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Traveling waves in one-dimensional networks of dynamical systems

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***Abstract*— Propagation of traveling waves along one-dimensional networks of identical dynamical systems is analysed by suitably defining a family of ordinary differential equations (ODEs) that describes the traveling wave itself. An ODE of reduced order is derived for computing reference solutions, which are then exploited to prove via implicit function theorem the existence of similar solutions in the original network. An example is included to illustrate the effectiveness of the proposed approach.**

I. INTRODUCTION

Networks of coupled dynamical systems arise in a wide variety of fields, ranging from physics to biology to neuroscience, just to name a few (see [1] and reference therein). For example, brain stimuli move among different areas of the brain by propagation of traveling waves across a network of neurons [2]; similarly, ring networks of neurons have been analysed as central pattern generators in the nervous system [3]. Analogous models have been proposed to describe some locomotory patterns such as crawling, where locomotion is achieved by propagation of a peristaltic wave along the animal body [4]. In physics, chains of masses linked by linear or nonlinear springs have been used to model propagation of information across networks [5] as well as in material science [6]. Networks of dynamical systems have been also proposed in computer science as a paradigm for analog dynamic processors arrays, e.g. cellular neural networks [7] have shown a great flexibility in generating patterns [8] and self-emerging phenomena [9].

Understanding the basic mechanisms which give rise to the emergence of such patterns represents a great challenge due to the high dimension of state space of these systems, to the rich family of behaviours that they can show and to the nontrivial interplay between local rules and global behaviour. Very often, as in all the above mentioned scenarios, the relevant evolution of the network dynamics is well described by the propagation of traveling waves across the nodes present in the network itself. Being able to model traveling wave propagation in networks can then be useful both for gaining insights on existing behaviours and for designing systems capable of sustaining waves having desired characteristics. Several approaches have been proposed for analysing this kind of problems. They include, for example, contraction analysis [10] and harmonic balance techniques [11].
An alternative way for the analysis of networks of dynamical systems is considering the finite number of nodes as a sampling from a continuous distribution of nodes with the same dynamics. In other words, one tries to build a partial differential equation whose solution evaluated at the location of each system is close to the output of the system itself. This allows for a dramatic reduction of the number of parameters that are present in the systems, being both the local dynamics and local interconnections collapsed into a single equation. On the other hand, it leads naturally to the problem of estimating how the partial differential equation solution is *close* (in some sense) to the one given by the original network. The reversed, and more diffused in the literature, viewpoint is obviously simulating a partial differential equation via a suitable network composed of a finite number of dynamical systems, see, e.g., [12]. It has been proved in [13] that the dynamics of a network of dynamical systems is in general a broader class with respect to the dynamics obtainable by partial differential equations. This is mainly due to the presence of phenomena, such as propagation failure, that are exclusive of spatially discrete systems.

In this paper, we exploit the continuous interpolation approach to study the existence of traveling waves and their propagation in chains, i.e. one dimensional networks, of nonlinear systems. Extensions to two- and three-dimensional cases can be easily obtained as discussed in the text. In particular, given a network, in Section II we introduce a family of associated Partial Differential Equations (PDEs) whose solutions approximate the network behaviour. We then obtain a family of Ordinary Differential Equations (ODEs) that describes the propagation of traveling waves in the corresponding PDEs. The ODE that provides the exact solution of the traveling wave problem is an infinite order ODE, so a *reference* finite order ODE is defined by truncating the higher order terms. Conditions for the “equivalence” between the nominal and the exact ODE solutions are obtained by the implicit function theorem in Section III and an example is included in Section IV to illustrate the proposed approach.

Notation. In the following we will denote as $\mathbb{R}$ the set of real numbers and as $\mathbb{R}^+$ and $\mathbb{R}_0^+$ the set, respectively, of positive and non-negative real numbers. The set of complex numbers is referred to as $\mathbb{C}$. Let f be a function $f : [0, T] \to \mathbb{R}$, then, we define its p-norm as

$$\|f\|_p = \left(\int_0^T |f|^p \right)^{\frac{1}{p}} .$$

We indicate with $L^2[0, T]$ the space of functions with $\|f\|_2 < \infty$. Given a normed space $\mathcal{N}$ and an operator

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$M : \mathcal{N} \to \mathcal{N}$, the norm $\|M\|$ is defined as

$$\|M\| = \sup_{\substack{f\in\mathcal{N}\\ f\neq 0}} \frac{\|M(f)\|}{\|f\|} \ .$$

Moreover, given two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ with the respective norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$, we denote by $\mathcal{X} \oplus \mathcal{Y}$ the space $\mathcal{X} \times \mathcal{Y}$ equipped with the direct sum norm

$$\|x \oplus y\| = \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}} \ . \tag{1}$$

Note that $\mathcal{X} \oplus \mathcal{Y}$ is a Banach space as well [14].

