. Preliminaries

A. Problem Class

Let $(t_0, x_0) \in [t_0, t_f] \times \mathbb{R}^n$ be fixed, and let $\xi(t) = \Xi(t, \omega)$ be a p-dimensional stationary Wiener process on $[t_0, t_f]$ where $\xi : [t_0, t_f] \times \Omega \to \mathbb{R}^p$ on the complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and the following correlation of increments property is satisfied for $W \succ \mathbf{0}^{p\times p}$,

$$E[(\xi(\tau_1) - \xi(\tau_2))(\xi(\tau_1) - \xi(\tau_2))^T] = W|\tau_1 - \tau_2|.$$

Let $\mathcal{U}_i \in L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_i})),\ 1 \le i \le N$ be Hilbert spaces of $\mathbb{R}^{m_i}$-valued, square-integrable processes $u_i \in \mathcal{U}_i$, where by their construction

$$E\left\{ \int_{t_0}^{t_f} u_i^T(\tau) u_i(\tau) d\tau \right\} < \infty.$$

Further, let the processes in $\mathcal{U}_i$ be adapted to the σ-field generated by $\xi(t)$, $\mathcal{F}_t$. Consider the problem of *Player* i choosing strategies $u_i \in \mathcal{U}_i$ so to influence the states $x(t) = X(t, \omega)$ of the following linear stochastic differential equation, which belongs to $L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^n))$ and is adapted to the σ-field generated by $\xi(t)$,

$$\begin{aligned} dx(t) &= A(t)x(t)dt + \sum_{i=1}^{N} B_i(t)u_i(t) + G(t)d\xi(t) \\ x_0 &= E\{x(t_0)\},\ x_0 \in \mathbb{R}^n,\ t \in [t_0, t_f] \end{aligned} \qquad (1)$$

where

$$A \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n\times n}),\ G \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n\times p})$$
$$B_i \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n\times m_i})\ 1 \le i \le N.$$

In particular, *Player i* chooses u_i to optimize the statistical characterization of the integral-quadratic cost functional $J_i[x, u; t_0, x_0]$ given below,

$$J_i = \int_{t_0}^{t_f} \left(x(\tau)^T Q_i(\tau) x(\tau) + \sum_{j=1}^{N} u_j(\tau)^T R_{ij}(\tau) u_j(\tau) \right) d\tau + x(t_f)^T Q_{if} x(t_f). \tag{2}$$

It is understood that $Q_i \in \mathcal{C}([t_0, t_f]; \mathbb{S}^n_+)$, $R_{ij} \in \mathcal{C}([t_0, t_f]; \mathbb{S}^{m_j}_{++})$, and $\mathcal{Q}_{if} \in \mathbb{S}^n_+$ for well-posedness of the problem. Suppose further that players choose their optimal control actions within the class of memoryless, full-observation strategies, or more precisely

$$\eta_i : [t_0, t_f] \times L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^n)) \to L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_i}))$$

and

$$u_i(t) = \eta_i(t, x(t)) = K_i(t) x(t). \tag{3}$$

When the process (1) is subjected to the controls of each player, where $K_i \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_i \times n})$ are the *admissible* control gains with respective compact, allowable sets of gains $\bar{K}_i \subset \mathbb{R}^{m_i \times n}$, it becomes

$$dx(t) = \left(A(t) + \sum_{i=1}^{N} B_i(t) K_i(t) \right) x(t) dt + G(t) d\xi(t)$$
$$x_0 = E\{x(t_0)\},\ x_0 \in \mathbb{R}^n,\ t \in [t_0, t_f] \tag{4}$$

and the costs (2) can be written as

$$J_i = \int_{t_0}^{t_f} \left(x(\tau)^T N_i(\tau) x(\tau) \right) d\tau + x(t_f)^T Q_{if} x(t_f) \tag{5}$$

where

$$N_i(\tau) = \sum_{j=1}^{N} K_j(\tau)^T R_{ij}(\tau) K_j(\tau) + Q_i(\tau).$$

Traditionally, the mathematical expectation of the cost (2) are optimized in the game, whereas cost cumulant control considers the optimization of higher-order statistics, the cumulants. These quantities are defined via the recursive relationship below,

$$\kappa^i_1(t_0) = E\{J_i\}$$
$$\kappa^i_r(t_0) = E\{J_i^r\} - \sum_{j=1}^{r-1} \begin{pmatrix} r-1 \\ j-1 \end{pmatrix} \kappa^i_j(t_0) E\{J_i^{r-j}\},\ r \ge 2.$$

B. Cost Cumulants

The cumulants of (5) associated with the process (4) have a special form, which is given in the following theorem.

***Theorem** 2.1:* (Cost Cumulants, N-Player Case) For the process (4), the r cost cumulants of (5) for *Player i* take the following form,

$$\kappa^i_k(\alpha) = x_0^T H^i_k(\alpha) x_0 + D^i_k(\alpha),\ 1 \le k \le r \tag{6}$$

where the $H^i_k(\alpha)$ and $D^i_k(\alpha)$ functions satisfy the system of differential equations,

$$\begin{aligned}
\frac{dH^i_1(\alpha)}{d\alpha} =& - \left(A(\alpha) + \sum_{j=1}^{N} B_j(\alpha) K_j(\alpha) \right)^T H^i_1(\alpha) \\
& - H^i_1(\alpha) \left(A(\alpha) + \sum_{j=1}^{N} B_j(\alpha) K_j(\alpha) \right) \\
& - N_i(\alpha) \triangleq \mathcal{F}_1(\mathbf{H}^i(\alpha), K_i(\alpha), K_{N-i}(\alpha)) \\
\frac{dH^i_k(\alpha)}{d\alpha} =& - \left(A(\alpha) + \sum_{j=1}^{N} B_j(\alpha) K_j(\alpha) \right)^T H^i_k(\alpha) \\
& - H^i_k(\alpha) \left(A(\alpha) + \sum_{j=1}^{N} B_j(\alpha) K_j(\alpha) \right) \\
& - 2 \sum_{j=1}^{k-1} \begin{pmatrix} k \\ j \end{pmatrix} H^i_j(\alpha) G(\alpha) W G^T(\alpha) H^i_{k-j}(\alpha), \\
& \triangleq \mathcal{F}_k(\mathbf{H}^i(\alpha), K_i(\alpha), K_{N-i}(\alpha)),\ 2 \le k \le r \\
\frac{dD^i_k(\alpha)}{d\alpha} =& - \mathrm{Tr}(H^i_k(\alpha) G(\alpha) W G^T(\alpha)) \\
& \triangleq \mathcal{G}_k(\mathbf{H}^i(\alpha)),\ \alpha \in [t_0,\ t_f]\ 1 \le k \le r.
\end{aligned} \tag{7}$$