II. Network of dynamical systems

In this section we introduce a procedure to investigate traveling waves propagation in networks of evenly distributed identical dynamical systems. For the sake of simplicity, hereafter we will introduce the procedure for dynamical systems settled in one-dimensional spatial configurations (chains), with periodic boundary conditions and first neighbours linear connections. Generalizations of this framework to networks with different types of connections are straightforward, as briefly discussed in the last section.

The main idea consists in the definition of a suitable ODE, whose periodic solutions can be related to the patterns showed by the original network. Some preliminary observations are in order.

The studied solutions are traveling waves, hence the outputs of the systems along the chain represent a sampling of their temporal evolutions. Moreover, since the number of systems equals the number of spatial samples, the network can only sustain real traveling waves with suitable shapes according to the Shannon theory [15]. In particular, the spatially discrete system can not show traveling wave solutions having spatial frequencies greater than one-half of the characteristic frequency z^{-1}, z being the distance between two nodes of the network.

Let us now consider N identical dynamical systems placed along a chain of total length l with pace $z = l/N$. Assume that they admit the following n-th order ODE model in their variables $\xi_j(t) \in \mathbb{R}$, $j = 1, \dots, N$:

$$\begin{aligned} &L_0(\mathcal{D}_t^n, \dots, \mathcal{D}_t, 1)\ \xi_j(t) = \\ &= F_0\left(\mathcal{D}_t^{n-1}\xi_j(t), \dots, \mathcal{D}_t\xi_j(t), \xi_j(t)\right) + \\ &+ h_{-1}\xi_{j-1}(t) + h_1\xi_{j+1}(t)\ , \end{aligned} \tag{2}$$

where $\mathcal{D}_t$ is the time derivative, L_0 expresses a linear combination of its arguments, $F_0 : \mathbb{R}^n \to \mathbb{R}$ is a scalar nonlinear function of ξ_j and its derivatives and $h_{-1}, h_1 \in \mathbb{R}$ are constant. Systems of the form (2) are an extension of the well known Lur'e model [16], which has been largely studied in the literature as central element of dynamical networks [11], [17]. In order to have a clear separation between the state space of the networked systems and their position along the chain, we introduce the spatial coordinate $x \in [0, l)$. We also assume as working hypothesis that the network is subjected to periodic boundary conditions, so that the equivalence relationship $x + zN \equiv x$ holds. Moreover, let us denote by x_j the j-th node position. It follows that $x_{j\pm1} \equiv (x_j \pm z) \mod l$, being $\mod$ the modulo operator.

Definition 1: A function $\xi(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an *interpolation* of the network state, if $\xi(x_j, t) = \xi_j(t)$, $\forall j = 1, \dots, N$, $\forall t \in \mathbb{R}$.

The interpolation property defines a class of equivalence. Among all the functions which satisfy this condition, we point out the subclass of the ones with the following property.

Definition 2: An interpolation $\xi(x,t)$ of the network state is said a *regular approximation* of the system solution, if it admits along the coordinate x the power series development of radius z

$$\begin{aligned} &\xi(x_{j\pm1}, t) = \sum_{i=0}^{+\infty} \frac{1}{i!} \frac{\partial^i}{\partial x^i} \xi(x_j, t)(\pm z)^i\ , \\ &\forall j = 1, \dots, N\ ,\ \forall t \in \mathbb{R}\ . \end{aligned}$$

Note that in terms of regular approximations, the dynamics of $\xi(x,t)$ can be represented by a PDE of infinite order in the space variable x. Indeed, denoting by $\mathcal{D}_t$ and $\mathcal{D}_x$ the partial derivatives with respect to time and space respectively, the above model can be formulated as:

$$\begin{aligned} &L_0(\mathcal{D}_t^n, \dots, \mathcal{D}_t, 1)\ \xi(x_j, t) = \\ &= F_0\left(\mathcal{D}_t^{n-1}\xi(x_j,t), \dots, \mathcal{D}_t\xi(x_j,t), \xi(x_j,t)\right) + \\ &+ \left(h_{-1}\xi(x_{j-1},t) + h_1\xi(x_{j+1},t)\right) = \qquad (3) \\ &= F_0\left(\mathcal{D}_t^{n-1}\xi(x_j,t), \dots, \mathcal{D}_t\xi(x_j,t), \xi(x_j,t)\right) + \\ &+ \sum_{i=0}^{+\infty} \frac{1}{i!} \mathcal{D}_x^i \xi(x_j,t)\left(h_1 + h_{-1}(-1)^i\right) z^i\ . \qquad (4) \end{aligned}$$