These functions satisfy the terminal conditions

$$\begin{aligned}
& H^i_1(t_f) = Q_{if},\ H^i_i(t_f) = \mathbf{0}^{n\times n},\ i \ge 2 \\
& D^i_1(t_f) = 0,\ D^i_2(t_f) = 1,\ D^i_j(t_f) = 0,\ j \ge 3.
\end{aligned} \tag{8}$$

Proof: See [5] ■

C. Notation

It is helpful in the following to introduce some notation to make restatements of the above equations easier in the development. This notation is heavily inspired by that used by Pham in [1]. Define variables $\mathbf{H}^i(\alpha)$ and $\mathbf{D}^i(\alpha)$ as below.

$$\mathbf{H}^i(\alpha) \triangleq (H^i_1(\alpha), \ldots, H^i_r(\alpha))$$
$$\mathbf{D}^i(\alpha) \triangleq (D^i_1(\alpha), \ldots,\ D^i_r(\alpha)),\ 1 \le i \le N.$$

In the following, the denotation K_{N-i} will be made on occasion and this refers to the set of control gains excluding the ith or more precisely,

$$\begin{aligned}
K_{N-i} &= \times_{j\ne i} K_j \\
&= \underbrace{K_1 \times K_2 \times \cdots \times K_{i-1} \times K_{i+1} \times \cdots \times K_N}_{N-1\ times}.
\end{aligned}$$

With this apparatus in place, the Cartesian product of *all* N control gains can be abbreviated by $K_i \times K_{N-i}$. It is to be understood that K^*_{N-i} refers to the situation when all players except the ith play their Nash strategy

Using these state variables, define the functions

$$\begin{aligned}&\mathcal{F}(\mathbf{H}^i(\alpha), K_i(\alpha), K_{N-i}(\alpha))\\&\triangleq \times_{j=1}^r \mathcal{F}_j(\mathbf{H}^i(\alpha), K_i(\alpha), K_{N-i}(\alpha))\\&\mathcal{G}(\mathbf{H}^i(\alpha)) \triangleq \times_{j=1}^r \mathcal{G}_j(\mathbf{H}^i(\alpha)),\ 1 \le i \le N.\end{aligned}$$

Let $\{\mathcal{F}_j(\cdot)\}_{j=1}^r$, and $\{\mathcal{G}_i(\cdot)\}_{j=1}^r$ in the above definitions be defined as beforehand in (7). Also a condensed form for the terminal conditions is introduced as below.

$$\mathbf{H}^i_f \triangleq (Q_{if}, \mathbf{0}^{n\times n}, \ldots, \mathbf{0}^{n\times n}),\ \mathbf{D}^i_f \triangleq (0, 1, 0, \ldots, 0)$$

Finally, denote the cost cumulant vectors $\boldsymbol{\kappa}^i(\alpha)$ as

$$\boldsymbol{\kappa}^i(\alpha) \triangleq (\kappa^i_1(\alpha), \ldots, \kappa^i_r(\alpha)),\ 1 \le i \le N.$$

Using this notation, the equations (7) and their associated terminal condition systems can be written concisely as

$$\begin{aligned}\frac{d\mathbf{H}^i(\alpha)}{d\alpha} &= \mathcal{F}(\mathbf{H}^i(\alpha), K_i(\alpha), K_{N-i}(\alpha))\\ \frac{d\mathbf{D}^i(\alpha)}{d\alpha} &= \mathcal{G}(\mathbf{H}^i(\alpha))\\ \mathbf{H}^i(t_f) &= \mathbf{H}^i_f,\ \mathbf{D}^i(t_f) = \mathbf{D}^i_f,\ \alpha \in [t_0, t_f].\end{aligned}$$

D. Target Cost Cumulants

Also, the target statistics for *Player i* can be written as follows,

$$\begin{aligned}\frac{d\tilde{\mathbf{H}}^i(\alpha)}{d\alpha} &= \mathcal{F}(\tilde{\mathbf{H}}^i(\alpha), \tilde{K}_i(\alpha), \tilde{K}_{N-i}(\alpha)),\\ \frac{d\tilde{\mathbf{D}}^i(\alpha)}{d\alpha} &= \mathcal{G}(\tilde{\mathbf{H}}^i(\alpha))\\ \tilde{\mathbf{H}}^i(t_f) &= \tilde{\mathbf{H}}^i_{f;\mathcal{E}^*_i},\ \tilde{\mathbf{D}}^i(t_f) = \tilde{\mathbf{D}}_{f;\epsilon^*_i},\ \alpha \in [t_0, t_f].\end{aligned} \tag{9}$$

where

$$\begin{aligned}&\tilde{H}^i_1(t_f) = Q_{if} + \mathcal{E}^*_i,\ \tilde{H}^i_j(t_f) = \mathbf{0}^{n\times n}, j \ge 2\\&\tilde{D}^i_1(t_f) = \epsilon^*_i,\ \tilde{D}^i_2(t_f) = 1, \tilde{D}^i_j(t_f) = 0,\ j \ge 3\end{aligned} \tag{10}$$

and the short-hand notation is used,

$$\begin{aligned}\mathbf{H}^i_{f;\mathcal{E}^*_i} &\triangleq (Q_{if} + \mathcal{E}^*_i, \mathbf{0}^{n\times n}, \ldots, \mathbf{0}^{n\times n})\\ \mathbf{D}_{f;\epsilon^*_i} &\triangleq (\epsilon^*_i, 1, 0, \ldots, 0),\ 1 \le i \le N.\end{aligned}$$

Here $\epsilon^*_i > 0$ are small perturbation constants, and $\mathcal{E}^*_i \succ \mathbf{0}^{n\times n}$ are positive-definite perturbation matrices. As with the cost cumulants, compose vectors of target cost cumulants $\tilde{\boldsymbol{\kappa}}^i(\alpha) \in \mathbb{R}^r$ defined below,

$$\tilde{\boldsymbol{\kappa}}^i(\alpha) \triangleq (\tilde{\kappa}^i_1(\alpha), \ldots, \tilde{\kappa}^i_r(\alpha)),\ 1 \le i \le N$$

where

$$\tilde{\kappa}^i_k(\alpha) = x_0^T \tilde{H}^i_k(\alpha) x_0 + \tilde{D}^i_k(\alpha),\ 1 \le k \le r. \tag{11}$$

III. Problem Formulation

The target set and admissible space of control gains characterizing linear controls for which the evolution equations are solvable are now presented.

Definition 3.1: (Target Sets)
Let $(t_0, \mathbf{H}^i(t_0), \mathbf{D}^i(t_0), \tilde{\mathbf{H}}^i(t_0), \tilde{\mathbf{D}}^i(t_0)) \in \mathcal{M}_i$, where $\mathcal{M}_i$ denotes the target set for the Player i which is a closed subset of

$$\begin{aligned}&[t_0, t_f] \times (\underbrace{\mathbb{R}^{n\times n} \times \cdots \times \mathbb{R}^{n\times n}}_{r\ times}) \times \mathbb{R}^r\\&\times (\underbrace{\mathbb{R}^{n\times n} \times \cdots \times \mathbb{R}^{n\times n}}_{r\ times}) \times \mathbb{R}^r.\end{aligned}$$

The combined target space is closed, $\cup_{i=1}^N \mathcal{M}_i$.

For given terminal conditions, the sets of admissible feedback gains are denoted as

$$\mathcal{K}^i_{t_f, \mathbf{H}^i(t_f), \mathbf{D}(t_f), \tilde{\mathbf{H}}^i(t_f), \tilde{\mathbf{D}}^i(t_f)},\ 1 \le i \le N.$$

and contain matrices $K_i \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m_i\times n})$ such that

$$(t_0, \mathbf{H}^i(t_0), \mathbf{D}^i(t_0), \tilde{\mathbf{H}}^i(t_0), \tilde{\mathbf{D}}^i(t_0)) \in \mathcal{M}_i.$$

is obtained at the end of the trajectories for the state equations (7) and (9). This is formally stated in the following definition.

Definition 3.2: (Admissible Feedback Gains)
Denote allowable sets of control gain values by $\bar{K}_i \subset \mathbb{R}^{m_i\times n}$ and let these sets be compact. For fixed $r \in \mathbb{N}$ let $\mathcal{K}^i_{t_f, \mathbf{H}^i(t_f), \mathbf{D}^i(t_f), \tilde{\mathbf{H}}^i(t_f), \tilde{\mathbf{D}}^i(t_f)} \triangleq \mathcal{K}^i(t_f)$ characterize a class of $\mathcal{C}([t_0, t_f]; \mathbb{R}^{m_i\times n})$ such that for $K_i \in \mathcal{K}^i(t_f),\ 1 \le i \le N$ the solutions to

$$\begin{aligned}\frac{d\mathbf{H}^i(\alpha)}{d\alpha} &= \mathcal{F}(\mathbf{H}^i(\alpha), K_i(\alpha), K_{N-i}(\alpha)),\\ \frac{d\mathbf{D}^i(\alpha)}{d\alpha} &= \mathcal{G}(\mathbf{H}^i(\alpha))\\ \mathbf{H}^i(t_f) &= \mathbf{H}^i_f,\ \mathbf{D}^i(t_f) = \mathbf{D}^i_f,\ 1 \le i \le N\end{aligned}$$

exist on $\alpha \in [t_0, t_f]$ and the initial values of the state trajectories satisfy

$$(t_0, \mathbf{H}^i(t_0), \mathbf{D}^i(t_0), \tilde{\mathbf{H}}^i(t_0), \tilde{\mathbf{D}}^i(t_0)) \in \mathcal{M}_i,\ 1 \le i \le N.$$

For general performance indices, consider scalar functions $g^i : \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}$ with vector arguments, which are denoted by $g^i(\boldsymbol{\kappa}, \tilde{\boldsymbol{\kappa}})$. For fixed $\tilde{\boldsymbol{\kappa}}$, the function becomes $g^i_{\tilde{\boldsymbol{\kappa}}} : \mathbb{R}^r \to \mathbb{R}$. Analogously for fixed $\boldsymbol{\kappa}$, the function becomes $g^i_{\boldsymbol{\kappa}} : \mathbb{R}^r \to \mathbb{R}$. Impose the following restrictions on $g^i_{\tilde{\boldsymbol{\kappa}}}(\boldsymbol{\kappa})$ and $g^i_{\boldsymbol{\kappa}}(\tilde{\boldsymbol{\kappa}})$ to ensure that the ensuing optimization problem is well-posed:

- The function $g^i_{\tilde{\boldsymbol{\kappa}}}$ is analytic on $\mathbf{dom}\ g^i_{\tilde{\boldsymbol{\kappa}}}$ and $g^i_{\boldsymbol{\kappa}}$ is analytic on $\mathbf{dom}\ g^i_{\boldsymbol{\kappa}}$
- The function $g^i_{\tilde{\boldsymbol{\kappa}}}$ is convex in $\boldsymbol{\kappa}$ and its domain $\mathbf{dom}\ g^i_{\tilde{\boldsymbol{\kappa}}}$ is a convex set
- The function $g^i_{\tilde{\boldsymbol{\kappa}}}$ is non-negative in $\boldsymbol{\kappa}$ on some neighborhood of $\tilde{\boldsymbol{\kappa}}$

Definition 3.3: (Performance Indices)
For $1 \le i \le N$ let Player i's performance index be

$$\phi^i(\mathbf{H}^i(t_0), \mathbf{D}^i(t_0), \tilde{\mathbf{H}}^i(t_0), \tilde{\mathbf{D}}^i(t_0)) = g^i(\boldsymbol{\kappa}^i(t_0), \tilde{\boldsymbol{\kappa}}^i(t_0)).$$

The Mayer-from MCCDS game is now formulated.

Definition 3.4: (Mayer MCCDS Game)
For every $\tilde{\boldsymbol{\kappa}}^i(t_0)$, let $g^i(\boldsymbol{\kappa}^i(t_0), \tilde{\boldsymbol{\kappa}}^i(t_0))$ be an analytic function, convex in $\boldsymbol{\kappa}^i(t_0)$, defined for positive values of its vector-valued arguments such that it is non-negative on some neighborhood of $\tilde{\boldsymbol{\kappa}}^i(t_0)$. Let $r \in \mathbb{N}$ be a fixed positive integer, where $\boldsymbol{\kappa}^i(t_0),\ \tilde{\boldsymbol{\kappa}}^i(t_0) \in \mathbb{R}^r$ are the vectors of initial cost cumulants and target initial cost cumulants, respectively, for Player i. Then the N-Player MCCDS game can be formulated as,

$$\begin{aligned}
&\min_{K_i \in \mathcal{K}^i(t_f)} \phi^i(\mathbf{H}^i(t_0), \mathbf{D}^i(t_0), \tilde{\mathbf{H}}^i(t_0), \tilde{\mathbf{D}}^i(t_0)) \\
&\text{subject to:} \\
&\frac{d\mathbf{H}^i(\alpha)}{d\alpha} = \mathcal{F}(\mathbf{H}^i(\alpha), K_i(\alpha), K_{N-i}(\alpha)) \\
&\frac{d\mathbf{D}^i(\alpha)}{d\alpha} = \mathcal{G}(\mathbf{H}^i(\alpha)) \\
&\frac{d\tilde{\mathbf{H}}^i(\alpha)}{d\alpha} = \mathcal{F}(\tilde{\mathbf{H}}^i(\alpha), \tilde{K}_i(\alpha), \tilde{K}_{N-i}(\alpha)) \\
&\frac{d\tilde{\mathbf{D}}^i(\alpha)}{d\alpha} = \mathcal{G}(\tilde{\mathbf{H}}^i(\alpha)) \\
&\mathbf{H}^i(t_f) = \mathbf{H}^i_f,\ \mathbf{D}^i(t_f) = \mathbf{D}^i_f \\
&\tilde{\mathbf{H}}^i(t_f) = \tilde{\mathbf{H}}^i_{f;\mathcal{E}_i^*},\ \tilde{\mathbf{D}}^i(t_f) = \tilde{\mathbf{D}}_{f;\epsilon_i^*}
\end{aligned} \tag{12}$$

where the initial values of the state trajectories satisfy

$$(t_0, \mathbf{H}^i(t_0), \mathbf{D}^i(t_0), \tilde{\mathbf{H}}^i(t_0), \tilde{\mathbf{D}}^i(t_0)) \in \mathcal{M}_i,\ 1 \le i \le N.$$

IV. Problem Solution

The following variables will be used in the derivation. Notice that since N pairs of differential equations (e.g. one per player) are specified, distinct dynamic programming variables are maintained for the associated set of terminal conditions. Define the block matrices $\boldsymbol{\mathcal{Y}}^j(\epsilon),\ \tilde{\boldsymbol{\mathcal{Y}}}^j(\epsilon) \in \mathbb{R}^{rn \times n}$ and the vectors $\boldsymbol{\mathcal{Z}}^j(\epsilon),\ \tilde{\boldsymbol{\mathcal{Z}}}^j(\epsilon) \in \mathbb{R}^r$ as below,

$$\begin{aligned}
\boldsymbol{\mathcal{Y}}^j(\epsilon) &= (\mathcal{Y}^j_1(\epsilon), \dots, \mathcal{Y}^j_r(\epsilon)),\ \tilde{\boldsymbol{\mathcal{Y}}}^j(\epsilon) = (\tilde{\mathcal{Y}}^j_1(\epsilon), \dots, \tilde{\mathcal{Y}}^j_r(\epsilon)) \\
\boldsymbol{\mathcal{Z}}^j(\epsilon) &= (\mathcal{Z}^j_1(\epsilon), \dots, \mathcal{Z}^j_r(\epsilon)),\ \tilde{\boldsymbol{\mathcal{Z}}}^j(\epsilon) = (\tilde{\mathcal{Z}}^j_1(\epsilon), \dots, \tilde{\mathcal{Z}}^j_r(\epsilon))
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{Y}^i_j(\epsilon) = H^i_j(\epsilon),\ \mathcal{Z}^i_j(\epsilon) = D^i_j(\epsilon) \\
&\tilde{\mathcal{Y}}^i_j(\epsilon) = \tilde{H}^i_j(\epsilon),\ \tilde{\mathcal{Z}}^i_j(\epsilon) = \tilde{D}^i_j(\epsilon),\ 1 \le j \le r,\ 1 \le i \le N.
\end{aligned}$$

The value functions are defined below. These functions give *Player i*'s value of the MCCDS game from whatever "displaced" terminal condition is considered.