In the following, we focus on a particular class of solutions, namely traveling waves. To this aim, let us consider regular approximations of the form

$$\xi(x,t) = \psi(kx - ct) = \psi(\tau)\ ,$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is a periodic function of period T in τ. Observe that, since ψ describes a traveling wave that propagates with phase velocity c/k, there is no ambiguity in the interpolation of the network samples. Moreover, with respect to ψ, equation (3) must hold at every point x. Denoting as $\mathcal{D}$ the derivative of ψ with respect to τ and observing that $\mathcal{D}_t\xi(x,t) = -c\mathcal{D}\psi(kx - ct)$ and $\mathcal{D}_x\xi(x,t) = k\mathcal{D}\psi(kx - ct)$, in terms of regular approximation, it follows that

$$\begin{aligned} &\xi(x \pm z, t) = \psi(kx \pm kz - ct) = \psi(\tau \pm kz) = \\ &= \psi(\tau) \pm zk\mathcal{D}\psi(\tau) + \frac{1}{2}z^2k^2\mathcal{D}^2\psi(\tau) + \ldots + \\ &+ \frac{1}{(n-1)!}(\pm z)^{n-1}k^{n-1}\mathcal{D}^{n-1}\psi(\tau) + \\ &+ \sum_{i=n}^{+\infty} \frac{1}{i\,!}(\pm z)^i k^i \mathcal{D}^i\psi(\tau)\ . \end{aligned}$$

Let us then define the spatial shift operator parametrized by z,

$$S(z)\psi(\tau) = h_1\psi(\tau + kz) + h_{-1}\psi(\tau - kz)\ ,$$

the linear parametric operator

$$L_1(\mathcal{D}^{n-1},\dots,\mathcal{D},1;z)\psi(\tau) = (h_1+h_{-1})\psi(\tau)+ \\ +(h_1-h_{-1})zk\mathcal{D}\psi(\tau)+\frac{1}{2}(h_1+h_{-1})z^2k^2\mathcal{D}^2\psi(\tau)+\dots+ \\ +\frac{1}{(n-1)!}\left(h_1+h_{-1}(-1)^{n-1}\right)z^{n-1}k^{n-1}\mathcal{D}^{n-1}\psi(\tau)$$

and their composition

$$G(\mathcal{D}^{n-1},\dots,\mathcal{D},1,S(z))\psi(\tau) = \\ = z^{-n}\left(S(z)-L_1(\mathcal{D}^{n-1},\dots,\mathcal{D},1;z)\right)\psi(\tau) = \\ = \sum_{i=n}^{+\infty}\frac{1}{i!}\left(h_1+h_{-1}(-1)^i\right)z^{i-n}k^i\mathcal{D}^i\psi(\tau)\ .$$

It is worth noticing that $G(\mathcal{D}^{n-1},\dots,\mathcal{D},1,S(z))\psi(\tau)\to(1/n!)k^n\mathcal{D}^n\psi(\tau)$ as $z\to 0$. Then, (3) becomes

$$L_0\left((-c)^n\mathcal{D}^n,\dots,-c\mathcal{D},1\right)\psi(\tau) = \\ = F_0\left((-c)^n\mathcal{D}^n\psi(\tau),\dots,\mathcal{D}\psi(\tau),\psi(\tau)\right)+ \\ +L_1(\mathcal{D}^{n-1},\dots,\mathcal{D},1;z)\psi(\tau)+ \\ +z^nG(\mathcal{D}^{n-1},\dots,\mathcal{D},1,S(z))\psi(\tau)\ ,$$

which, for the sake of simplicity, can be rewritten in compact form as

$$L(\mathcal{D};c,z)\psi(\tau) = F_0(\mathcal{D},\psi(\tau);c)+z^nG(\mathcal{D},S(z))\psi(\tau) \tag{5}$$

where

$$L(\mathcal{D};c,z)\psi(\tau) = (L_0(\mathcal{D};c)-L_1(\mathcal{D};z))\psi(\tau)\ . \tag{6}$$

Therefore, the study of traveling waves propagation in the original network (2) has boiled down to the investigation of the periodic solutions of the associated ODE (5), which is derived from the general interpolating PDE (4).

Definition 3: The n-th order *reference ODE* describing the original network is defined as

$$L(\mathcal{D};c,z)\psi(\tau) = F_0(\mathcal{D},\psi(\tau);c)\ . \tag{7}$$

In the following section we will introduce a procedure to study the existence of periodic solutions of (5), when the dynamical system represented by (7) admits limit cycles.

III. Existence of periodic solutions

Let us consider the complete ODE (5). For a fixed pair (c,z), the operator $L(\mathcal{D};c,z)$ produces a linear combination of the input function and its derivatives. Then, if we apply such an operator to periodic functions, the corresponding kernel turns out being characterized only by the period and, in particular, there may exist at most n different frequencies related to it. Since $L(\mathcal{D};c,z)$ does not change the period, it is locally invertible when dealing with periodic functions that do not belong to the kernel. Then, let us denote by $L^{-1}(\mathcal{D};c,z)$ its inverse operator and consider the following parametric form of (5):

$$\psi(\tau) = \Pi(\psi(\tau);\varepsilon) = \\ = L^{-1}(\mathcal{D};c,\varepsilon)\big(F_0(\mathcal{D},\psi(\tau);c)+\varepsilon^nG(\mathcal{D},S(\varepsilon))\psi(\tau)\big) \tag{8}$$