Definition 4.1: (Value Function, MCCDS Game)
Let the dynamic programming variables be defined

$$\begin{aligned}
&(\epsilon, \boldsymbol{\mathcal{Y}}^i(\epsilon), \boldsymbol{\mathcal{Z}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Y}}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Z}}}^i(\epsilon)) \\
&\in [t_0, t_f] \times (\mathbb{S}^n)^r \times \mathbb{R}^r \times (\mathbb{S}^n)^r \times \mathbb{R}^r
\end{aligned}$$

and let
$\mathcal{V}^i(\epsilon, \boldsymbol{\mathcal{Y}}^i(\epsilon), \boldsymbol{\mathcal{Z}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Y}}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Z}}}^i(\epsilon))$ be scalar functions

$$\mathcal{V}^i : [t_0, t_f] \times (\mathbb{S}^n)^r \times \mathbb{R}^r \times (\mathbb{S}^n)^r \times \mathbb{R}^r \to \mathbb{R}$$

such that for $\mathcal{K}^i(\epsilon) \neq \emptyset$

$$\begin{aligned}
&\mathcal{V}^i(\epsilon, \boldsymbol{\mathcal{Y}}^i(\epsilon), \boldsymbol{\mathcal{Z}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Y}}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Z}}}^i(\epsilon)) \\
&= \min_{K_i \in \mathcal{K}^i(\epsilon)} \phi^i(\mathbf{H}^i(t_0), \mathbf{D}^i(t_0), \tilde{\mathbf{H}}^i(t_0), \tilde{\mathbf{D}}^i(t_0))
\end{aligned}$$

where $\mathcal{K}^i(\epsilon) \triangleq \mathcal{K}^i_{\epsilon, \boldsymbol{\mathcal{Y}}^i(\epsilon), \boldsymbol{\mathcal{Z}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Y}}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Z}}}^i(\epsilon)}$.

The playable set of each value function is the set of all feasible displaced terminal conditions, such that the MCCDS game is solvable on a reduced time-horizon $[t_0, \epsilon]$, where $\epsilon \in (t_0, t_f]$. .

Definition 4.2: (Playable Set, MCCDS Game)
Define the playable sets as the sets of terminal values from which there exists a control that can take the system to the target set. More formally, this is for each $1 \le i \le N$,

$$\mathcal{Q}_i = \{(\epsilon, \boldsymbol{\mathcal{Y}}^i(\epsilon), \boldsymbol{\mathcal{Z}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Y}}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Z}}}^i(\epsilon)) \mid \mathcal{K}^i(\epsilon) \neq \emptyset\}.$$

The solution concept follows the Nash equilibrium idea, which is now presented.

Definition 4.3: (Nash Equilibrium, MCCDS Game)
Consider a set of gains $\{K_j^*\}_{j=1}^N$ such that $K_j^* \in \mathcal{K}^j(t_f), 1 \le j \le N$. The control gains $\{K_j^*\}_{j=1}^N$ constitute a Nash equillibrium solution to the N-Player CDS game, if for $K_j = K_j^*,\ j \neq i$ and $1 \le i \le N$ it is true that

$$\begin{aligned}
&\phi^i(\mathbf{H}^i(t_0; K_i^*, K_{N-i}^*), \mathbf{D}^i(t_0; K_i^*, K_{N-i}^*), \tilde{\mathbf{H}}^i(t_0), \tilde{\mathbf{D}}^i(t_0)) \\
&\le \phi^i(\mathbf{H}^i(t_0; K_i, K_{N-i}^*), \mathbf{D}^i(t_0; K_i, K_{N-i}^*), \tilde{\mathbf{H}}^i(t_0), \tilde{\mathbf{D}}^i(t_0))
\end{aligned}$$

The following MCCDS Game verification lemma can be established analogously as the lemma presented in [2]. Essentially, this lemma provides sufficient conditions whereby a set of linear control inputs to (1) characterized by gains $\{K_i\}_{i=1}^N$ can be verified to be the Nash solution to the MCCDS game.

Lemma 4.4: (HJB Verification, MCCDS Game)
Suppose $(\epsilon, \boldsymbol{\mathcal{Y}}^i(\epsilon), \boldsymbol{\mathcal{Z}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Y}}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Z}}}^i(\epsilon))$ are points in the playable sets $\mathcal{Q}_i$ for $1 \le i \le N$ where the non-increasing, scalar functions

$$\mathcal{W}^i(\epsilon, \boldsymbol{\mathcal{Y}}^i(\epsilon), \boldsymbol{\mathcal{Z}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Y}}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Z}}}^i(\epsilon))$$

are differentiable. Suppose that for a nominal pair (K_i^*, K_{N-i}^*), the function $\mathcal{W}^i(\epsilon, \boldsymbol{\mathcal{Y}}^i(\epsilon), \boldsymbol{\mathcal{Z}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Y}}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Z}}}^i(\epsilon))$ satisfies, for $1 \le i \le N$,

$$\begin{aligned}
&-\frac{\partial \mathcal{W}^i(\epsilon, \boldsymbol{\mathcal{Y}}^i(\epsilon), \boldsymbol{\mathcal{Z}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Y}}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Z}}}^i(\epsilon))}{\partial \epsilon} \\
&= \min_{K_i \in \bar{K}_i} \Bigg\{ \frac{\partial \mathcal{W}^i(\epsilon, \boldsymbol{\mathcal{Y}}^i(\epsilon), \boldsymbol{\mathcal{Z}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Y}}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Z}}}^i(\epsilon))}{\partial \tilde{\boldsymbol{\mathcal{Z}}}^i(\epsilon)} \mathcal{G}(\tilde{\boldsymbol{\mathcal{Y}}}^i(\epsilon)) \\
&+ \frac{\partial \mathcal{W}^i(\epsilon, \boldsymbol{\mathcal{Y}}^i(\epsilon), \boldsymbol{\mathcal{Z}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Y}}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Z}}}^i(\epsilon))}{\partial \boldsymbol{\mathcal{Z}}(\boldsymbol{\epsilon})} \mathcal{G}(\boldsymbol{\mathcal{Y}}^i(\epsilon)) \\
&+ \frac{\partial \mathcal{W}^i(\epsilon, \boldsymbol{\mathcal{Y}}^i(\epsilon), \boldsymbol{\mathcal{Z}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Y}}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Z}}}^i(\epsilon))}{\partial \text{vec}(\tilde{\boldsymbol{\mathcal{Y}}}^i(\epsilon))} \\
&\cdot \text{vec}(\mathcal{F}(\tilde{\boldsymbol{\mathcal{Y}}}^i(\epsilon), \tilde{K}_i(\epsilon), \tilde{K}_{N-i}(\epsilon))) \\
&+ \frac{\partial \mathcal{W}^i(\epsilon, \boldsymbol{\mathcal{Y}}^i(\epsilon), \boldsymbol{\mathcal{Z}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Y}}}^i(\epsilon), \tilde{\boldsymbol{\mathcal{Z}}}^i(\epsilon))}{\partial \text{vec}(\boldsymbol{\mathcal{Y}}^i(\epsilon))} \\
&\cdot \text{vec}(\mathcal{F}(\boldsymbol{\mathcal{Y}}^i(\epsilon), K_i(\epsilon), K_{N-i}^*(\epsilon))) \Bigg\},\ K_i(\epsilon) = K_i^*(\epsilon)
\end{aligned}$$

with the boundary condition,

$$\begin{aligned}&\mathcal{W}^i(\epsilon,\boldsymbol{\mathcal{Y}}^i(\epsilon),\boldsymbol{\mathcal{Z}}^i(\epsilon),\tilde{\boldsymbol{\mathcal{Y}}}^i(\epsilon),\tilde{\boldsymbol{\mathcal{Z}}}^i(\epsilon))\\ &=\phi^i(\mathbf{H}^i(t_0),\mathbf{D}^i(t_0),\tilde{\mathbf{H}}^i(t_0),\tilde{\mathbf{D}}^i(t_0)) \\ &(\epsilon,\boldsymbol{\mathcal{Y}}^i(\epsilon),\boldsymbol{\mathcal{Z}}^i(\epsilon),\tilde{\boldsymbol{\mathcal{Y}}}^i(\epsilon),\tilde{\boldsymbol{\mathcal{Z}}}(\epsilon))\in\mathcal{M}_i.\end{aligned}\tag{13}$$

Under these conditions, it must be true that $1 \leq i \leq N$

$$\begin{aligned}&\mathcal{W}^i(\epsilon,\boldsymbol{\mathcal{Y}}^i(\epsilon),\boldsymbol{\mathcal{Z}}^i(\epsilon),\tilde{\boldsymbol{\mathcal{Y}}}^i(\epsilon),\tilde{\boldsymbol{\mathcal{Z}}}^i(\epsilon))\\ &=\mathcal{V}^i(\epsilon,\boldsymbol{\mathcal{Y}}^i(\epsilon),\boldsymbol{\mathcal{Z}}^i(\epsilon),\tilde{\boldsymbol{\mathcal{Y}}}^i(\epsilon),\tilde{\boldsymbol{\mathcal{Z}}}^i(\epsilon))\end{aligned}$$

and that (K_i^*, K_{N-i}^*) is a Nash equilibrium.

Lemma 4.4 is now used to establish the form of the Nash solution to the MCCDS game.

Theorem 4.5: (N-Player MCCDS Nash Solution) Consider the LQG stochastic optimal control problem involving the process (1) and the costs (2). Then Player i's Nash equilibrium solution is characterized by the optimal gain

$$K_i^*(\alpha) = -R_{ii}^{-1}(\alpha)B_i^T\left(H_1^{i*}(\alpha)+\sum_{j=2}^{r}\gamma_j^i(\alpha)H_j^{i*}(\alpha)\right)\tag{14}$$

with

$$\gamma_j^i(\alpha)=\left(\frac{\dfrac{\partial g^i(\boldsymbol{\kappa}^{i*}(\alpha),\tilde{\boldsymbol{\kappa}}^i(\alpha))}{\partial\kappa_j^i(\alpha)}}{\dfrac{\partial g^i(\boldsymbol{\kappa}^{i*}(\alpha),\tilde{\boldsymbol{\kappa}}^i(\alpha))}{\partial\kappa_1^i(\alpha)}}\right),\ 1\leq i\leq N,\ 2\leq j\leq r$$

and where the jth optimal cost cumulant for Player i is defined by

$$\kappa_j^{i*}(\alpha) = x_0^T H_j^{i*}(\alpha)x_0 + D_j^{i*}(\alpha)$$

and the jth target cost cumulant for Players i is,

$$\tilde{\kappa}_j^i(\alpha) = x_0^T\tilde{H}_j^i(\alpha)x_0+\tilde{D}_j^i(\alpha)$$

where $1 \leq j \leq r$ and $1 \leq i \leq N$. The optimal state variables $\mathbf{H}^{i*}(\alpha)$ and $\mathbf{D}^{i*}(\alpha)$ follow the equations of motion

$$\begin{aligned}\frac{d\mathbf{H}^{i*}(\alpha)}{d\alpha}&=\mathcal{F}(\mathbf{H}^{i*}(\alpha),K_i(\alpha),K_{N-i}(\alpha))\\ \frac{d\mathbf{D}^{i*}(\alpha)}{d\alpha}&=\mathcal{G}(\mathbf{H}^{i*}(\alpha))\\ \mathbf{H}^{i*}(t_f)&=\mathbf{H}_f^i,\ \mathbf{D}^{i*}(t_f)=\mathbf{D}_f^i\end{aligned}$$

and the target variables are

$$\begin{aligned}\frac{d\tilde{\mathbf{H}}^{i}(\alpha)}{d\alpha}&=\mathcal{F}(\tilde{\mathbf{H}}^{i}(\alpha),\tilde{K}_i(\alpha),\tilde{K}_{N-i}(\alpha))\\ \frac{d\tilde{\mathbf{D}}^{i}(\alpha)}{d\alpha}&=\mathcal{G}(\tilde{\mathbf{H}}(\alpha))\\ \tilde{\mathbf{H}}^{i}(t_f)&=\tilde{\mathbf{H}}^i_{f;\mathcal{E}_i^*},\ \tilde{\mathbf{D}}^{i}(t_f)=\tilde{\mathbf{D}}_{f;\epsilon_i^*}.\end{aligned}$$

Proof: See [5] ■

V. SIMULATION RESULTS

In [4], a four-story structure is considered. A cost density-shaping game will be posed using this system, which has two disturbances - the ground excitation, and also uncertainties in the system. These system uncertainties pertain to the stiffness and damping matrices, and also those dealing with the control input itself.