where we have introduced a parameter ε in place of z in order to be able to vary it continuously. Assume that, for $\varepsilon=0$, ψ_0 is a periodic solution of (8) with period T_0 and angular frequency $\omega_0=2\pi/T_0$, that is ψ_0 is a fixed point of $\Pi(\,\cdot\,;0)$. Observe that the output of Π, when it is fed with a periodic function of period $T=2\pi/\omega$, is still a periodic function, that admits a unique representation in terms of the functional basis of $L^2\,[0,T]$ provided by the powers of $e^{\jmath\omega}$, i.e. in terms of Fourier series development. Notice also that, by introducing in (8) the time transformation $\tau\mapsto\tau^*=\tau/T$, the periodic function $\psi(\tau)$ of period T reduces to a new periodic function $u(\tau^*)=\psi(T\tau^*)\in L^2\,[0,1]$. However, since the linear maps L, G and also the nonlinear function F_0 depend on multiple time derivatives of their arguments, it follows that their periodic outputs depend on the original angular frequency of the related periodic inputs.

Proposition 4: Any bounded periodic function can be represented as a pair $(u,\omega)\in\mathcal{Q}=L^2\,[0,1]\times\mathbb{R}_0^+$, where u specifies its shape and $\omega=2\pi/T$ takes into account its original period. Any non constant bounded periodic function admits infinite isolated representations in $\mathcal{Q}$.

Proof. The only non trivial point is given by the infinite isolated representations. Let us consider a non constant periodic function of period $T=2\pi/\omega$ and observe that all its equivalent descriptions can only be originated by different way of expressing it as the concatenation of a certain unit pattern. Then, all the alternative representations can only be obtained by repeating the original pattern an integer number n of times and recurring to the related angular velocity equal to ω/n. This implies the existence of infinite descriptions, which are also isolated along the ω coordinate. To complete the proof, it is sufficient to notice that all the considered operators are time invariant and thus time shifts can be neglected. □

Remark 5: The space $\mathcal{Q}=L^2\,[0,1]\oplus\mathbb{R}_0^+$ provided with the direct sum norm (1) is a Banach space.

It is worth observing, that we can represent the parametric operator Π as the map $\Pi^*:\mathcal{Q}\times\mathbb{R}\to\mathcal{Q}$ defined by

$$\Pi^*(u,\omega;\varepsilon) = [\hat{u},\hat{\omega}]^T = \\ = \begin{bmatrix} L^{-1}(\omega;c,\varepsilon)\left(F_0(u,\omega;c)+\varepsilon^nG(\omega,\varepsilon)u\right) \\ \omega \end{bmatrix}\ , \tag{9}$$

where L, G and F_0 acts on the shape of the input according to the value of ω. Some considerations on Π^* are in order. It is straightforward to observe that G and L^{-1} (where L can be inverted) are linear in u. Moreover, they are continuous functions with respect to ω and ε. Notice also that F_0 is the composition of two different functions. Indeed, given the original F_0 in (2), we have that F_0 in (9) is such that $F_0(u,\omega;c)=F_0\left(\Delta(\omega)u;c\right)$, where $\Delta:\mathbb{R}\to\mathbb{C}^n$ is the column operator defined as

$$\Delta(\omega) = \begin{bmatrix}\omega^{n-1}\mathcal{D}^{n-1} & \dots & \omega\mathcal{D} & 1\end{bmatrix}^T \tag{10}$$

and it is responsible for the multiple time derivatives of u.

Proposition 6: If F_0 in (2) is a continuous nonlinearity in u, then $\Pi^*(u,\omega;\varepsilon)$ turns out to be a continuous operator in u, ω and ε.

Proof. The operator $F_0(u,\omega)$ is continuous with respect to its arguments, since the original nonlinearity is such and Δ acts linearly on u and it is continuous on ω. Furthermore, the continuity with respect to ε is trivial. Then, $\Pi^*(u,\omega;\varepsilon)$ is a continuous operator as well, because it is a composition of functions and operators which satisfy that property. □

Let us assume that F_0 is differentiable in $q_0 = (u_0,\omega_0)$, which represents the fixed point of (8) for $\varepsilon = 0$, and let $DF_0(u_0,\omega_0)$ be its derivative in that point:

$$\begin{aligned} DF_0(q_0) &= [D_u F_0(u_0,\omega_0), D_\omega F_0(u_0,\omega_0)] = \\ &= \big[JF_0(\Delta(\omega_0)u_0)\; D\Delta(\omega_0)u_0 \;, \\ &\quad JF_0(\Delta(\omega_0)u_0)\; J\Delta(\omega_0)\; u_0 \big] \;, \end{aligned}$$

where JF_0 represents the Jacobian row vector of the original F_0 in (2), $J\Delta$ is the Jacobian column vector of the function (10) with respect to ω and $D\Delta$ is the derivative with respect to u of the function $\Delta(\omega)u$. Similar considerations hold for L and G.