First consider the parameters $k = 350 \times 10^6$ N/m, $m = 1.05\times10^6$ kg, and $c = 1.575\times10^6$ Ns/m. Using these values, the stiffness, damping, and mass matrices (denoted as K, C, M respectively) can be defined as

$$K=\begin{bmatrix}4k & -2k & 0 & 0\\ -2k & 3k & -k & 0\\ 0 & -k & 2k & -k\\ 0 & 0 & -k & k\end{bmatrix},$$
$$C=\begin{bmatrix}2c & -c & 0 & 0\\ -c & 2c & -c & 0\\ 0 & -c & 2c & -c\\ 0 & 0 & -c & c\end{bmatrix},\ M=2m\cdot\boldsymbol{I}^{4\times4}.$$

In terms of these matrices, the system matrices are

$$A=\begin{bmatrix}\mathbf{0}^{4\times4} & \boldsymbol{I}^{4\times4}\\ -M^{-1}K & -M^{-1}C\end{bmatrix},$$
$$B=\begin{bmatrix}\mathbf{0}^{4\times4}\\ -M^{-1}B_{ch}\end{bmatrix},\ G=\begin{bmatrix}\mathbf{0}^{4\times1}\\ F_w\end{bmatrix}$$

with B_{ch} and F_w given as

$$B_{ch}=\boldsymbol{I}^{4\times4}+\begin{bmatrix}\mathbf{0}^{3\times1} & -\boldsymbol{I}^{3\times3}\\ 0 & \mathbf{0}^{1\times3}\end{bmatrix},\ F_w=\frac{1}{m}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}.$$

To represent how uncertainties impact the dynamics of the system, the following matrix is used,

$$\begin{bmatrix}\mathbf{0}^{4\times4} & \mathbf{0}^{4\times4} & \mathbf{0}^{4\times4}\\ -M & -M & -M^{-1}B_{ch}\end{bmatrix}$$

The dynamics of the 4-story structure with the ground acceleration $\ddot{x}_g(t)$ due to the earthquake, and accounting for system uncertainties, are given by the following model, with initial condition $x_0 = E\{x(t_0)\}$,

$$\begin{aligned}dx(t)&=\Big(Ax(t)+Bu(t)+Dw(t)\Big)dt+G\ddot{x}_g(t),\\ z_1(t)&=H_1x(t)+G_1u(t)\\ z_2(t)&=H_2x(t)+G_2u(t),t\in[t_0,t_f].\end{aligned}$$

Above, the regulated outputs $z_1(t)$ and $z_2(t)$ are characterized by H_1, H_2 and G_1, G_2 given below as

$$H_1=10^6\begin{bmatrix}\boldsymbol{I}^{8\times8}\\ \mathbf{0}^{4\times8}\end{bmatrix},\ G_1=\begin{bmatrix}\mathbf{0}^{8\times4}\\ \boldsymbol{I}^{4\times4}\end{bmatrix}$$
$$H_2=\begin{bmatrix}0.1K & \mathbf{0}^{4\times4}\\ \mathbf{0}^{4\times4} & -0.1C\\ \mathbf{0}^{4\times4} & \mathbf{0}^{4\times4}\end{bmatrix},\ G_2=\frac{1}{8\times10^5}\begin{bmatrix}\mathbf{0}^{8\times4}\\ \boldsymbol{I}^{4\times4}\end{bmatrix}.$$

Now, form a 2-Player MCCDS game as follows. Let *Player 1* be the control designer, who is interested in choosing $u(t) = K_1(t)x(t)$ to optimize the statistical characterization (the probability density) of the random payoff,

$$J_1 = \int_{t_0}^{t_f} z_1^T(\tau) z_1(\tau) d\tau$$

On the other hand, let *Player 2* be the second disturbance (e.g. the system uncertainties), who is interested in choosing $w(t) = K_2(t)x(t)$ to optimize the statistical characterization (the probability density) of the random payoff,

$$J_2 = \int_{t_0}^{t_f} \left(\delta^2 w^T(\tau) w(\tau) - z_2^T(\tau) z_2(\tau) \right) d\tau, \;\; \delta = 20$$

This formulation for costs is taken from the generalization of H_2/H_∞ control with multi-objective, multi-cumulant control attributed to Diersing [4]. Motivate the selections for performance indices by introducing normalized cost and target cost variates for $j = 1, 2$,

$$\begin{aligned} Z_i &= \frac{J_i - \kappa_1^i(t_0)}{\kappa_2^i(t_0)^{1/2}} \\ \tilde{Z}_i &= \frac{J_i - \tilde{\kappa}_1^i(t_0)}{\tilde{\kappa}_2^i(t_0)^{1/2}} \\ &= \underbrace{\left(\frac{\kappa_2^i(t_0)}{\tilde{\kappa}_2^i(t_0)} \right)^{1/2}}_{a_i} \cdot \underbrace{\frac{J - \kappa_1^i(t_0)}{\kappa_2^i(t_0)^{1/2}}}_{Z_i} + \underbrace{\frac{\kappa_1^i(t_0) - \tilde{\kappa}_1^i(t_0)}{\tilde{\kappa}_2^i(t_0)^{1/2}}}_{b_i} \\ &= a_i Z_i + b_i. \end{aligned}$$

with best Gaussian density approximations,

$$p_{Z_i}(z) \approx \frac{1}{\sqrt{2\pi}} \cdot \exp\left(\frac{-z^2}{2} \right), \;\; p_{\tilde{Z}_i}(\tilde{z}) \approx a_i p_{Z_i}(a_i z + b_i).$$