In order to study the existence of fixed points of (8) as ε varies, let us introduce the operator $H(q,\varepsilon) : \mathcal{Q}\times\mathbb{R}\to\mathcal{Q}$ defined as $H(q,\varepsilon) = q - \Pi^*(q;\varepsilon)$ with $q = (u,\omega)\in\mathcal{Q}$. According to Proposition 6 and the assumptions about the differentiability property, H is defined and continuous on Banach spaces, $H(q_0,0) = (0,0)$ and H is differentiable in $(q_0,0)$. In particular, the derivative $DH(q_0,0)$ is a linear operator defined by the matrix

$$\begin{aligned} &DH(q_0,\varepsilon) = \\ &= \begin{bmatrix} 1 - L^{-1}(\omega_0;c,\varepsilon)\, D_u F_0(u_0,\omega_0) & 0 \\ J(L^{-1})(\omega_0;c,\varepsilon)\, J\Delta(\omega_0)\, D_\omega F_0(u_0,\omega_0) & 1 \end{bmatrix} . \end{aligned} \tag{11}$$

Proposition 7: Assume that

$$L^{-1}(\omega_0;c,0) D_u F_0(u_0,\omega_0) \neq 1 \; . \tag{12}$$

Then, the inverse map of the linear operator $DH(q_0,\varepsilon)$ is bounded for sufficiently small ε.

Proof. The proof directly follows by checking the singular value for the matrix operator in (11), also considering that $L^{-1}(\omega_0;c,0)$ is bounded, because of the hypothesis about the existence of the nominal solution ψ_0. □

Then, under condition (12) and according to the Implicit Function Theorem [18], we can state that there exists a continuous function $q(\varepsilon) = (u(\varepsilon),\omega(\varepsilon))$ such that its graph is contained in a sufficiently small ball centered in $((0,0),r)$ and such that $H(q(\varepsilon),\varepsilon) = H(q_0,0) = (0,0)$. The above reasoning proves that, when the reference ODE has a bounded periodic solution for $\varepsilon = 0$, the map $\Pi(\,\cdot\,;\varepsilon)$ has a solution, which is close to it for a sufficiently small $\varepsilon < \bar{\varepsilon}$.

Proposition 8: Assume that L^{-1} admits the power series development

$$\begin{aligned} L^{-1}(u,\omega;\varepsilon) &= L^{-1}(u,\omega;0) + D_\varepsilon(L^{-1})(u,\omega;0)\varepsilon + \\ &+ \frac{1}{2} D^2_\varepsilon(L^{-1})(u,\omega;0)\varepsilon^2 + \dots \; . \end{aligned} \tag{13}$$

Then, for a sufficiently small ε the reference ODE (7) has a periodic solution which is a better approximation than ψ_0.

Proof. Under assumption (12) for a sufficiently small ε there exists a periodic solution of the complete ODE (5). Since the term with the lowest order in ε belongs to $L^{-1}(u,\omega;\varepsilon)F_0(u,\omega)$, being G a $O(\varepsilon^n)$, it follows that the reference ODE with $\varepsilon \neq 0$ represents a most suited operator to describe the complete ODE when ε is sufficiently small. □

Proposition 9: Let us assume that the reference ODE admits a periodic solution $\psi_0(\tau)$ for $z = 0$ and that condition (12) is satisfied. Then, there exists a new network, counting a sufficiently large number N^* of identical systems, such that the reference ODE (7) with $z^* = l/N^*$ has a periodic solution $\psi_{z^*}(\tau)$, which is close to an existing periodic solution of the complete network ODE (5), i.e. the admissible traveling waves are approximated by $\psi_{z^*}(kx - ct)$.

Proof. The proof is a direct consequence of the previous results by choosing N^* such that $z^* = l/N^* < \bar{\varepsilon}$. □

IV. An Illustrative Example

In this section we show the effectiveness of the proposed approach by considering an example where the features of the solution of the reference ODE can be obtained by analytical tools, whereas the direct analysis of the original network dynamics is not trivial.

Let then consider a network composed of N third-order dynamical systems with local nonlinearities, where each system receives as input the sum of its first neighbours outputs. The dynamics of the $j-$th system can be thus written as

$$\dddot{\xi}_j + \alpha\ddot{\xi}_j + \beta\dot{\xi}_j + \gamma\xi_j = \varphi(\xi_j) + \delta(\xi_{j+1} + \xi_{j-1}) \;, \tag{14}$$

where $\varphi(\cdot)$ indicates the nonlinear function that acts on the local dynamics. Comparing (14) with the generic expression (2) we have $L_0(\mathcal{D}^n_t,\dots,\mathcal{D}_t,1) = \mathcal{D}^3_t + \alpha\mathcal{D}^2_t + \beta\mathcal{D}_t + \gamma$, $F_0(\mathcal{D}^{n-1}_t\xi_j,\dots,\mathcal{D}_t\xi_j,\xi_j) = \varphi(\xi_j)$ and $h_1 = h_{-1} = \delta$. Note that equation (16) includes as special cases well known systems like Coullet systems [19] and the Genesio-Tesi system [20].

Introducing the interpolating function $\xi(x,t)$ such that $\xi_j(t) = \xi(jz,t)$, the partial differential equation associated with (14) boils down to

$$\frac{\partial^3\xi}{\partial t^3} + \alpha\frac{\partial^2\xi}{\partial t^2} + \beta\frac{\partial\xi}{\partial t} + (\gamma - 2\delta)\xi = \varphi(\xi) + \delta\frac{\partial^2\xi}{\partial x^2}z^2 \;, \tag{15}$$

where we have truncated the Taylor expansion of $\xi_{j\pm1}$ at the first non-vanishing spatial derivative. We are now able to study the traveling wave solutions of (15) by setting $\psi(kx - ct) := \xi(x,t)$, thus obtaining

$$-c^3\psi''' + \left(\alpha c^2 - \delta k^2 z^2\right)\psi'' - \beta c\psi' + (\gamma - 2\delta)\psi = \varphi(\psi) \;, \tag{16}$$

where the primes indicate derivatives with respect to the moving coordinate $\tau = kx - ct$. We note that the presence of periodic boundary conditions imposes constraints on the possible values that the wave number k can assume. In fact, given the total length l of the chain and the number N of dynamical systems, the periodic solution must have an integer number of period fitted in l, so that $k = s\left(T_\tau / l\right), \quad s = 1, 2, \ldots$ where T_τ is the period of the periodic solution of the ODE (16). In the following simulations we always choose $s = 1$, i.e. we make the hypothesis that only one period of the ODE solution is fitted in the total length of the chain. Moreover, as working hypothesis, we assume that the systems of the network are naturally stable when isolated ($\delta = 0$). Then, we investigate if traveling waves may arise when the connections are turned on ($\delta \neq 0$). We consider nonlinearities that vanish in the origin in order to have a fixed point at $\xi_j = 0$. Necessary conditions for stability of the linear part indicate that each isolated systems has a stable fixed point in the origin if $\alpha > 0, \gamma > 0, \beta > \gamma/\alpha$. Let us then consider, for example, the choice $\alpha = 1$, $\beta = 1.2$, $\gamma = 1$ so that the isolated nodes are stable.

Let us now focus on the reference ODE (16). When $\delta = 0$ the equation admits stable solutions only with $c < 0$ because only in this case the leading term of (14) and (16) have the same sign. In this case, the ODE (16) shows a stable fixed point $\psi(\tau) = 0$ and both the solutions of (16) and (14) converge towards the origin. A positive interconnection gain $\delta > 0$ preserves the stability of this solution and all the systems still converge toward the origin. On the other hand, a negative interconnection gain $\delta < 0$ makes the stable fixed point lose its stability and the solution of the ODE converges towards a stable limit cycle.
One can easily estimate some of the characteristics of this oscillating solution by noticing that the dynamical system (16) can be expressed in Lur'e form, which counts on well established theories for its analysis, such as describing function and harmonic balance techniques [16]. Let us then assume that the periodic solution of (16) can be well approximated by a pure harmonic function $\psi(\tau) \simeq B_\tau \cos(\omega_\tau \tau)$. Then, the nonlinear block φ can be substituted by its *describing function* defined as $N(B_\tau, \omega_\tau) := B_\tau^{-1}(B + jA)$, where B and A are defined as $\varphi(B_\tau \cos \omega_\tau \tau) \simeq B \cos(\omega_\tau \tau) + A \sin(\omega_\tau \tau)$. The harmonic balance technique asserts that B_τ and ω_τ are such that

$$L(j\omega_\tau) N(B_\tau, \omega_\tau) = 1 \;, \tag{17}$$

where $L(j\omega_\tau) \in \mathbb{C}$ is the frequency response of the linear part of the system [16]. If $\varphi(\cdot)$ is a static nonlinearity, as in this example, then $A = 0$, $N(B_\tau, \omega_\tau) = N(B_\tau) \in \mathbb{R}$ and (17) implies $L(j\omega_\tau) \in \mathbb{R}$, i.e. the frequency ω_τ does not depend on the nonlinearity. By imposing this condition in (16), we obtain $\omega_\tau = \sqrt{\beta/c^2}$. Moreover, once the frequency ω_τ is known, the oscillation amplitude can be estimated by solving (17) in the only unknown B_τ. In the following we will focus only on the cubic nonlinearity $\varphi(\xi_i) = \xi_i^3$. The describing function for the cubic nonlinearity is [16]

$$N(B_\tau, \omega_\tau) = \frac{3}{4} B_\tau^2 \;.$$

Notice that this describing function is monotonically increasing in B_τ, which is a shared feature among several