Consider the performance index for *Player 1* first,

$$\begin{aligned} & KLD(p_{Z_1}(z), p_{\tilde{Z}_1}(z)) \\ &= \int_{-\infty}^{\infty} p_{Z_1}(z) \log\left(\frac{p_{Z_1}(z)}{p_{\tilde{Z}_1}(z)} \right) dz \\ &= \frac{1}{2} \left(\frac{\kappa_2^1(t_0)}{\tilde{\kappa}_2^1(t_0)} - 1 - \log\left(\frac{\kappa_2^1(t_0)}{\tilde{\kappa}_2^1(t_0)} \right) + \frac{(\kappa_1^1(t_0) - \tilde{\kappa}_1^1(t_0))^2}{\tilde{\kappa}_2^1(t_0)} \right) \\ &= g_1\left(\begin{bmatrix} \kappa_1^1(t_0) \\ \kappa_2^1(t_0) \end{bmatrix}, \begin{bmatrix} \tilde{\kappa}_1^1(t_0) \\ \tilde{\kappa}_2^1(t_0) \end{bmatrix} \right). \end{aligned}$$

Next, consider the performance index for *Player 2*,

$$\begin{aligned} & HD^2(p_{Z_2}(z), p_{\tilde{Z}_2}(z)) \\ &= 1 - \int_{-\infty}^{\infty} \sqrt{p_{Z_2}(z) p_{\tilde{Z}_2}(z)} dz \\ &= 1 - \sqrt{2} \cdot \frac{(\kappa_2^2(t_0)\tilde{\kappa}_2^2(t_0))^{\frac{1}{4}}}{\sqrt{\kappa_2^2(t_0) + \tilde{\kappa}_2^2(t_0)}} \cdot \exp\left(-\frac{(\kappa_1^2(t_0) - \tilde{\kappa}_1^2(t_0))^2}{4(\kappa_2^2(t_0) + \tilde{\kappa}_2^2(t_0))} \right) \\ &= g_2\left(\begin{bmatrix} \kappa_1^2(t_0) \\ \kappa_2^2(t_0) \end{bmatrix}, \begin{bmatrix} \tilde{\kappa}_1^2(t_0) \\ \tilde{\kappa}_2^2(t_0) \end{bmatrix} \right). \end{aligned}$$

In the above expressions, the cumulants of each player's cost are computed according to (6) and (7) under optimal MCCDS controls of the form (14). The optimal gains involve the target cumulants for each player, which have been computed according to (11) and (9), with $\mathcal{E}^* = \mathbf{0}^{8\times 8}$ and $\epsilon^* = 1.0 \times 10^{-9}$. For *Player 1* and *Player 2*, these equations are solved using the control gains below to drive the target cost cumulants for each player, where $\mu_2^1 = 1.0 \times 10^{-5}$ and $\mu_2^2 = 0$.

$$\tilde{K}_i(\alpha) = -R_{ii}^{-1} B^T (\tilde{H}_1^i(\alpha) + \mu_2^i \tilde{H}_2^i(\alpha))$$

By numerical simulation, it can be verified that the 2-Player MCCDS controls approximately realize the target mean-variance approximations to the target densities for the random costs J_1 and J_2.

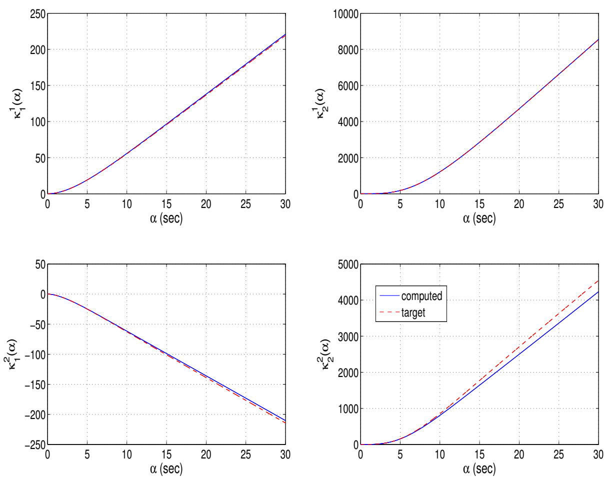

Fig. 1: Cumulant Trajectories, 2-Player MCCDS Game

REFERENCES

[1] K. D. Pham,"Statistical Control Paradigms for Structural Vibration Suppression", *Ph.D. Dissertation*, Department of Electrical Engineering, University of Notre Dame, 2004.

[2] M. J. Zyskowski, M.K. Sain, and R.W. Diersing, "State-Feedback, Finite-Horizon, Cost Density-Shaping Control for the LQG Framework", accepted to the *Journal of Optimization Theory and Applications*, September 2010.

[3] M. J. Zyskowski, M. K. Sain, and R. W. Diersing, "Maximum Bhattacharyya Coefficient, Cost Density-Shaping: A New Cumulant-Based Control Paradigm with Applications to Seismic Protection", *Proceedings of the 5th World Conference on Structural Control and Monitoring*, 5WCSCM-10402, July 2010.

[4] R. W. Diersing and M. K. Sain, "Multi-cumulant, multi-objective structural control: a circuit analog", *Journal of Analog Circuits and Signal Processing*, Vol. 56, pp. 117-126, 2008.

[5] M. J. Zyskowski,"Cost Density-Shaping for Stochastic Optimal Control", *Ph.D. Dissertation*, Department of Electrical Engineering, University of Notre Dame, 2010.