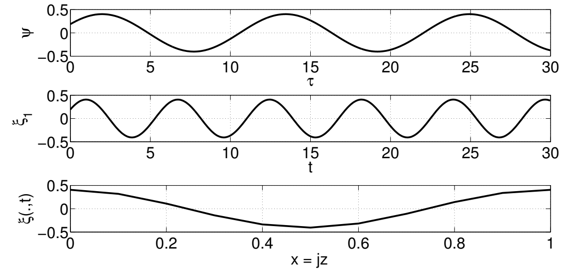

Fig. 1. Comparison between (a) reference ODE (16) solution $\psi(\tau)$, (b) $\xi_1(t)$ given by the distributed dynamics (14) and (c) final spatial profile $\xi(jz, 30)$, $j = 1, \ldots 11$.

nonlinearities (polynomial of arbitrary degree, hyperbolic and so on...) and thus the qualitative behaviour of the network is quite not affected by the exact shape of the nonlinearity. Within this hypothesis, the harmonic balance equation (17) provides the following estimate for the oscillation amplitude

$$B_\tau = \sqrt{\frac{4}{3}\left[-\left(\alpha c^2 - \delta k^2 z^2\right)\omega_\tau + \gamma - 2\delta\right]} \;. \tag{18}$$

Consider now the case $\delta = -0.2$, $l = 1$, $N = 11$ and $c = -2$ as illustrative example. The magnitude of the interconnection gain δ is sufficient to make the fixed point of (16) lose its stability and an Hopf bifurcation occours. The harmonic balance estimates for the frequency and the amplitude of oscillation read $\omega_\tau = 0.5477$, $B_\tau = 0.4017$, whereas periodic boundary conditions impose $k = 11.4715$. Note that obtaining these estimates directly from (14) would have been not trivial.

To check the quality of these predictions, numerical simulations have been performed as follows:

1) given the system parameters α, β, γ, δ, c, l and N, equation (16) is numerically integrated with random initial condition close to the origin, thus obtaining $\psi(\tau)$;
2) we sample $\psi(\tau)$ every kz unit of the moving coordinate τ;
3) we numerically integrate the distributed system (14) to obtain $\xi_j(t)$;
4) we check that the amplitude B_τ, the wave speed c and the wave vector k estimated from the latter simulation are in sufficient agreement with the ones imposed at the beginning.

In Fig. 1 we compare the solutions given by the reference ODE (16) with the distributed system (14) solution. Note that $\psi(\tau)$ has a period twice as long as that of $\xi_j(t)$, in complete agreement with the hypothesis $c = -2$. Indeed, the estimated c is -1.994, within less than one percent error. The spatial wave profile obtained by fixing $t = \bar{t}$ in the distributed system solution and plotting $\xi_j(\bar{t})$ at location $z = (j-1)z$ is reported at the bottom, showing that a full period is fitted in the total length l, in agreement with the choice of k. The full spatiotemporal profile is reported in Fig. 2. A discrepancy exists between ODE and distributed system solutions amplitude. Indeed, the amplitude of $\xi_j(t)$ is 0.406, whereas that of

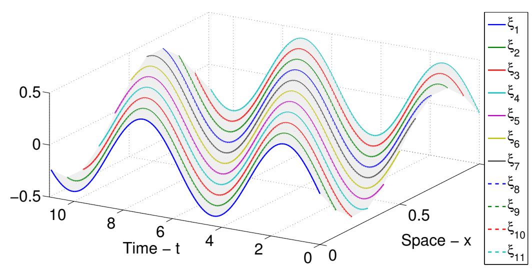

Fig. 2. Spatiotemporal profile of the traveling wave. A light shaded surface has been included to help the interpretation.

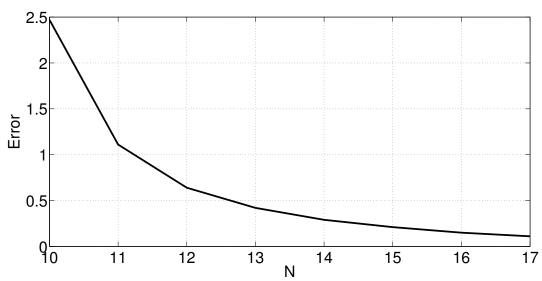

Fig. 3. Percentage error on oscillation amplitude as a function of the number N of chain elements.

$\psi(\tau)$ is 0.402 (about 1% error). This discrepancy is directly related to the approximation introduced by neglecting the exact ODE higher order dynamics to obtain the reference ODE (16) and can be reduced by increasing the number N of systems in the chain. In Fig. 3 the error on the amplitude is plotted as a function of N, showing a monotone behaviour, as expected.

V. Final remarks

In this paper we have analysed the propagation of traveling waves along one-dimensional chains of dynamical systems. We have proposed an approach based on the definition of a partial differential equation whose solutions evaluated at the position of each dynamical system are close to its output. Then a traveling wave solution has been imposed, thus obtaining a *reference* ODE. Conditions for the equivalence between the solutions of the distributed system and the reference ODE have been derived by implicit function theorem to provide useful informations about the existence of a class of traveling wave solutions in the original network. An example has been included to show the effectiveness of this approach. Similar qualitative results still hold for different local dynamics and interconnections, although they are not shown here due to space constraints. The presented results, here used for analysing the possible behaviour of an existing network, could also be exploited for designing the local dynamics or interconnections with the goal of obtaining a network capable of sustaining traveling waves with certain desired characteristics.

Although we have considered networks with first-neighbours interactions, long range coupling would only require considering higher order spatial derivatives in the derivation of the associated PDE. Similarly, networks in two or three dimensions can be studied by simply considering PDEs in two- or three-dimensional spaces. In both these extensions the only change is in the definition of the PDE, while the remaining part of the results still holds.

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References

[1] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, "Complex networks: Structure and dynamics," *Physics Reports*, vol. 424, no. 4-5, pp. 175 – 308, 2006.

[2] E. Chen, K. M. Stiefel, T. J. Sejnowski, and T. H. Bullock, "Model of traveling waves in a coral nerve network," *Journal of Comparative Physiology A: Neuroethology, Sensory, Neural, and Behavioral Physiology*, vol. 194, pp. 195–200, 2008.

[3] Y. Horikawa and H. Kitajima, "Bifurcation and stabilization of oscillations in ring neural networks with inertia," *Physica D: Nonlinear Phenomena*, vol. 238, no. 23-24, pp. 2409 – 2418, 2009.

[4] R. M. Alexander, *Principles of animal locomotion*. Princeton University Press, 2006.

[5] G. B. Ermentrout and N. Kopell, "Inhibition-produced patterning in chains of coupled nonlinear oscillators," *SIAM Journal on Applied Mathematics*, vol. 54, no. 2, pp. 478–507, 1994.

[6] R. Calleja and Y. Sire, "Travelling waves in discrete nonlinear systems with non-nearest neighbour interactions," *Nonlinearity*, vol. 22, no. 11, p. 2583, 2009.

[7] T. Roska and L. Chua, "The CNN universal machine: an analogic array computer," *Circuits and Systems II: Analog and Digital Signal Processing, IEEE Transactions on*, vol. 40, no. 3, pp. 163 –173, 1993.

[8] T. Nishio and Y. Nishio, "Periodic pattern formation and its applications in cellular neural networks," *Circuits and Systems I: Regular Papers, IEEE Transactions on*, vol. 55, no. 9, pp. 2736 –2742, 2008.

[9] P. Arena, S. Baglio, L. Fortuna, and G. Manganaro, "Self-organization in a two-layer CNN," *Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on*, vol. 45, no. 2, pp. 157 –162, 1998.

[10] W. Lohmiller and J.-J. E. Slotine, "Contraction analysis for nonlinear systems," *Automatica*, vol. 34, no. 6, pp. 883–696, 1998.

[11] M. Gilli, F. Corinto, and P. Checco, "Periodic oscillations and bifurcations in cellular nonlinear networks," *Circuits and Systems I: Regular Papers, IEEE Transactions on*, vol. 51, no. 5, pp. 948 – 962, 2004.

[12] T. Kozek, L. Chua, T. Roska, D. Wolf, R. Tetzlaff, F. Puffer, and K. Lotz, "Simulating nonlinear waves and partial differential equations via CNN. I. Basic techniques," *Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on*, vol. 42, no. 10, pp. 807–815, 1995.

[13] M. Gilli, T. Roska, L. O. Chua, and P. P. Civalleri, "CNN dynamics represents a broader class than PDEs," *International Journal of Bifurcation and Chaos*, vol. 12, no. 10, pp. 2051–2068, 2002.

[14] R. Megginson, *An introduction to Banach space theory*, ser. Graduate texts in mathematics. Springer, 1998, vol. 183.

[15] C. Gasquet, R. D. Ryan, and P. Witomsk, *Fourier Analysis and Applications: Filtering, Numerical Computation, Wavelets*. New York: Springer-Verlag, 1999.

[16] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, 2002.

[17] M. Gilli, "Analysis of periodic oscillations in finite-dimensional CNNs through a spatio-temporal harmonic balance technique," *International Journal of Circuit Theory and Applications*, vol. 25, no. 4, pp. 279 – 288, 1997.

[18] E. Zeidler, *Nonlinear Functional Analysis and its Applications: Part 1: Fixed-Point Theorems*. Springer, 1985.

[19] J. C. Sprott, *Chaos and time-series analysis*. Oxford University Press, 2003.

[20] R. Genesio and A. Tesi, "Harmonic balance methods for the analysis of chaotic dynamics in nonlinear systems," *Automatica*, vol. 28, no. 3, pp. 531 – 548, 1992.